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Homotopy Perturbation Method for Solving System of Generalized Abel's Integral Equations

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Abstract

In this paper, a user friendly algorithm based on the homotopy perturbation method (HPM) is proposed to solve a system of generalized Abel's integral equations. The stability of the solution under the influence of noise in the input data is analyzed. It is observed that the approximate solutions converge to the exact solutions. Illustrative numerical examples are given to demonstrate the efficiency and simplicity of the proposed method in solving such types of systems of Abel's integral equations.

Keywords: Homotopy Perturbation Method, System of Generalized Abel's Integral Equations, Abel's Kernel.

MSC (2010) No.: 45E10, 45F15

1. Introduction

The Volterra integral equations were studied by Traian Lalescu in his thesis, Sur les equations de Volterra, written under the supervision of Emile Picard. Lalescu (1912) wrote the first book ever on integral equations. These equations find applications in demography, the study of viscoelastic materials, and in insurance mathematics through renewal equation. Volterra integral equations arise in many problems pertaining to mathematical physics like heat conduction problems. Many interesting problems of mechanics and physics lead to an integral equation in which the kernel $K(x,t)$ is of convolution type, that is, $K(x,t)=k(x-t)$, where k is a certain function of one variable. Recently, systems of Volterra integral equations of convolution type have attracted the attention of many authors and several different methods have been developed to solve these numerically. In particular, we consider the following system of singular Volterra integral equation of convolution type

$$c(x)\ y(x) + \lambda\int_0^x k(x-t)\,a\,y(t)\,dt = f(x), \qquad 0 \le x \le 1, \tag{1}$$

where $c(x) = (c_{ij}(x))_{ij}$, $a = (a_{ij})_{ij}$ are square matrices of order n,

$$y(x) = [y_1(x),\ y_2(x),\ \ldots,\ y_n(x)]^T \text{ and } f(x) = [f_1(x),\ f_2(x),\ldots,\ f_n(x)]^T \tag{2}$$

are column vectors and $\lambda = (\lambda_{ii})_{ii}$ and $k(x-t) = (k_{ii}(x-t))_{ii}$ are diagonal matrices of order n.

In equation (1), the functions k and f are given and y is the vector function of the solution of the system (1) to be determined. Here, we assume that the system (1) has a unique solution. If the domain of definition of the kernel is infinite, or if the kernel has a singularity within its domain of definition, then the integral equation is said to be singular. In certain cases, the kernel is only weakly singular as the singularity may be transformed away by a change of variable. In the Volterra integral equations system of convolution type, if at least one of the integral equations is singular, then the system is called system of singular Volterra integral equations of convolution type. There are several numerical method for solving equation (1) for example, Galerkin, collocation, Taylor series and Taylor polynomials Methods, [Burton (2005), Maleknejad and Kajani (2004), Maleknejad and Aghazadeh (2005) and Yalsinbas (2002)]. Recently a number of algorithms have been proposed based on Legendre wavelets [Malekkejad and Kajani (2003)] Chebyshev polynomials [Malekkejad et al. (2007)], Bernstein polynomials [Pandey et al. (2009)], Expansion method [Rabbani (2007)], Power Series method [Tahmasbi and Fard (2008)] and almost Bernstein operational matrix method [Singh et al. (2010)].

For the special case when the kernel $k(x-t) = \dfrac{1}{(x-t)^{\alpha}}$, where is a diagonal matrix of order n with all the entries lying in (0, 1): the system (1) reduces to a system of generalized Abel's integral equations.

In this paper, we have developed a simple algorithm based on homotopy perturbation method for the numerical solution of system of generalized Abel's integral equations.

2. Basic Idea of Homotopy Perturbation Method

In this method, using the homotopy technique of topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter". This method became very popular amongst the scientists and engineers, even though it involves continuous deformation of a simple problem into a more difficult problem under consideration. Most of the perturbation methods depend on the existence of a small perturbation parameter but many nonlinear problems have no small perturbation parameter at all. Many new methods have been proposed in the late nineties to solve such nonlinear equation devoid of such small parameters [Dehghan and Shakeri (2008, 2008), Ganji and Rajabi (2006), He (1999, 1999), Lio (1999, 1997)]. Late 1990s saw a surge in applications of homotopy theory in the scientific and engineering computations [Abbasbandy (2006, 2007), Aminikhah and Salahi (2009), Biazar et al. (2009), Biazar and Eslami (2010)].When the homotopy theory is coupled with perturbation theory it provides a powerful mathematical tool [Ganji et al. (2007), He (1998, 2004), Shakeri and Dehghan (2008)]. A review of recently developed methods of nonlinear analysis can be found in He (2000). To illustrate the basic concept of HPM, consider the following nonlinear functional equation

$$A(u) = f(r), \quad r \in \Omega, \text{ with the boundary conditions: } B\left(u, \frac{\partial u}{\partial n}\right) = 0, \; r \in \partial\Omega, \tag{3}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytic function, and $\partial\Omega$ is the boundary of the domain Ω. The operator A is decomposed as $A = L + N$, where L is the linear and N is the nonlinear operator. Hence, Equation (3) can be written as

