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On Factorization of a Special type of Vandermonde Rhotrix

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Abstract

Vandermonde matrices have important role in many branches of applied mathematics such as combinatorics, coding theory and cryptography. Some authors discuss Vandermonde rhotrices in the literature for its mathematical enrichment. Here, we introduce a special type of Vandermonde rhotrix and obtain its LR factorization, namely left and right triangular factorization which is further used to obtain the inverse of the rhotrix.

Keywords: Vandermonde Matrix; Vandermonde Rhotrix; Special Vandermonde Rhotrix; Left and Right Triangular Rhotrix

AMS Classification: 15A09, 15A23

1. INTRODUCTION

Vandermonde matrix V is an $l \times m$ matrix with terms of a geometric progression in each row; that is

$$V=\begin{bmatrix}1 & \alpha_1 & \alpha_1^2 & . & \alpha_1^{m-1}\\ 1 & \alpha_2 & \alpha_2^2 & . & \alpha_2^{m-1}\\ 1 & \alpha_3 & \alpha_3^2 & . & \alpha_3^{m-1}\\ . & . & . & . & .\\ 1 & \alpha_l & \alpha_l^2 & . & \alpha_l^{m-1}\end{bmatrix}. \tag{1.1}$$

Due to wide range of applications of Vandermonde matrices in different areas of mathematical sciences as well as other sciences, they have attained much importance; see Lacan and Fimes (2004), Lin and Costello (2004) and Sharma and Rehan (2014). The solutions of the linear system of equations $Vx=b$ have been studied by Bj¨orck and Pereyra (1970) and Tang and Golub (1981). The solutions of the equation $x=V^{-1}b$ lead to the factorizations of V^{-1}, such as Lower and Upper factorizations and 1-banded factorizations. Oruc and Phillips (2000) obtained formula for the LU factorization of V and expressed the matrices L and U as a product of 1-banded matrices. Yang (2004, 2005) modified the results of Oruc and obtained a simpler formula. In recent literature, special generalized Vandermonde matrices have attain great amount of attention. Demmel and Koev (2005) studied totally positive generalized Vandermonde matrices and gave formulae for the entries of the bidiagonal factorization and the LDU factorization. Yang and Holtti (2004) discussed various types of generalized Vandermonde matrices. Li and Tan (2008) discussed the LU factorization of special class of generalized Vandermonde matrices which was introduced by Liu (1968). This matrix arises while solving the equation

$$a_m=c_1a_{m-1}+c_2a_{m-2}+.....+c_pa_{m-p}\,,\quad m\geq p,\ (p \text{ fixed}), \tag{1.2}$$

where $c_1,c_2,...,c_p$ are constants and $c_p\neq 0.$ If Equation (1.2) has distinct real roots $v_1,v_2,...,v_q$ with multiplicities $u_1,u_2,...,u_q$ respectively and $\sum_{i=1}^{q}u_i=m,$ then the corresponding generalized Vandermonde matrix has the following form:

$$V'_{\{q;u_1,u_2,..,u_q\}}=\begin{bmatrix}1 & 0 & ... & 0 & ... & 1 & 0 & ... & 0\\ v_1 & v_1 & ... & v_1 & ... & v_q & v_q & ... & v_q\\ : & : & : & : & : & : & : & : & :\\ v_1^{m-1} & (m-1)v_1^{n-1} & ... & (m-1)^{u_1-1}v_1^{m-1} & ... & v_q^{m-1} & (m-1)v_q^{m-1} & ... & (m-1)^{u_q-1}v_q^{m-1}\end{bmatrix}. \tag{1.3}$$

