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Some Results on f-Simultaneous Chebyshev Approximation

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Abstract

Let X be a Hausdorff topological vector space and f be a real valued continuous function on X. In this paper we introduce and study the concept of $f-$simultaneous approximation of a nonempty subset K of X as a generalization to the problem of simultaneous approximation. Further we present some results regarding $f-$simultaneous approximation in the quotient space.

Keywords: Hausdorff topological vector space; f-best simultaneous approximation; f-simultaneous Chebyshev; simultaneous approximation; quotient space

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1. Introduction

Let K be a subset of a Hausdorff topological vector space X and f be a real valued continuous function on X. For $x \in X$, set $F_K(x) = \inf\limits_{k\in K} f(x-k)$. A point $k_0 \in K$ is called $f-$best approximation to x in K if $F_K(x) = f(x-k_0)$. The set $P_K^f(x) = \{k_0 \in K : F_K(x) = f(x-\kappa_0)\}$ denotes the set of all $f-$best approximations to x in K. Note that this set may be empty. The set K is said to be $f-$proximinal ($f-$Chebyshev) if for each $x \in X$, $P_K^f(x)$ is non-empty (singleton). The notion of $f-$best approximation in a vector space X was given by Breckner and Brosowski and in a Hausdorff topological vector space X by Narang. For a Hausdorff locally convex topological vector space and a continuous sublinear functional f on X, Breckner, Brosowski, and Govindarajulu proved certain results on best approximation relative to the functional f. By using the existence of elements of $f-$best approximation some results on fixed point were proved by Pai and Veermani.

As a generalization to the problem of simultaneous approximation (see Saidi and Singer), we introduce the concept of best $f-$simultaneous approximation as follows:

Definition 1.

Let f be a real valued continuous function on a Hausdorff topological vector space X. A subset A of X is called $f-$bounded if there exists $M > 0$ such that $|f(x)| \leq M$ every $x \in A$.

Note that $f-$bounded sets need not be bounded in the classical sense, for example if $f(x) = e^{-x}$, the set $[0, \infty)$ is an $f-$bounded subset of real numbers.

Definition 2.

Let X be a Hausdorff topological real vector space, f be a real valued continuous function on X, and K be a non-empty subset of X. A point $k_0 \in K$ is called $f-$best simultaneous approximation in K if there exists an $f-$bounded subset A of X such that

$$F_K(A) = \inf_{k\in K} \sup_{a\in A} |f(a-k)| = \sup_{a\in A} |f(a- \ k_0)| .$$

The set of all $f-$best simultaneous approximations to an $f-$bounded subset A of X in K is denoted by

$$P_K^f(A) = \left\{ k \in K : F_K(A) = \sup_{a\in A} |f(a- \ k)| \right\} .$$

The set K is called $f-$simultaneously proximinal $(f-$ simultaneously Chebyshev$)$ if for each $f-$bounded set A in X, $P_K^f(A) \neq \phi$ (singleton) .

We note that if $f(x) = \|x\| \ (f(x) = \|x\| + \epsilon)$, then the concept of $f-$best approximation is precisely best approximation, i.e. best ϵ-approximation (see Khalil, Rezapour, Singer and others).

A set K is said to be inf-compact at a point $x \in X$, (see Pai and Veermani), if each minimizing sequence in K (i.e. $f(x - k_n) \to F_K(x)$) has a convergent subsequence in K. The set K is called inf-compact if it is inf-compact at each $x \in X$. A subset K of X is called $f-$compact, (see Moghaddam), if for every sequence $\{k_n\}$ in K, there exist a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ and $k_0 \in K$ such that $f(k_{n_i} - k_0) \to 0$. It is easy to see that if K is $f-$compact or $\inf$-compact, then K is $f-$simultaneously proximinal.

In this paper we introduce and study the concept of $f-$simultaneous approximation of a subspace K of a Hausdorff topological real vector space X, and existence and uniqueness. Certain results regarding $f-$simultaneous approximation in quotient spaces is obtained by generalizing some of the results in Moghaddam.

Throughout this paper X is a Hausdorff topological real vector space and f is a real valued continuous function on X.

2. f-Simultaneous Approximation

In this section we give some characterization of $f-$proximinal sets in X. We begin with the following definitions:

Definition 3.