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega.$$

We construct a homotopy $v(r,p): \Omega \times [0,1] \to R,$ satisfying

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \quad p \in [0,1], \; r \in \Omega. \tag{4}$$

Hence,

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0, \tag{5}$$

where u_0 is an initial approximation for the solution of equation (3). As

$$H(v,0) = L(v) - L(u_0) \quad \text{and} \quad H(v,1) = A(v) - f(r),$$

it shows that $H(v,p)$ continuously traces an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(v,1)$. The embedding parameter p increases monotonously from zero to one as the trivial linear part $L(u)=0$ deforms continuously to the original problem $A(u)=f(r)$. The embedding parameter $p\in[0,1]$ can be considered as an expanding parameter [He (1999)] to obtain

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{6}$$

The solution is obtained by taking the limit as p tends to 1 in equation (6). Hence,

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{7}$$

The series (7) converges for most cases and the rate of convergence depends on $A(u) - f(r)$, He (1999).

3. Method of Solution

We consider the following system of generalized Abel's integral equation

$$cy(x) = f(x) + \lambda \int_0^x \frac{a\,y(t)}{(x-t)^{\alpha}}dt, \qquad 0 \le x \le 1, \tag{8}$$

which is equivalent to the following set of n equations

$$c_i y_i(x) = f_i(x) + \lambda_i \int_0^x \sum_{j=1}^{n} \frac{a_{ij}\; y_j(t)}{(x-t)^{\alpha_{ij}}}dt, \quad 1 \le i \le n. \tag{9}$$

To solve the above equation, a convex homotopy is constructed as

$$(1-p)\left[L_i(x) - y_{i0}(x)\right] + p\left[L_i(x) - f_i(x) - \lambda_i \int_0^x \sum_{j=1}^{n} \frac{a_{ij}\; L_{ji}(t)}{(x-t)^{\alpha_{ij}}}dt\right] = 0. \tag{10}$$

We seek the solution of (10) in the following form,

$$L_i(x) = \sum_{j=0}^{\infty} p^j\, L_{ij}(x),\; i = 1,2,3,\ldots n, \tag{11}$$

where $L_{ij}(x), i, j = 1,2,3,...$, are the functions to be determined. After choosing the initial approximations $L_{i0}(x) = y_{i0}(x)$ appropriately for $1 \le i \le n$, the following iterative scheme is used to evaluate $L_{ij}(x)$.
Substituting equation (11) in (10) and equating the coefficients of p with the same power, we get

$$p^1 \quad : \quad L_{i1}(x) = \lambda_i \int_0^x \left(\sum_{j=1}^{n} \frac{a_{ij} L_{j0}(t)}{(x-t)^{\alpha_{ij}}} \right) dt ,$$

$$p^2 \quad : \quad L_{i2}(x) = \lambda_i \int_0^x \left(\sum_{j=1}^{n} \frac{a_{ij} L_{j1}(t)}{(x-t)^{\alpha_{ij}}} \right) dt ,$$

$$p^3 \quad : \quad L_{i3}(x) = \lambda_i \int_0^x \left(\sum_{j=1}^{n} \frac{a_{ij} L_{j2}(t)}{(x-t)^{\alpha_{ij}}} \right) dt ,$$

$$\vdots$$

$$p^m : \quad L_{im}(x) = \lambda_i \int_0^x \left(\sum_{j=1}^{n} \frac{a_{ij} L_{j(m-1)}(t)}{(x-t)^{\alpha_{ij}}} \right) dt , \tag{12}$$

Hence, the solution vector y of equation (8) is given by $y = (y_1, y_2, y_3, ... y_n)$, where

$$y_i(x) = \lim_{p \to 1} L_i(x) = \lim_{m \to \infty} \sum_{j=0}^{m} L_{ij}(x). \tag{13}$$

4. Stability Analysis

We consider the stability of the solution components y_i as given by (13) under the influence of noise in the input data $f(x)$. That is, we wish to investigate the effect on the solution $y(x)$ when the input $f(x)$ is corrupted with noise $\delta f(x)$ where $\delta f(x)$ is unknown apart from some restriction on its magnitude relative to $f(x)$.