A special class of generalized Vandermonde matrices $V'_{R\{2;1,m-1\}}$ is defined by Li and Tan (2008) as follows: For $u_1=1,u_2=m-1,q=2,V'_{R\{2;1,m-1\}}$ is the transpose of $V'_{\{q;u_1,u_2,..,u_q\}}$ and given as

$$V'_{R\{2;1,m-1\}} = \begin{bmatrix} 1 & v_1 & v_1^2 & . & v_1^{m-1} \\ 1 & v_2 & v_2^2 & . & v_2^{m-1} \\ 0 & v_2 & 2v_2^2 & . & (m-1)v_2^{m-1} \\ . & . & . & . & . \\ 0 & v_2 & 2^{m-2}v_2^2 & . & (m-1)^{m-2}v_2^{m-1} \end{bmatrix}. \tag{1.4}$$

The study of rhotrices is introduced in the literature of mathematics by Ajibade (2003). Rhotrix is a mathematical object, which is in some ways between 2×2- dimensional and 3×3 -dimensional matrices. The dimension of a rhotrix is the number of entries in the horizontal or vertical diagonal of the rhotrix and is always an odd number. A rhotrix of dimension 3 is defined as

$$R(3) = \left\langle \begin{matrix} & a_{11} & \\ a_{31} & a_{21} & a_{12} \\ & a_{32} & \end{matrix} \right\rangle, \tag{1.5}$$

where $a_{11}, a_{12}, a_{21}, a_{31}, a_{32}$ are real numbers. Sani (2007) extended the dimension of a rhotrix to any odd number $n \geq 3$ and gave the row-column multiplication and inverse of a rhotrix as follows:

Let

$$Q(3) = \left\langle \begin{matrix} & b_{11} & \\ b_{31} & b_{21} & b_{12} \\ & b_{32} & \end{matrix} \right\rangle,$$

Then

$$R(3)oQ(3) = \left\langle \begin{matrix} & a_{11}b_{11} + a_{12}b_{31} & \\ a_{31}b_{11} + a_{32}b_{31} & a_{21}b_{21} & a_{11}b_{12} + a_{12}b_{32} \\ & a_{31}b_{12} + a_{32}b_{32} & \end{matrix} \right\rangle.$$

Also,

$$(R(3))^{-1} = \frac{1}{a_{11}a_{32} - a_{31}a_{12}} \left\langle \begin{matrix} & a_{32} & \\ -a_{31} & \frac{a_{11}a_{32} - a_{31}a_{12}}{a_{21}} & -a_{12} \\ & a_{11} & \end{matrix} \right\rangle,$$

provided $a_{21}(a_{11}a_{32}-a_{31}a_{12})\neq 0$. Conversion of a rhotrix to a coupled matrix is discussed by Sani (2008). For instance, for a 5–dimensional rhotrix

$$R(5)=\left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle,$$

we have two coupled matrices

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and

$$c=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}.$$

Algebra and analysis of rhotrices are discussed in the literature by Ajibade (2003), Sani (2004), Sani (2007), Aminu (2010), Tudunkaya and Makanjuola (2010), Absalom et al. (2011), Sharma and Kanwar (2011), Sharma and Kanwar (2012a, 2012b, 2012c), Kanwar (2013), Sharma and Kanwar (2013), Sharma and Kumar (2013), Sharma et al. (2013a, 2013b), Sharma and Kumar (2014a, 2014b, 2014c) and Sharma et al. (2014). Sharma et al. (2013b) have introduced Vandermonde rhotrix which is defined as

$$V_n=\left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & a_0 & & & \\ & & 1 & 1 & a_2 & a_1 & a_0^2 & & \\ & . & . & . & . & . & . & . & \\ 1 & . & & a_{\frac{n+1}{2}} & a_{\frac{n-1}{2}} & a_{\frac{n-3}{2}}^{\frac{n-3}{2}} & & . & a_0^{\frac{n-1}{2}} \\ & . & & . & . & . & & . & \\ & & & . & . & . & . & & \\ & & & a_{n-1}^{\frac{n-3}{2}} & a_{n-2}^{\frac{n-3}{2}} & a_{n-3}^{\frac{n-1}{2}} & & & \\ & & & & a_{n-1}^{\frac{n-1}{2}} & & & & \end{array} \right\rangle. \tag{1.6}$$

In the present paper, we introduce special type of generalized Vandermonde rhotrices and factored it into L_5 and R_5 rhotrices namely left and right triangular rhotrices. Further, we factor the rhotrices L_5 and R_5 as product of left and right triangular rhotrices. As an application, we obtain the inverse of the rhotrix considered with the help of the above said factorization.