A function $f : X \to \mathbb{R}$ is called
(1) absolutely subadditive if $|f(x+y)| \leq |f(x)| + |f(y)|$ for all $x, y \in X$.
(2) absolutely homogeneous if $f(\alpha x) = |\alpha| \, f(x)$, for all $x \in X$ and all $\alpha \in \mathbb{R}$.

Definition 4.

A subset K of X is called $f-$closed if for all sequences $\{k_m\}$ of K and for all $x \in X$ such that $f(x - k_m) \to 0$, we have $x \in K$.

Theorem 1.

Let K be a subset of X. Then,
(1) $F_{K+y}(A+y) = F_K(A)$, for all $f-$bounded sets $A \subset X$, $y \in X$.
(2) $P^f_{K+y}(A+y) = P^f_K(A) + y$, for all $f-$bounded sets $A \subset X$, $y \in X$.
(3) K is $f-$simultaneously proximinal ($f-$simultaneously Chebyshev) if and only if $K+y$ is $f-$simultaneously proximinal ($f-$simultaneously Chebyshev) for every $y \in X$.

Moreover if f is an absolutely homogeneous function, then
(4) $F_{\lambda K}(\lambda A) = |\lambda| \, F_K(A)$, for all $f-$bounded sets $A \subset X$ and $\lambda \in \mathbb{R}$.
(5) $P^f_{\lambda K}(\lambda A) = \lambda P^f_K(A)$, for all $f-$bounded sets $A \subset X$ and $\lambda \in \mathbb{R}$.
(6) K is $f-$simultaneously proximinal ($f-$ simultaneously Chebyshev) if and only if λK is $f-$simultaneously proximinal ($f-$simultaneously Chebyshev), $\lambda \in \mathbb{R}$.

Proof:

(1) Let $A \subset X$, $f-$bounded set. Then

$$F_{K+y}(A+y) = \inf_{w \in K} \sup_{a \in A} |f((a+y) - (w+y))| = F_K(A).$$

(2) The equation

$$\sup_{a \in A} |f(a - k_0)| = \inf_{k \in K} \sup_{a \in A} |f((a+y) - (k+y))| = \inf_{k \in K} \sup_{a \in A} |f(a-k)| ,$$

implies that $k_0 + y \in P^f_{K+y}(A+y)$ if and only if $k_0 \in P^f_K(A)$. Thus

$$P^f_{K+y}(A+y) = P^f_K(A) + y.$$

(3) This follows immediately from part two.
(4) Let $A \subset X$ be an $f-$bounded set, $\lambda \in \mathbb{R}$. Then

$$F_{\lambda K}(\lambda A) = \inf_{k \in K} \sup_{a \in A} |f(\lambda a - \lambda k)| = |\lambda| \inf_{k \in K} \sup_{a \in A} |f(a-k)| = |\lambda| \, F_K(A).$$

(5) If $\lambda = 0$, we are done. If $\lambda \neq 0$ and $k_0 \in P^f_{\lambda K}(\lambda A)$, then $k_0 \in \lambda K$ and

$$\sup_{a\in A} |f(\lambda a - k_0)| = \inf_{k\in K}\sup_{a\in A} |f(\lambda a - \lambda k)| .$$

This implies that

$$\sup_{a\in A} \left| f(a - \frac{1}{\lambda}k_0) \right| = F_K(A),$$

which implies that $\frac{1}{\lambda}k_0 \in P^f_K(A)$.

(6) This follows immediately from part 5. □

Theorem 2.

Let f be an absolutely homogeneous real valued function on X and M be a subspace of X. Then,

(1) $F_M(\lambda A) = |\lambda|\, F_M(A)$, for all f-bounded sets $A \subset X$ and $\lambda \in \mathbb{R} - \{0\}$.

(2) $P^f_M(\lambda A) = \lambda P^f_M(A)$, for all f-bounded sets $A \subset X$ and $\lambda \in \mathbb{R} - \{0\}$.