Assuming $\tilde{y}(x)$ to be the solution of equation (8) under the influence of noise

$$\delta f(x) = (\delta f_1(x), \delta f_2(x), \delta f_3(x), ..., \delta f_n(x)),$$

then

$$\tilde{y}(x) = f(x) + \delta f(x) + \lambda \int_0^x \frac{a\,\tilde{y}(t)}{(x-t)^{\alpha}} dt, \quad 0 \le x \le 1. \tag{14}$$

Choosing the initial approximation $\tilde{L}_0(x) = y_0(x) + \varepsilon_0(x) = L_0 + \varepsilon_0$, where

$$\tilde{L}_0 = \left(\tilde{L}_{10}, \tilde{L}_{20}, \tilde{L}_{30}, ..., \tilde{L}_{n0}\right),\ y_0(x) = \left(y_{10}, y_{20}, y_{30}, ..., y_{n0}\right) \text{ and } \varepsilon_0(x) = \left(\varepsilon_{10}, \varepsilon_{20}, \varepsilon_{30}, ..., \varepsilon_{n0}\right) = \delta f(x),$$

we have, by equation (12)

$$\tilde{L}_1(x) = \lambda \int_0^x \frac{a\left(y_0(t) + \in_0 (t)\right)}{(x-t)^{\alpha}} dt = L_1(x) + \varepsilon_1(x),$$

$$\tilde{L}_2(x) = L_2(x) + \varepsilon_2(x),$$

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$$\tilde{L}_m(x) = L_m(x) + \varepsilon_m(x),$$

where the various components of $L_m(x)$ are given by equation (12) and

$$\varepsilon_m(x) = \lambda \int_0^x \frac{a \in_{m-1}(t)}{(x-t)^{\alpha}} dt, \quad m = 1, 2, 3, \cdots.$$

Thus, the perturbed solution $\tilde{y}(x)$ is given by $\tilde{y}(x) = \lim_{m\to\infty} \sum_{j=0}^{m} \tilde{L}_j(x)$.

The effect of the noise term $\varepsilon_0(x)$ in the input data $f(x)$ deviates the solution by

$$\delta y(x) = \tilde{y}(x) - y(x) = \lim_{m\to\infty} \sum_{i=0}^{m} \left[\tilde{L}_i(x) - L_i(x)\right] = \lim_{n\to\infty} \sum_{i=0}^{n} \varepsilon_i(x). \tag{15}$$

From equation (15), we conclude that $\delta y(x)$ and $\delta f(x)$ are connected via the following generalized Abel integral equation

$$c\delta y(x) = \delta f(x) + \lambda \int_0^x \frac{\delta y(t)}{(x-t)^{\alpha}} dt. \tag{16}$$

Thus, we have proved the following theorem:

Theorem:

The presence of the noise $\delta f(x)$ in the input function $f(x)$ perturbs the solution vector $y(x)$ by an amount equivalent to the solution of the Abel integral equation (16) with input equal to the noise $\delta f(x)$ itself.

As $\delta f(x)$ is not known before hand, we take an upper bound for $\delta f(x)$. Let $\sup\limits_{0<x<1}|\delta f(x)| < \varepsilon$, then (16) reduces to $c\delta y(x) = \varepsilon + \lambda \int_0^x \frac{\delta y(t)}{(x-t)^\alpha} dt,$ which can be readily solved.

5. Illustrative Examples

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the homotopy perturbation method. The simplicity and accuracy of the proposed method is illustrated through the following numerical examples by computing the absolute error $E_i(x) = \left| y_i(x) - \tilde{y}_{im}(x) \right|, 1 \le i \le n,$ where $y_i(x)$ is the exact solution and $\tilde{y}_{im}(x)$ is the approximate solution of the problem when the series (13) is truncated at level $j = m.$

Example 1.

As the first examples, we consider the following system of generalized Abel's integral equations

$$\begin{cases} y_1(x) - y_2(x) + \int\limits_0^x \frac{1}{\sqrt{x-t}} \left(y_1(t) + y_2(t) \right) dt = x - \sqrt{x} + \frac{\pi x}{2} + \frac{4x^{3/2}}{3}, \\ 2y_2(x) - y_1(x) + \int\limits_0^x \frac{1}{\sqrt{x-t}} \left(y_2(t) \right) dt = 2\sqrt{x} - x + \frac{\pi x}{2}, \end{cases} \tag{17}$$

with the exact solutions $y_1(x) = x,$ and $y_2(x) = \sqrt{x}.$ This is a weakly singular system of Volterra integral equations of convolution type.