2. NOTATIONS AND SYMBOLS

The following notations are used in the present paper:

m	a positive integer
n	an odd number
$V'_{\{q;u_1,u_2,...,u_q\}}$	generalized Vandermonde matrix
$V'_{R\{2;1,m-1\}}$	special Vandermonde matrix
$R(3), Q(3)$	3-dimensional rhotrices
V_n	n-dimensional Vandermonde rhotrix
$V_{R\{2;1,n-1\}}$	n-dimensional special Vandermonde rhotrix
$L_n, L_n^{(1)}, L_n^{(2)}$	n -dimensional left triangular rhotrix
$R_n, R_n^{(1)}, R_n^{(2)}$	n -dimensional right triangular rhotrix
a_{ij}	entries in the rhotrix

3. LEFT AND RIGHT TRIANGULAR FACTORIZATION

In this section, we introduce special type of Vandermonde rhotrix. We factor $V_{R\{2;1,4\}}$ and $V_{R\{2;1,2\}}$ in terms of left and right triangular rhotrices in Theorem 3.1 and Theorem 3.2. Further factorization of left and right triangular rhotrices is given in Theorem 3.3.

Definition:

Let v_1, v_2 be two numbers such that v_2 has multiplicity $(n-1)$. Then the rhotrix

$$V_{R\{2;1,n-1\}} = \left\langle \begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & v_1 & & & & & \\
 & & & & 0 & 0 & v_2 & v_2 & v_1^2 & & & & \\
 & & & . & 0 & v_2 & v_2 & 2v_2^2 & v_2^2 & . & & & \\
 & & . & . & . & . & . & . & . & . & . & & \\
 & . & . & . & . & . & . & . & . & . & . & . & \\
0 & 0 & . & . & . & . & . & . & . & . & . & . & v_1^{\frac{n-1}{2}} \\
 & v_2 & v_2 & . & . & . & . & . & . & . & . & . & \\
 & & 2^{n-2}v_2^2 & . & . & . & . & . & . & . & . & & \\
 & & & . & . & . & . & . & . & . & & & \\
 & & & & . & . & . & . & . & & & & \\
 & & & & & . & . & . & & & & & \\
 & & & & & & (\frac{n-1}{2})^{n-2} v_2^{\frac{n-1}{2}} & & & & & &
\end{array} \right\rangle, \tag{3.1}$$

is defined as special generalized Vandermonde rhotrix.

Theorem 3.1.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix. Then $V_{R\{2;1,4\}}$ can be factored as $V_{R\{2;1,4\}} = L_5 R_5$, where L_5 is a left triangular rhotrix and R_5 is a right triangular rhotrix and the entries of L_5 are

$$\begin{aligned}
&a_{ij} = 1, i = 1,2,3,5,\ j = 1;\\
&a_{ij} = 0, i = j, i \neq 1;\\
&a_{ij} = 0,\ j = 2,3,\ j = 1;\\
&a_{41} = 0, a_{32} = v_2 - v_1;\\
&a_{ij} = v_2,\ j = 2, i = j+3;\\
&a_{ij} = v_2, i = 2+j,\ j = 2;\\
&a_{ij} = 8v_2^2 - \frac{(2v_2^2 - v_1^2)}{v_2 - v_1}, i = 2+j,\ j = 3,
\end{aligned}$$

and the entries of R_5 are

$$\begin{aligned}
&a_{ij} = 0, i = 3,4,5;\ j = 1;\\
&a_{ij} = 0,\ j = 2, i = j+3;\\
&a_{ij} = 1, i = 3,4;\ j = 2;\\
&a_{ij} = 1, i = 1,2,\ j = 1;\\
&a_{ij} = v_1^{\ 2}, i = 1,\ j = 2+i;\\
&a_{ij} = v_1, i = 1,\ j = i+1;\\
&a_{ij} = \frac{2v_2^{\ 2} - v_1^{\ 2}}{v_2 - v_1}, i = j, i = 3;\\
&a_{ij} = v_2, i = 2,\ j = 2;\\
&a_{ij} = 1,\ j = 3, i = j+2.
\end{aligned}$$