Proof:

(1) Let $A \subset X$ be an f-bounded set and $\lambda \neq 0 \in \mathbb{R}$. Then,

$$F_M(\lambda A) = \inf_{m\in M}\sup_{a\in A} |f(\lambda a - m)| = |\lambda| \inf_{m'\in M}\sup_{a\in A} \left| f(a - m') \right| = |\lambda|\, F_M(A).$$

(2) Let $m_0 \in P^f_M(\lambda A)$. Then,

$$\begin{aligned} \sup_{a\in A} |\lambda| \left| f(a - \frac{1}{\lambda}m_0) \right| &= \sup_{a\in A} |f(\lambda a - m_0)| \\ &= \inf_{m\in M}\sup_{a\in A} |f(\lambda a - m)| \\ &= \inf_{m'\in M}\sup_{a\in A} |\lambda| \left| f(a - m') \right| . \end{aligned}$$

Therefore,

$$\sup_{a\in A} \left| f(a - \frac{1}{\lambda}m_0) \right| = \inf_{m'\in M}\sup_{a\in A} \left| f(a - m') \right| = F_M(A),$$

for all $\lambda \in \mathbb{R} - \{0\}$, which implies that $\frac{1}{\lambda}m_0 \in P^f_M(A)$, and so $m_0 \in \lambda\, P^f_M(A)$. □

For a subset K of X, let us define $\widehat{K_F}$ such that

$$\widehat{K_F} = \left\{ A \subset X : F_K(A) = \sup_{a\in A} f(a) \right\}.$$

Using this we prove the following theorem characterizing $f-$simultaneously proximinal sets.

Theorem 3.

Let K be a subspace of X. Then K is $f-$simultaneously proximinal in X if and only if every $f-$bounded subset A of X can be written as $B + k$ for some $k \in K$ and $B \in \widehat{K_F}$.

Proof:

Suppose the condition hold. Let $A \subset X$ be an $f-$bounded subset of X. By assumption there exists $k_0 \in K$ and $B \in \widehat{K_F}$ such that $A = B + k_0$. Hence $A - k_0 \in \widehat{K_F}$. Therefore,

$$\begin{aligned}\sup_{a\in A}|f(a-k_0)| &= F_K(A-k_0)\\ &= \inf_{k\in K}\sup_{a\in A}|f(a-k_0-k)|\\ &= \inf_{k'\in K}\sup_{a\in A}|f(a-k')| = F_K(A).\end{aligned}$$

Hence, K is $f-$simultaneously proximinal.

Conversely, suppose K is $f-$simultaneously proximinal and $A \subset X$ be an $f-$bounded subset of X. Then there exists $k_0 \in K$ such that

$$\sup_{a\in A}|f(a-k_0)| = \inf_{k\in K}\sup_{a\in A}|f(a-k)| = \inf_{k'\in K}\sup_{a\in A}\left|f(a-(k^{'}+k_0))\right|$$

where $k = k' + k_0$. Hence,

$$\sup_{a\in A}|f(a-k_0)| = F_K(A-k_0).$$

Consequently, $A - k_0 \in \widehat{K_F}$. So there exists $B \in \widehat{K_F}$ such that $A - k_0 = B$ or $A = B + k_0$. $\square$

Theorem 4.

Let f be a real valued continuous function on X such that $x = 0$ if and only if $f(x) = 0$. If K is $f-$simultaneously proximinal, then K is $f-$closed.

Proof:

Let $\{k_m\}$ be a sequence of K and $x \in X$, such that $f(x - k_m) \to 0$. Taking $A = \{x\}$, we have

$$F_K(A) = \inf_{k\in K}\sup_{a\in A}|f(a-k)| \leq |f(x-k_m)| \to 0.$$

Since K is $f-$simultaneously proximinal, there exists $k_0 \in K$ such that

$$F_K(A) = |f(x-k_0)| = 0.$$

Hence, $f(x - k_0) = 0$. Using assumption it follows that $x - k_0 = 0$. Therefore, $x = k_0 \in K$ and K is $f-$closed. $\square$

3. $f-$Simultaneous Approximation in Quotient Space

Let M be a closed subspace of X. Then a function $\widetilde{f} : (X/M) \to \mathbb{R}$ can be defined as follows:

$$\widetilde{f}(x+M) = \inf_{y\in M}|f(x+y)|.$$

Proposition 1.

Let M be a closed subspace of X. If A is $f-$bounded in X, then A/M is $\widetilde{f}-$bounded in X/M.

Proof:

Let A be an $f-$bounded subset in X. Since M is a subspace, for $x+M \in A/M$

$$\left|\widetilde{f}(x+M)\right| = \inf_{y\in M} |f(x+y)| \leq |f(x)|.$$

Consequently since A is an $f-$bounded subset of X, it follows that A/M is $\widetilde{f}-$bounded in X/M. $\square$

Theorem 5.

Let M a closed subspace of X. If B is $\widetilde{f}-$bounded in X/M, then there exists an $f-$bounded subset A of X such that $B = A/M$.