Choosing the initial approximations $L_{10}(x) = x + \frac{\pi x}{2} + \frac{4x^{3/2}}{3},$ $L_{20}(x) = \sqrt{x} + \frac{\pi x}{4}$ and using the iterative scheme (12), we obtain the various iterates as follows:

$$p^1: \quad L_{11}(x) = -\frac{\pi x}{2} - \frac{4x^{3/2}}{3} - \pi x^{3/2} - \frac{\pi x^2}{2},$$

$$L_{21}(x) = -\frac{\pi x}{4} - \frac{\pi x^{3/2}}{6},$$

$$p^2: \quad L_{12}(x) = \pi x^{3/2} + \frac{\pi x^2}{2} + \frac{7\pi^2 x^2}{16} + \frac{8\pi x^{5/2}}{15},$$

$$L_{22}(x) = \frac{\pi x^{3/2}}{6} + \frac{\pi^2 x^2}{32},$$

$$p^3: \quad L_{13}(x) = -\frac{7\pi^2 x^2}{16} - \frac{8\pi x^{5/2}}{15} - \frac{\pi^2 x^{5/2}}{2} - \frac{\pi^2 x^3}{6}, \ldots.$$

$$L_{23}(x) = -\frac{\pi^2 x^{5/2}}{60} - \frac{\pi^2 x^2}{32}, \ldots.$$

Figures 1 and 2 show the absolute error between the exact solution $y_i(x)$ and the approximate solution $\tilde{y}_{i28}(\mathrm{x})$ obtained by truncating (13) at level $n = 28$ for $i = 1, 2$, respectively, for example.

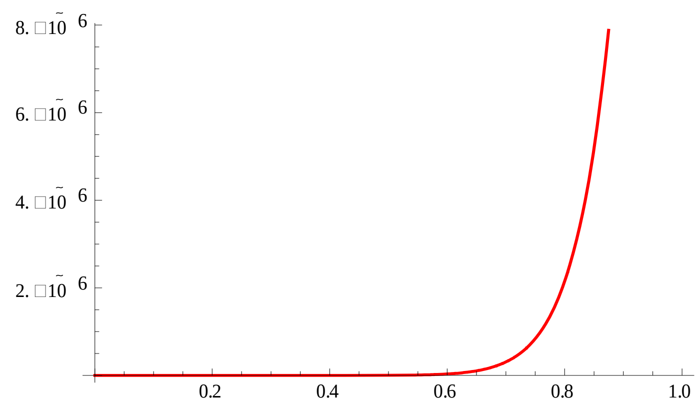

Figure 1. The absolute error $E_1(x)$ for Example 1

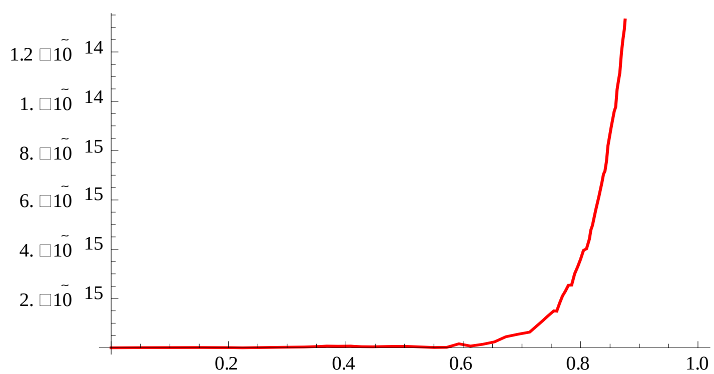

Figure 2. The absolute error $E_2(x)$ for Example 1

Example 2.

In this example, the following system of generalized Abel's integral equations of the second kind is considered.

$$\begin{cases} 2y_1(x)-y_2(x)+\int_0^x \frac{2y_1(t)}{\sqrt{x-t}}\,dt+\int_0^x \frac{y_2(t)}{(x-t)^{1/3}}\,dt+=4\sqrt{x}-x^{1/3}+\frac{2\pi x}{3\sqrt{3}}, \\ y_2(x)-y_1(x)+\int_0^x \frac{y_2(t)}{(x-t)^{1/4}}\,dt=x^{1/3}+e^{\pi x}\,erfc\,(\sqrt{\pi x})-1+\frac{x^{13/12}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{25}{12}\right)}, \end{cases} \tag{18}$$

with the exact solutions $y_1(x)=1-e^{\pi x}\,erfc\,(\sqrt{\pi x})$ and $y_2(x)=x^{1/3}$.

Taking, $L_{10}(x)=2\sqrt{x}+\frac{\pi x}{3\sqrt{3}}$ and $L_{20}(x)=x^{1/3}+\frac{x^{13/12}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{25}{12}\right)}$, the iteration formula (12) gives the various $L_{ij}(x)$ as

$$p^1: \quad L_{11}(x)=-\pi x-\frac{\pi x}{3\sqrt{3}}-\frac{4\pi x^{3/2}}{9\sqrt{3}}-\frac{16\pi x^{7/4}}{63\sqrt{3}},$$

$$L_{21}(x)=-x^{13/12}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)\left[\frac{1}{\Gamma\left(\frac{25}{12}\right)}+\frac{x^{3/4}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{17}{6}\right)}\right],$$