Proof:

Let $V_{R\{2;1,4\}}$ be 5-dimensional special Vandermonde rhotrix defined as

$$V_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & v_1 & \\ 0 & 0 & v_2 & v_2 & v_1^{\ 2} \\ & v_2 & v_2 & 2v_2^{\ 2} & \\ & & 8v_2^{\ 2} & & \end{array} \right\rangle. \tag{3.2}$$

Two coupled matrices of (3.2) of order 3×3 and 2×2 are

$$A = \begin{bmatrix} 1 & v_1 & v_1^2 \\ 1 & v_2 & 2v_2^2 \\ 0 & v_2 & 8v_2^2 \end{bmatrix}$$

and

$$B=\begin{bmatrix}1 & v_2\\ 0 & v_2\end{bmatrix}.$$

Now, we factor the matrix A using LU decomposition (see, Horn and Johnson (2013)) method as follows: Consider

$$\begin{bmatrix}1 & v_1 & v_1^2\\ 1 & v_2 & 2v_2^2\\ 0 & v_2 & 8v_2^2\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}1 & u_{12} & u_{13}\\ 0 & 1 & u_{23}\\ 0 & 0 & 1\end{bmatrix}.$$

This gives,

$$\begin{bmatrix}1 & v_1 & v_1^2\\ 1 & v_2 & 2v_2^2\\ 0 & v_2 & 8v_2^2\end{bmatrix}=\begin{bmatrix}l_{11} & l_{11}u_{12} & l_{11}u_{13}\\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23}\\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33}\end{bmatrix}. \tag{3.3}$$

On solving (3.3), we get

$$\begin{aligned}
&l_{11}=l_{21}=1, l_{31}=0;\\
&l_{22}=v_2-v_1, l_{32}=v_2;\\
&l_{33}=8v_2^2-\frac{v_2(2v_2^2-v_1^2)}{v_2-v_1};\\
&u_{12}=v_1, u_{13}=v_1^2;\\
&u_{23}=\frac{2v_2^2-v_1^2}{v_2-v_1}.
\end{aligned}$$

Therefore, the matrix A becomes

$$A=\begin{bmatrix}1 & 0 & 0\\ 1 & v_2-v_1 & 0\\ 0 & v_2 & 8v_2^2-\dfrac{v_2(2v_2^2-v_1^2)}{v_2-v_1}\end{bmatrix}\begin{bmatrix}1 & v_1 & v_1^2\\ 0 & 1 & \dfrac{2v_2^2-v_1^2}{v_2-v_1}\\ 0 & 0 & 1\end{bmatrix}=A_1A_2.$$

Now, we factor the matrix B using the same method as follows:

Consider

$$\begin{bmatrix} 1 & v_2 \\ 0 & v_2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}.$$

This gives,

$$\begin{bmatrix} 1 & v_2 \\ 0 & v_2 \end{bmatrix} = \begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} \end{bmatrix}. \tag{3.4}$$

On solving (3.4), we get

$$\begin{aligned} &l_{11} = u_{22} = 1, l_{21} = 0; \\ &l_{22} = v_2, u_{11} = 1, u_{12} = v_1. \end{aligned}$$

Therefore, matrix B becomes

$$B = \begin{bmatrix} 1 & 0 \\ 0 & v_2 \end{bmatrix} \begin{bmatrix} 1 & v_2 \\ 0 & 1 \end{bmatrix} = B_1 B_2.$$