Proof:

Let B be a nonempty $\widetilde{f}-$bounded in X/M. Let $C = \bigcup\limits_{b\in B} b$.

Claim: $B = \{\overline{x} = x + M : x \in C\}$. Indeed if $b \in B$, then $b = x_b + M$ for some $x_b \in X$. But M is a subspace. Thus $x_b = x_b + 0 \in x_b + M \subset C$. Hence $b = x_b + M \in \{\overline{x} = x + M : x \in C\}$ and $B \subseteq \{\overline{x} = x + M : x \in C\}$. Similarly if $x \in C$, then $x \in b_x + M$ for some $b_x + M \in B$. This implies that $x = b_x + m_x$ for some $m_x \in M$. Hence $x + M = b_x + m_x + M = b_x + M \in B$. Therefore, $\{\overline{x} = x + F : x \in C\} \subseteq B$.

Now clearly C is not bounded unless M is trivial. Note that B is $\widetilde{f}-$bounded. So there exists $K > 0$ such that $\left|\widetilde{f}(b)\right| \leq K$ for all $b \in B$. Consider the set $A = \{x \in C : |f(x)| \leq K + 1\} \subseteq C$.

Now we claim that for all $x \in C$,

$$\overline{x} \cap A = (x + M) \cap A \neq \phi.$$

Given $x \in C$. Since

$$\left|\widetilde{f}(x+M)\right| = \inf_{m\in M} |f\,(x+m)| \leq K,$$

there exists $m_x \in M$ such that $|f\,(x + m_x)| < K+1$. But $x+m_x \in x+M \subseteq C$. Hence $x+m_x \in (x + M) \cap A \neq \phi$. Claim $B = A/M$. Since $A \subseteq C$, we have $A/F \subseteq \{\overline{x} = x + F : x \in C\} = B$. To show the other inclusion, let $b \in B = \{\overline{x} = x + M : x \in C\}$. Then $b = x_b + M$ for some $x_b \in C$. But $(x_b + M) \cap A \neq \phi$. Thus there exists $a \in A$ such that $a = x_b + m_a \in x_b + M$. Therefore, $b = x_b + M = (x_b + m_a) + M = a + M \in A/M$. Hence $B \subseteq A/M$. Consequently $A/M = B$. $\square$

Theorem 6.

Let K be a subspace of X and M be a closed $f-$proximinal subspace of K. If k_0 is a point of $f-$best simultaneous approximation to $A \subset X$ in K, then $k_\circ + M$ is an $\widetilde{f}-$best simultaneous approximation to A/M in K/M.

Proof:

Suppose $k_0 + M$ is not $\widetilde{f}-$best simultaneous approximation to A/M in K/M. Then, for at least $k \in K$, say $k_1 \in K$, we have

$$\sup_{a\in A}\widetilde{f}(a-k_1+M) < \sup_{a\in A}\widetilde{f}(a-k_0+M).$$

Since

$$\sup_{a\in A}\widetilde{f}(a-k_0+M) = \sup_{a\in A}\inf_{m\in M}|f(a-k_0+m)| \le \sup_{a\in A}|f(a-k_0)|,$$

we have

$$\sup_{a\in A}\widetilde{f}(a-k_1+M) = \sup_{a\in A}\inf_{m\in M}|f(a-k_1+m)| < \sup_{a\in A}|f(a-k_0)|.$$

But M is $f-$proximinal, so for some $m_0 \in M$ we have

$$\sup_{a\in A}|f(a-k_1+m_0)| = \sup_{a\in A}\inf_{m\in M}|f(a-k_1+m)| < \sup_{a\in A}|f(a-k_0)|.$$

Since $M \subset K$, it follows that $k_1 - m_0 \in K$. Therefore, k_0 not $f-$best simultaneous approximation to A in K, which is a contradiction. □

Corollary 1.

Let K be a subspace of X and M is a closed $f-$proximinal subspace of K. If K is $f-$simultaneously proximinal in X, then K/M is $\widetilde{f}-$simultaneously proximinal in X/M.

Proof:

Let B be an $\widetilde{f}-$bounded subset of X/M. Then, by Theorem 5, there exists $f-$bounded subset $A \subset X$ such that $B = A/M$. If K is $f-$simultaneously proximinal in X, then there exists at least $k_0 \in K$ such that k_0 is $f-$best simultaneous approximation to A in K. By Theorem 6, $k_0 + M$ is an $\widetilde{f}-$best simultaneous approximation to A/M in K/M, so K/M is $\widetilde{f}-$simultaneously proximinal in X/M. □

Theorem 7.