$$p^2: \quad L_{12}(x)=\frac{4\pi x^{3/2}}{3}+\frac{4\pi x^{3/2}}{9\sqrt{3}}+\frac{16\pi x^{7/4}}{63\sqrt{3}}+\frac{\pi^2x^2}{6\sqrt{3}}+\frac{8x^{5/2}}{45}\sqrt{\frac{\pi}{3}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2+\frac{16\pi^{3/2}x^{9/4}\Gamma\left(\frac{11}{4}\right)}{63\sqrt{3}\,\Gamma\left(\frac{13}{4}\right)},$$

$$L_{22}(x)=x^{11/6}\left(\Gamma\left(\frac{3}{4}\right)\right)^2\Gamma\left(\frac{4}{3}\right)\left[\frac{1}{\Gamma\left(\frac{17}{6}\right)}+\frac{x^{3/4}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{43}{12}\right)}\right],$$

$$p^3: \quad L_{13}(x) = -\frac{\pi^2 x^2}{2} - \frac{\pi^2 x^2}{6\sqrt{3}} - \frac{8\pi^2 x^{5/2}}{45\sqrt{3}} - \frac{64\pi^2 x^{11/4}}{693\sqrt{3}} - \frac{8x^{5/2}}{45}\sqrt{\frac{\pi}{3}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2 - \frac{\pi^{3/2}x^3}{18\sqrt{3}}\left(\Gamma\left(\frac{3}{4}\right)\right)^2$$
$$-\frac{16\pi^{3/2}x^{9/4}\Gamma\left(\frac{11}{4}\right)}{63\sqrt{3}\Gamma\left(\frac{13}{4}\right)} - \frac{\pi x^{13/4}\left(\Gamma\left(\frac{3}{4}\right)\right)^3}{3\sqrt{3}\,\Gamma\left(\frac{17}{4}\right)},$$

$$L_{23}(x) = -x^{31/12}\left(\Gamma\left(\frac{3}{4}\right)\right)^3 \Gamma\left(\frac{4}{3}\right)\left[\frac{1}{\Gamma\left(\frac{43}{12}\right)} + \frac{x^{3/4}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{13}{3}\right)}\right], \ldots .$$

Figures 3 and 4 show the absolute errors between the exact solution $y_i(x)$ and the approximate solution $\tilde{y}_{i30}(\mathrm{x})$ for $i = 1, 2$, respectively, for Example 2.

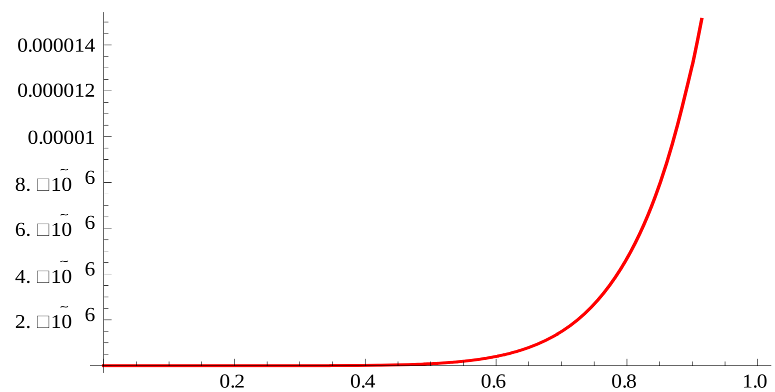

Figure 3. The absolute error $E_1(x)$ for Example 2

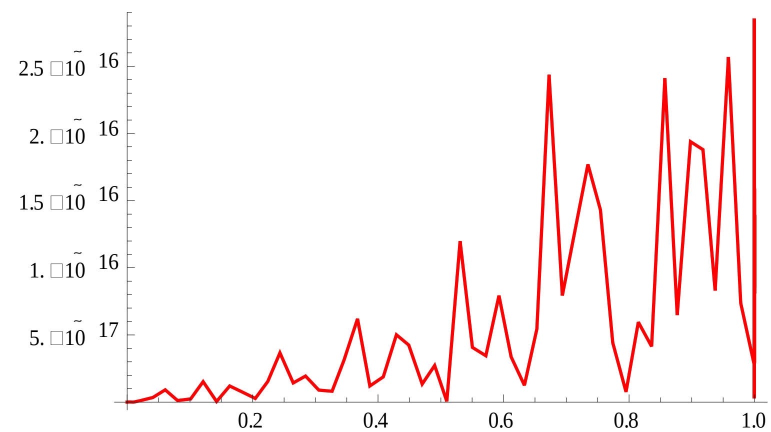

Figure 4. The absolute error $E_2(x)$ for Example 2

Example 3.