Now, using matrices A_1, B_1, we get the rhotrix L_5 as

$$L_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & v_2 - v_1 & 0 & 0 \\ & v_2 & v_2 & 0 & \\ & & 8v_2^2 - \dfrac{v_2(2{v_2}^2 - {v_1}^2)}{v_2 - v_1} & & \end{array} \right\rangle$$

and using matrices A_2, B_2, we get the rhotrix R_5 as

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & 1 & v_2 & {v_1}^2 \\ & 0 & 1 & \dfrac{2{v_2}^2 - {v_1}^2}{v_2 - v_1} & \\ & & 1 & & \end{array} \right\rangle.$$

Therefore, (3.2) can be factored in the product of two rhotrices as

$$V_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & v_2 - v_1 & 0 & 0 \\ & v_2 & v_2 & 0 & \\ & & 8v_2^2 - \dfrac{v_2(2{v_2}^2 - {v_1}^2)}{v_2 - v_1} & & \end{array} \right\rangle . \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & 1 & v_2 & {v_1}^2 \\ & 0 & 1 & \dfrac{2{v_2}^2 - {v_1}^2}{v_2 - v_1} & \\ & & 1 & & \end{array} \right\rangle = L_5 R_5, \tag{3.5}$$

where the rhotrix L_5 is a left triangular rhotrix and R_5 is a right triangular rhotrix.

∎

Theorem 3.2.

Let $V_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix. Then $V_{R\{2;1,2\}}$ can be factored as $V_{R\{2;1,2\}} = L_3 R_3,$ where L_3 is a left triangular rhotrix and R_3 is a right triangular rhotrix. The entries of L_3 are

$$\begin{aligned} a_{ij} &= 1, i = 1, 2, 3, \ j = 1; \\ a_{12} &= 0; \\ a_{ij} &= v_2 - v_1, \ j = 2, i = j + 1, \end{aligned}$$

and the entries of R_3 are

$$\begin{aligned} a_{ij} &= 1, i = 1, 2, 3, \ j = 1, 2; \\ a_{ij} &= v_1, i = 1, \ j = i + 1; \\ a_{ij} &= 0, i = j + 2, \ j = 1. \end{aligned}$$

Proof:

Let $V_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix defined as

$$V_{R\{2;1,2\}} = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle . \tag{3.6}$$

Then, we factor the rhotrix (3.6) using similar argument as in Theorem 3.1. Therefore,

$$V_{R\{2;1,2\}} = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & v_2 - v_1 & \end{array} \right\rangle . \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & v_1 \\ & 1 & \end{array} \right\rangle . \tag{3.7}$$

Using (3.6) in (3.7), we get

$$\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & v_2 - v_1 & \end{array} \right\rangle . \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & v_1 \\ & 1 & \end{array} \right\rangle .$$

From row-column multiplication of rhotrices, we get

$$\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & v_1 \\ & v_2 & \end{array} \right\rangle ,$$

which verifies the result.

■

Theorem 3.3.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix. Then, $V_{R\{2;1,4\}}$ can be factored as

$$V_{R\{2;1,4\}} = L_5^{(1)} R_5^{(1)} L_5^{(2)} R_5^{(2)} ,$$

where the entries of $L_5^{(1)}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > j, j = 1, i \ne 2, 3;$$
$$a_{ij} = 1, i > j, i = 2, 3, 4, j = 1, 2;$$
$$a_{ij} = \frac{v_2 - v_1}{v_2}, i = j + 1, j = 2;$$
$$a_{ij} = 0, j = 1, i = j + 3;$$
$$a_{ij} = -1, i = 2 + j, j = 3,$$

the entries of $R_5^{(1)}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > j, j = 1, i = 3, 4, 5;$$
$$a_{ij} = 1, i > j, i = 2, 4, 5, j = 1, 2, 3;$$
$$a_{ij} = \frac{v_2}{v_2 - v_1}, i = j + 1, j = 2;$$
$$a_{52} = 0,$$