Let K be a subspace of X and M is a closed $f-$proximinal subspace of K. If K/M is $\widetilde{f}-$simultaneously proximinal in X/M, then K is $f-$simultaneously proximinal in X.

Proof:

Let A be an $f-$ bounded subset of X. By Proposition 1, A/M is $\widetilde{f}-$bounded in X/M. Since K/M is $\widetilde{f}-$simultaneously proximinal in X/M, then there exists $k_0 + M \in K/M$ such that $k_0 + M$ is $\widetilde{f}-$best simultaneous approximation to A/M from K/M, so

$$\begin{aligned}
\sup_{a\in A}\widetilde{f}(a-k_0+M) &= \inf_{k\in K}\sup_{a\in A}\widetilde{f}(a-k+M) \\
&= \inf_{k\in K}\sup_{a\in A}\inf_{m\in M}|f(a-k+m)| \\
&\le \inf_{k\in K}\sup_{a\in A}|f(a-k+m)| \\
&= \inf_{k\in K}\sup_{a\in A}|f(a-k')|, \qquad (1)
\end{aligned}$$

where, $k' = k - m \in K$. Since M is $f-$proximinal, there exists $m_0 \in M$ such that

$$\sup_{a\in A}|f(a-k_0-m_0)| = \sup_{a\in A}\inf_{m\in M}|f(a-k_0+m)| = \sup_{a\in A}\widetilde{f}(a-k_0+M). \tag{2}$$

Consequently, combining (1) and (2) since $M \subset K$, it follows that

$$\begin{aligned}\sup_{a\in A}|f(a-k_0-m_0)| &\leq \inf_{k'\in K}\sup_{a\in A}|f(a-k')| \\ &\leq \sup_{a\in A}|f(a-k_0-m_0)|\end{aligned}$$

Hence,

$$\sup_{a\in A}|f(a-k_0+m_0)| = \inf_{k\in K}\sup_{a\in A}|f(a-k')|$$

So $k_0 + m_0$ is an $f-$best simultaneous approximation to A from K and K is $f-$simultaneously proximinal in X. $\square$

Theorem 8.

Let W and M be two subspaces of X. If M is a closed $f-$proximinal subspace of X, then the following assertions are equivalent:
(1) W/M is $\widetilde{f}-$simultaneously proximinal in X/M,
(2) $W + M$ is $f-$simultaneously proximinal in X.

Proof:

(1) $\Rightarrow$ (2). Since $(W+M)/M = W/M$ and M are $f-$simultaneously proximinal, using Theorem 7, it follows that $W + M$ is $f-$simultaneously proximinal in X.

(2) $\Rightarrow$ (1). Since $W + M$ is $f-$simultaneously proximinal and $M \subseteq W + M$, by Corollary 1, $(W + M)/M = W/M$ is simultaneously $f-$proximinal. $\square$

Theorem 9.

Let K, M be two subspaces of X such that, $M \subset K$. If M is closed $f-$simultaneously proximinal in X and K is f-simultaneously Chebyshev in X, then, K/M is $\widetilde{f}-$simultaneously Chebyshev in X/M.

Proof:

Suppose not. Then there exists A, $f-$bounded subset of X such that $A/M \in X/M$ is $\widetilde{f}-$bounded and $k_1 + M$, $k_2 + M \ \in P^{\widetilde{f}}_{K/M}(A/M)$ such that $k_1 + M \neq k_2 + M$. Thus $k_1 - k_2 \notin M$. Since M is an $f-$simultaneously proximinal in X, then

$$P^f_M(A-k_1) \neq \phi, \text{ and } P^f_M(A-k_2) \neq \phi.$$

Let $m_1 \in P^f_M(A-k_1)$, and $m_2 \in P^f_M(A-k_2)$. By Theorem 7, $k_1+ m_1$ and $k_2 + m_2$ are $f-$best simultaneous approximations to A from K. Since K is $f-$simultaneously Chebyshev in X, then $k_1 + m_1 = k_2 + m_2$ and hence $k_1 - k_2 = m_1 - m_2 \in M$, which is a contradiction. $\square$

Definition 5.