In this example, we consider the following system of generalized Abel's integral equations

$$\begin{cases} 35y_1(x)-y_2(x)+\int_0^x \frac{7y_1(t)}{(x-t)^{2/5}}\,dt+\int_0^x \frac{5y_2(t)}{(x-t)^{2/7}}\,dt=35\sin x-x+\frac{49x^{12/7}}{12}+\frac{175x^{8/5}}{24}\,{}_1F_2\left[\{1\},\left\{\frac{13}{10},\frac{9}{5}\right\},-\frac{x^4}{4}\right], \\ 5y_2(x)-y_1(x)+\int_0^x \frac{(y_2(t))}{(x-t)^{1/5}}\,dt=5x-\sin x+\frac{25x^{9/5}}{36}, \end{cases} \tag{19}$$

with the exact solutions $y_1(x)=\sin x$ and $y_2(x)=x.$

What follows is self explanatory.

$$p^0: \quad L_{10}(x)=\sin x+\frac{7x^{12/7}}{60}+\frac{5x^{8/5}}{24}\,{}_1F_2\left[\{1\},\left\{\frac{13}{10},\frac{9}{5}\right\},-\frac{x^4}{4}\right],$$

$$L_{20}(x)=x+\frac{5x^{9/5}}{36},$$

$$p^1: \quad L_{11}(x)=-\frac{7x^{12/7}}{60}-\frac{7x^{81/35}\,\Gamma\left(\frac{3}{5}\right)\Gamma\left(\frac{19}{7}\right)}{300\,\Gamma\left(\frac{116}{35}\right)}-\frac{5x^{88/35}\,\Gamma\left(\frac{5}{7}\right)\Gamma\left(\frac{14}{5}\right)}{252\,\Gamma\left(\frac{123}{35}\right)}-\frac{5x^{8/5}}{24}\,{}_1F_2\left[\{1\},\left\{\frac{13}{10},\frac{9}{5}\right\},-\frac{x^4}{4}\right]$$

$$-\frac{x^{11/5}\left(\Gamma\left(\frac{3}{5}\right)\right)^2}{25\,\Gamma\left(\frac{16}{5}\right)}\,{}_1F_2\left[\{1\},\left\{\frac{8}{5},\frac{21}{10}\right\},-\frac{x^4}{4}\right],$$

$$L_{21}(x)=-\frac{5x^{9/5}}{36}-\frac{x^{13/5}\left(\Gamma\left(\frac{4}{5}\right)\right)^2}{25\,\Gamma\left(\frac{18}{5}\right)},\ldots\ .$$

Figures 5 and 6 show the absolute errors between the exact solution $y_i(x)$ and the approximate solution $\tilde{y}_{i10}(\mathrm{x})$ obtained by truncating (13) at level $n=10$ for $i=1,2,$ respectively, for Example 3.

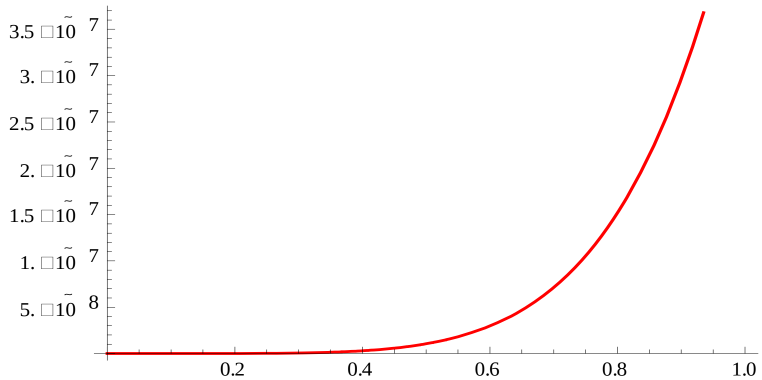

Figure 5. The absolute error $E_1(x)$ for Example 3

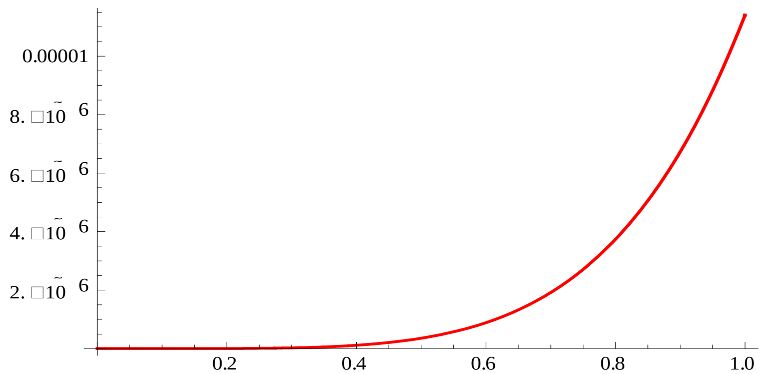

Figure 6. The absolute error $E_2(x)$ for Example 3

Example 4.