the entries of $L_5^{(2)}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > 1, i = 3, 4, 5; j = 1;$$
$$a_{ij} = 0, i > 1, i = 3, j = i;$$
$$a_{ij} = 1, i > j, i = 2, 3, 4, j = 1, 2;$$

$$
\begin{aligned}
&a_{22} = a_{52} = 0;\\
&a_{42} = 1;\\
&a_{ij} = \frac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} - 8v_2{}^2, i = 2 + j, j = 3,
\end{aligned}
$$

and the entries of $R_5^{(2)}$ are

$$
\begin{aligned}
&a_{ij} = 1, i = j, i = 1;\\
&a_{ij} = v_1, i < j;\\
&a_{ij} = v_1{}^2, i < j;\\
&a_{ij} = 0, i > j, j = 1, i = 3,4,5;\\
&a_{42} = a_{22} = v_2;\\
&a_{53} = 1, j = 3, i = j + 2;\\
&a_{ij} = 0, i > 1, i = 3,4,5, j = 1,2;\\
&a_{ij} = (2v_2{}^2 - v_1{}^2), i = j, j = 3.
\end{aligned}
$$

Proof:

Let $V_{R\{2;1,4\}}$ be a special type of Vandermonde rhotrix as defined in (3.2). From (3.5), we have

$$
V_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & v_2 - v_1 & 0 & 0 \\ & v_2 & v_2 & 0 & \\ & & 8v_2^2 - \dfrac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} & & \end{array} \right\rangle . \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & 1 & v_2 & v_1{}^2 \\ & 0 & 1 & \dfrac{2v_2{}^2 - v_1{}^2}{v_2 - v_1} & \\ & & 1 & & \end{array} \right\rangle = L_5 R_5.
$$

Using the same arguments as used in Theorem 3.1, we further factor L_5 and R_5.

$$
L_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & \dfrac{v_2}{v_2 - v_1} & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & 1 & 0 & \\ 0 & 0 & \dfrac{v_2 - v_1}{v_2} & 0 & 0 \\ & 1 & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle . \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & \dfrac{v_2}{v_2 - v_1} & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle, \tag{3.8}
$$

which are left and right triangular rhotrices. Therefore,

$$L_5 = L_5^{(1)} R_5^{(1)}.$$

Similarly,

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & v_2 - v_1 & v_2 & v_1{}^2 \\ & 0 & v_2 & 2v_2{}^2 - v_1{}^2 & \\ & & \frac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} - 8v_2{}^2 & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & \\ & & \frac{v_2(2v_2{}^2 - v_1{}^2)}{v_2 - v_1} - 8v_2{}^2 & & \end{array} \right\rangle . \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & v_1 & \\ 0 & 0 & v_2 - v_1 & v_2 & v_1{}^2 \\ & 0 & v_2 & 2v_2{}^2 - v_1{}^2 & \\ & & 1 & & \end{array} \right\rangle, \qquad (3.9)$$

which are left and right triangular rhotrices. Therefore,

$$R_5 = L_5^{(2)} R_5^{(2)}.$$

Hence,

$$V_{R\{2;1,4\}} = L_5^{(1)} R_5^{(1)} L_5^{(2)} R_5^{(2)}.$$

■

4. Application of Factorization of Special Vandermonde Rhotrix

In this section, we apply the factorization of special Vandermonde rhotrix to find the inverse of the rhotrix. The inverse of $V_{R\{2;1,4\}}$ in terms of the inverses of L_5, R_5 is given in Theorem 4.1. We also obtain the inverse of $V_{R\{2;1,2\}}$ in Theorem 4.2. We find the inverse of $V_{R\{2;1,4\}}$ in terms of $L_5^{(1)}, R_5^{(1)}, L_5^{(2)}, R_5^{(2)}$ in Theorem 4.3.