A subset K of X is called $f-$quasi-simultaneously Chebyshev if $P_K^f(A)$ is nonempty and f-compact set in X for all $f-$bounded subsets of X.

Theorem 10.

Let M be a closed $f-$simultaneously proximinal subspace of X and K is $f-$quasi-simultaneously Chebyshev of X such that $M \subset K$. Then, K/M is $\widetilde{f}-$quasi-simultaneously Chebyshev in X/M.

Proof:

Since K is $f-$simultaneously proximinal in X, By Corollary 1, K/M is $\widetilde{f}-$simultaneously proximinal in X/M. Let B be an $\widetilde{f}-$bounded subset of X/M. Then, by Theorem 5, $B = A/M$ for an $f-$bounded subset A of X. If $(k_n + M)$ a sequence in $P_{K/M}^{\widetilde{f}}(A/M)$, by the proof of Theorem 7, for every n, there exists $m_n \in M$ such that $k_n + m_n = k'_n \in P_K^f(A)$. But since M is a subspace, we have

$$k'_n + M = k_n + m_n + M = k_n + M.$$

Since K is $f-$quasi-simultaneously Chebyshev in X, the sequence $\{k_n\}$ has a subsequence $\{k_{ni}\}$ such that $f(k_{ni} - k_0) \to 0$ for some $k_0 \in P_K^f(A)$. But

$$\widetilde{f}(k_{ni} - k_0 + M) \leq |f(k_{ni} - k_0)| \to 0.$$

Therefore,

$$\widetilde{f}(k_{ni} - k_0 + M) \to 0$$

and

$$\widetilde{f}((k_{ni} + M) - (k_0 + M)) \to 0.$$

Hence, $P_{K/M}^{\widetilde{f}}(A/M)$ is $\widetilde{f}$-compact and K/M is $\widetilde{f}-$quasi-simultaneously Chebyshev. This complete the proof. $\square$

Definition 6.

A topological vector space X is said to have the $f-$ property if every $f-$bounded sequence in X has an $f-$ convergent subsequence, where f is a real valued continuous function on X.

Note that the space $X = l^2$ has the $f-$property for every projection $f : X \to \mathbb{R}$, and if $f(x) = \|x\|$, then every finite dimensional Banach space has the $f-$property.

Proposition 2.

Let f be an absolutely homogeneous subadditive continuous real valued function on a topological vector space X and K be an $f-$closed subspace of X. Then, for any $f-$bounded subset A of X, $P_K^f(A)$ is $f-$closed.

Proof:

Let K be an $f-$closed subspace of X and A be an $f-$bounded subset of X. If $\{k_m\}$ is a sequence in $P_K^f(A)$ and $x \in X$ such that $f(k_m - x) \to 0$, then $x \in K$ since K is $f-$closed.

Further,

$$\begin{aligned}\inf_{k\in K}\sup_{a\in A}|f(a-k)| &= \sup_{a\in A}|f(a-k_m)| \\ &= \sup_{a\in A}|f((a-x)-(k_m-x))| \\ &\geq \sup_{a\in A}||f\,(a-x)|-|f\,(k_m-x)||\,.\end{aligned}$$

Taking the limit as $m \to \infty$, we get

$$\inf_{k\in K}\sup_{a\in A}|f(a-k)| \geq \sup_{a\in A}|f\,(a-x)|\,.$$

Consequently,

$$\inf_{k\in K}\sup_{a\in A}|f(a-k)| = \sup_{a\in A}|f\,(a-x)|\,.$$

Then, $x \in P_K^f(A)$ and $P_K^f(A)$ is $f-$closed. $\square$

Theorem 11.

Let f be a real valued sub-additive continuous function on a topological vector space X that has the $f-$property and M be a closed subspace of X. If W is a subspace of X such that $W+M$ is $f-$closed, then the following assertions are equivalent:
(1) W/M is $\widetilde{f}-$simultaneously quasi-Chebyshev in X/M.
(2) $W+M$ is $f-$simultaneously quasi-Chebyshev in X.