Now we consider the following system of generalized Abel's integral equations

$$\begin{cases} 100y_1(x)-y_2(x)-y_3(x)+\dfrac{5}{\pi}\displaystyle\int_0^x \frac{y_1(t)}{\sqrt{(x-t)}}\,dt+\int_0^x \frac{25y_2(t)}{(x-t)^{1/3}}\,dt+\int_0^x \frac{4y_2(t)}{(x-t)^{1/5}}\,dt=100\sqrt{x}-x^{1/3}-x^2+\dfrac{5x}{2}+\dfrac{50\pi x}{3\sqrt{3}}+\dfrac{125\,x^{14/5}}{63}, \\ 36y_2(x)-y_1(x)-y_3(x)+\displaystyle\int_0^x \frac{9y_2(t)}{(x-t)^{1/4}}\,dt++\int_0^x \frac{4y_3(t)}{(x-t)^{1/3}}\,dt=36\,x^{1/3}-\sqrt{x}-x^2+\dfrac{27\,x^{8/3}}{10}+\dfrac{9\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)x^{13/12}}{\Gamma\left(\frac{25}{12}\right)}, \\ 44y_3(x)-y_1(x)-y_2(x)+\displaystyle\int_0^x \frac{11y_1(t)}{(x-t)^{1/6}}\,dt+\int_0^x \frac{4y_3(t)}{(x-t)^{1/7}}\,dt=44x^2-\sqrt{x}-x^{1/3}+\dfrac{343\,x^{20/7}}{195}+\dfrac{11\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)x^{4/3}}{2\Gamma\left(\frac{7}{3}\right)}, \end{cases} \tag{20}$$

with the exact solutions $y_1(x)=\sqrt{x}$, $y_2(x)=x^{1/3}$ and $y_3(x)=x^2$.

$$p^0: \quad L_{10}(x) = \sqrt{x} + \frac{x}{40} + \frac{\pi\, x}{6\sqrt{3}} + \frac{5\,x^{14/5}}{252},$$

$$L_{20}(x) = x^{1/3} + \frac{3\,x^{8/3}}{40} + \frac{x^{13/12}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{4\,\Gamma\left(\frac{25}{12}\right)},$$

$$L_{30}(x) = x^2 + \frac{343\,x^{20/7}}{8580} + \frac{\sqrt{\pi}\,x^{4/3}\,\Gamma\left(\frac{5}{6}\right)}{8\,\Gamma\left(\frac{7}{3}\right)},$$

$$p^1: \quad L_{11}(x) = -\frac{x}{40} - \frac{\pi x}{6\sqrt{3}} - \frac{x^{3/2}}{90\sqrt{3}} - \frac{x^{3/2}}{600\pi} - \frac{2\pi x^{7/4}}{63\sqrt{3}} - \frac{5x^{14/5}}{252} - \frac{3\sqrt{\pi}\,x^{10/3}\,\Gamma\left(\frac{2}{3}\right)}{560\,2^{1/3}\,\Gamma\left(\frac{7}{6}\right)} - \frac{\sqrt{\pi}\,\Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{5}{6}\right)x^{32/15}}{200\Gamma\left(\frac{47}{15}\right)}$$
$$-\frac{x^{33/10}\Gamma\left(\frac{19}{5}\right)}{1008\sqrt{\pi}\,\Gamma\left(\frac{43}{10}\right)} - \frac{343\Gamma\left(\frac{4}{5}\right)\Gamma\left(\frac{27}{7}\right)x^{128/35}}{214500\,\Gamma\left(\frac{163}{35}\right)},$$

$$L_{21}(x) = -\frac{3x^{8/3}}{40} - \frac{\sqrt{\pi}\,x^2}{144}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right) - \frac{x^{13/12}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{4}{3}\right)}{4\Gamma\left(\frac{25}{12}\right)} - \frac{x^{11/6}\left(\Gamma\left(\frac{3}{4}\right)\right)^2\Gamma\left(\frac{4}{3}\right)}{16\Gamma\left(\frac{17}{6}\right)} - \frac{3x^{41/12}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{11}{3}\right)}{160\Gamma\left(\frac{53}{12}\right)}$$
$$-\frac{343x^{74/21}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{27}{7}\right)}{77220\Gamma\left(\frac{95}{21}\right)},$$

$$L_{31}(x) = -\frac{9x^{11/6}}{2200} - \frac{\sqrt{3}\,\pi\,x^{11/6}}{110} - \frac{343x^{20/7}}{8580} - \frac{\sqrt{\pi}\,x^{4/3}\Gamma\left(\frac{5}{6}\right)}{8\Gamma\left(\frac{7}{3}\right)} - \frac{\sqrt{\pi}\,x^{46/21}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{6}{7}\right)}{88\Gamma\left(\frac{67}{21}\right)} - \frac{5x^{109/30}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{19}{5}\right)}{1008\,\Gamma\left(\frac{139}{30}\right)}$$
$$-\frac{2x^{26/7}\left(\Gamma\left(\frac{6}{7}\right)\right)^2}{121\Gamma\left(\frac{33}{7}\right)}, \ldots .$$

Figures 7 and 9 show the absolute errors between the exact solution $y_i(x)$ and the approximate solution $\tilde{y}_{i7}(\mathrm{x})$ obtained by truncating (13) at level $n = 7$ for $i = 1, 2, 3$, respectively, for Example 4.