Theorem 4.1.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix. Then, $V^{-1}{}_{R\{2;1,4\}} = R_5^{-1} L_5^{-1}$, where the entries of R_5^{-1} are

$$
\begin{aligned}
&a_{ij} = 1, i = 1, 2,\ j = 1;\\
&a_{ij} = 0, i = 1, 2,\ j = 2, 3;\\
&a_{ij} = 0, i = j,\ j = 3, a_{41} = 0;\\
&a_{ij} = \frac{1}{v_1 - v_2}, i = 3,\ j = 1;\\
&a_{ij} = \frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2}, j = 1, i = j + 4;\\
&a_{ij} = -\frac{1}{v_1 - v_2}, i = j + 1,\ j = 2;\\
&a_{ij} = \frac{1}{v_2}, j = 2, i = 2 + j;\\
&a_{ij} = \frac{-1}{v_1^2 - 8v_1v_2 + 6v_2^2}, j = 2, i = j + 3;\\
&a_{ij} = \frac{-(v_1 - v_2)}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3}, j = 3, i = j + 2,
\end{aligned}
$$

and L_5^{-1} has entries

$$
\begin{aligned}
&a_{ij} = 1, i \ge j, i = 1, 2,\ j = 1;\\
&a_{ij} = 1,\ j = 2, i = 3, 4;\\
&a_{ij} = 0, i = 3, 4, 5,\ j = 1;\\
&a_{52} = 0, a_{53} = 1;\\
&a_{ij} = -v_1, i = 1,\ j = i + 1;\\
&a_{ij} = \frac{v_1^2v_2 - 2v_1v_2^2}{v_2 - v_1}, i = 1,\ j = i + 2;\\
&a_{ij} = -v_2, i = j,\ j = 2;\\
&a_{ij} = \frac{-(v_1^2 - 2v_2^2)}{v_1 - v_2}, i = j,\ j = 3.
\end{aligned}
$$

Proof:

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix as defined in (3.2). Then the inverse of $V_{R\{2;1,4\}}$ is

$$
V^{-1}_{R\{2;1,4\}} = \left\langle \begin{array}{ccccc}
 & & \frac{6v_2^2}{v_1^2 - 8v_1v_2 + 6v_2^2} & & \\
 & -8\frac{v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} & 1 & \frac{v_1^2 - 8v_1v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} & \\
\frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} & 0 & \frac{8v_2}{v_1^2 - 8v_1v_2 + 6v_2^2} & -1 & -\frac{v_1^2 - 2v_1v_2}{v_1^2v_2 - 8v_1v_2 + 6v_2^3} \\
 & \frac{-1}{v_1^2 - 8v_1v_2 + 6v_2^2} & \frac{1}{v_2} & \frac{1}{v_2}\frac{v_1^2 - 2v_2^2}{v_1^2 - 8v_1v_2 + 6v_2^3} & \\
 & & -\frac{1}{v_2}\frac{(v_1 - v_2)}{v_1^2 - 8v_1v_2 + 6v_2^2} & &
\end{array} \right\rangle. \tag{4.1}
$$

Now,

$$L_5^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & \frac{1}{v_1 - v_2} & 1 & 0 & \\ \frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} & 0 & \frac{-1}{v_1 - v_2} & 0 & 0 \\ & -\frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} & \frac{1}{v_2} & 0 & \\ & & -\frac{v_1 - v_2}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3} & & \end{array} \right\rangle \tag{4.2}$$

and

$$R_5^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & -v_1 & \\ 0 & 0 & 1 & -v_2 & -\frac{v_1{}^2v_2 - 2v_1v_2{}^2}{v_1 - v_2} \\ & 0 & 1 & -\frac{v_1{}^2 - 2v_2{}^2}{v_1 - v_2} & \\ & & 1 & & \end{array} \right\rangle. \tag{4.3}$$