Proof:

(1) $\Rightarrow$ (2) Since M is $f-$simultaneously proximinal by Theorem 8, $W+M$ is $f-$simultaneously proximinal in X. Let A be an arbitrary $f-$bounded set in X. Then $P_{W+M}^f(A) \neq \phi$. Now to show that $P_{W+M}^f(A)$ is $f-$compact, we need to show that every sequence in $P_{W+M}^f(A)$ has an $f-$convergent subsequence. Let $\{g_n\}_{n=1}^{\infty}$ be an arbitrary sequence in $P_{W+M}^f(A)$. Then by Theorem 6, for each $n > 1$, $g_n + M \in P_{(W+M)/M}^{\widetilde{f}}(A/M)$. Since $P_{(W+M)/M}^{\widetilde{f}}(A/M)$ is $\widetilde{f}-$compact, one can choose $g_0 \in W+M$ with $g_0 + M \in P_{(W+M)/M}^{\widetilde{f}}(A/M)$ and $\{g_{n_k}+M\}_{k=1}^{\infty}$ is $\widetilde{f}-$convergent to g_0+M for some subsequence $\{g_{n_k}+M\}_{k=1}^{\infty}$ of $\{g_n+M\}_{n=1}^{\infty}$. That means,

$$\widetilde{f}\,(g_0-g_{n_k}+M) = \inf_{m\in M}|f\,(g_0-g_{n_k}-m)| \to 0.$$

Now, since M is $f-$proximinal in X, there exists $m_{n_k} \in M$ such that $m_{n_k} \in P_M^f\,(g_0-g_{n_k})$, for every $k \geq 1$, and hence

$$|f\,(g_0-g_{n_k}-m_{n_k})| = \inf_{m\in M}|f\,(g_0-g_{n_k}-m)|\,.$$

Therefore,

$$\lim_{k\to\infty} f\,(g_0-g_{n_k}-m_{n_k}) = 0.$$

On the other hand, $\{g_{n_k}\}_{k=1}^{\infty}$ is an $f-$bounded sequence because $g_n \in P_{W+M}^f(A)$. In fact $|f\,(g_n)| \leq 2\sup\limits_{a\in A}|f(a)|$. Since M has the $f-$property, with out loss of generality, we may assume that for some $m_0 \in M$, $f(m_{n_k} - m_0) \to 0$. Let $g' = g_0 - m_0$. Then, $g' \in W + M$ and

$$\begin{aligned} f(g' - g_{n_k}) &= f(g_0 - m_0 - g_{n_k}) \\ &\leq f(g_0 - g_{n_k} - m_{n_k}) + f(m_{n_k} - m_0), \end{aligned}$$

$\forall\ k \geq 1$. Thus, $\lim\limits_{k\to\infty} f(g' - g_{n_k}) = 0$. Since $\{g_{n_k}\}_{k=1}^{\infty} \in P^f_{W+M}(A)$, for every $k \geq 1$, and $P^f_{W+M}(A)$ is $f-$closed, since $W + M$ is $f-$closed by Proposition 19, we conclude that $g' \in P^f_{W+M}(A)$. Hence, $P^f_{W+M}(A)$ is $f-$compact.

(2) $\Rightarrow$ (1) Since M and $W + M$ are $f-$simultaneously proximinal and $M \subseteq W + M$, then $(W + M)/M = W/M$ is $\widetilde{f}-$simultaneously proximinal in X/M.

Now, let A be an arbitrary $f-$bounded set in X. Then, $P^{\widetilde{f}}_{W/M}(A/M)$ is non-empty. So from the hypothesis we have $W + M$ is $f-$simultaneously quasi-Chebyshev in X, and hence $P^f_{W+M}(A)$ is $f-$compact in X. Using Theorem 6, we conclude that

$$P^{\widetilde{f}}_{(W+M)/M}(A/M) = \pi\left(P^f_{W+M}(A)\right),$$

where $\pi : X \to X/M$, $\pi(x) = x + M$, is continuous. Consequently $P^{\widetilde{f}}_{W/M}(A/M)$ is $\widetilde{f}-$compact. Therefore, W/M is $f-$simultaneously quasi-Chebyshev in X. $\square$

Note that Theorem 11 is still true if the restriction $W + M$ is $f-$closed is replaced by the condition that the function $f(x) = 0$ if and only if $x = 0$ and use Theorem 4 to prove that $W + M$ is $f-$closed.

4. Conclusions

In this paper we introduce and study the concept of $f-$simultaneous approximation of a nonempty subset K of Hausdorff topological vector space X, existence and uniqueness as a generalization to the problem of simultaneous approximation in the sense that if the function f is taken to be the usual norm, the problem is turned out to be precisely the problem of best approximation in the usual sense. Further, we obtain some results regarding $f-$simultaneous approximation in the quotient space.

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