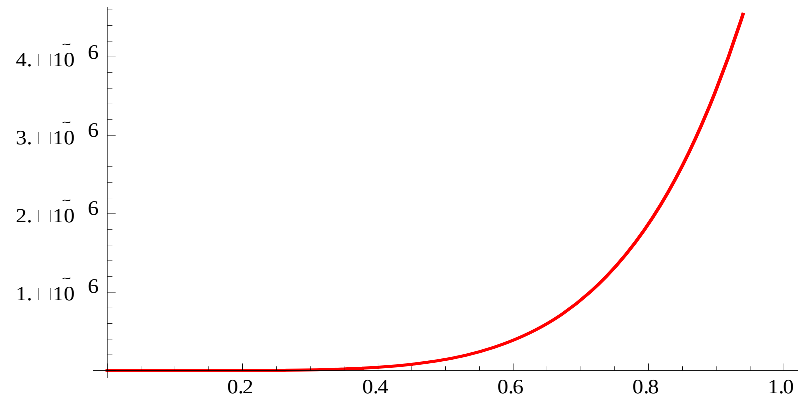

Fig.7. The absolute error $E_1(x)$ for Ex.4.

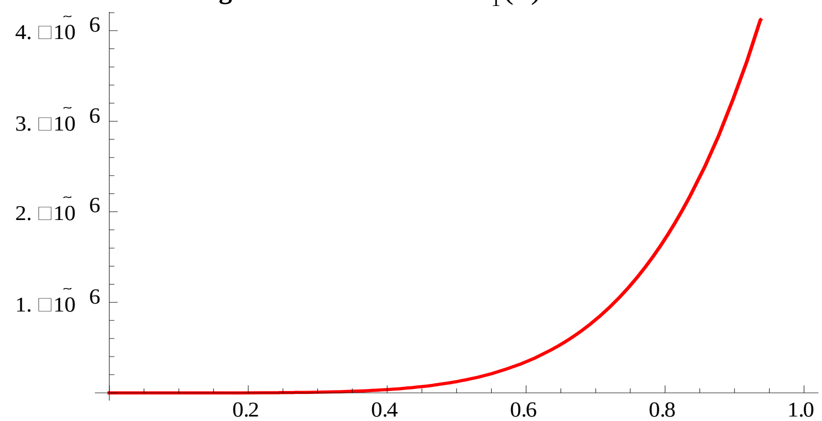

Figure 8. The absolute error $E_2(x)$ for Example 4

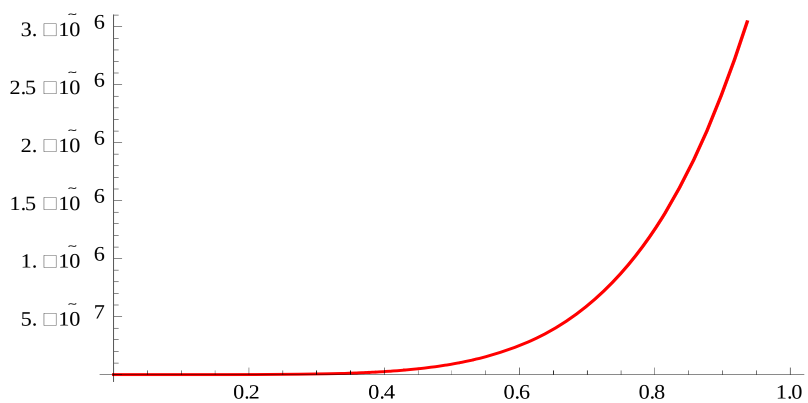

Figure 9. The absolute error $E_3(x)$ for Example 4

5. Conclusion

The Homotopy perturbation method proposed is a new and efficient algorithm for the numerical solution of system of generalized Abel's integral equations. It is proved that the change $\delta y(x)$, in the solution $y(x)$ caused by the presence of noise $\delta f(x)$ in the observable data $f(x)$, is the solution of the generalized Abel integral equation with input data equal to the noise $\delta f(x)$ itself. From the given numerical examples, and Figs 1-9, we conclude that the method is accurate and easy to implement for solving systems of generalized Abel's integral equations especially of the second kind.

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