On multiplying (4.2) and (4.3), we get

$$R_5^{-1}L_5^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & -v_1 & \\ 0 & 0 & 1 & -v_2 & -\frac{v_1{}^2v_2 - 2v_1v_2{}^2}{v_1 - v_2} \\ & 0 & 1 & -\frac{v_1{}^2 - 2v_2{}^2}{v_1 - v_2} & \\ & & 1 & & \end{array} \right\rangle . \left\langle \begin{array}{ccccc} & & 1 & & \\ & \frac{1}{v_1 - v_2} & 1 & 0 & \\ \frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} & 0 & \frac{-1}{v_1 - v_2} & 0 & 0 \\ & -\frac{1}{v_1^2 - 8v_1v_2 + 6v_2^2} & \frac{1}{v_2} & 0 & \\ & & -\frac{v_1 - v_2}{v_1^2v_2 - 8v_1v_2^2 + 6v_2^3} & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & & \frac{6v_2{}^2}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & & \\ & -8\frac{v_2}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & 1 & \frac{v_1{}^2 - 8v_1v_2}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & \\ \frac{1}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & 0 & \frac{8v_2}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & -1 & -\frac{v_1{}^2 - 2v_1v_2}{v_1{}^2v_2 - 8v_1v_2 + 6v_2{}^2} \\ & \frac{-1}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & \frac{1}{v_2} & \frac{1}{v_2}\frac{v_1{}^2 - 2v_2{}^2}{v_1{}^2 - 8v_1v_2 + 6v_2{}^3} & \\ & & -\frac{1}{v_2}\frac{(v_1 - v_2)}{v_1{}^2 - 8v_1v_2 + 6v_2{}^2} & & \end{array} \right\rangle$$

$$= V^{-1}{}_{R\{2;1,4\}}.$$

Therefore,

$$V^{-1}{}_{R\{2;1,4\}} = R_5^{-1} L_5^{-1}.$$

▪

Theorem 4.2.

Let $V_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix. Then $V^{-1}_{R\{2;1,2\}} = R_3^{-1} L_3^{-1}$, where the entries of R_3^{-1} are

$$a_{ij} = 1, i = 1, j = 1;$$
$$a_{ij} = -v_1, i = 1, j = i + 1;$$
$$a_{ij} = 1; j = 1, i = j + 1;$$
$$a_{ij} = 0, i = j + 2, j = 1;$$
$$a_{ij} = 1, j = 2, i = j + 1,$$

and entries of L_3^{-1} are

$$a_{ij} = 1, i = 1, j = 1;$$
$$a_{ij} = 0, i = 1, j = i + 1;$$
$$a_{ij} = 1, i = j + 1, j = 1;$$
$$a_{ij} = \frac{1}{v_1 - v_2}, i = j + 2, j = 1;$$
$$a_{ij} = \frac{-1}{v_1 - v_2}, i = j + 1, j = 2.$$

Proof:

Let $V^{-1}{}_{R\{2;1,2\}}$ be a 3-dimensional special Vandermonde rhotrix as defined in (3.5). Then,

$$V^{-1}{}_{R\{2;1,2\}} = \left\langle \begin{array}{ccc} & \frac{-v_2}{v_1 - v_2} & \\ \frac{1}{v_1 - v_2} & 1 & \frac{v_1}{v_1 - v_2} \\ & \frac{-1}{v_1 - v_2} & \end{array} \right\rangle. \tag{4.4}$$

Now,

$$L_3^{-1} = \left\langle \begin{array}{ccc} & 1 & \\ \dfrac{1}{v_1 - v_2} & 1 & 0 \\ & \dfrac{-1}{v_1 - v_2} & \end{array} \right\rangle \tag{4.5}$$

and

$$R_3^{-1} = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -v_1 \\ & 1 & \end{array} \right\rangle. \tag{4.6}$$

On multiplying (4.5) and (4.6), we get

$$\begin{aligned} R_3^{-1} L_3^{-1} &= \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -v_1 \\ & 1 & \end{array} \right\rangle \left\langle \begin{array}{ccc} & 1 & \\ \dfrac{1}{v_1 - v_2} & 1 & 0 \\ & \dfrac{-1}{v_1 - v_2} & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & \dfrac{-v_2}{v_1 - v_2} & \\ \dfrac{1}{v_1 - v_2} & 1 & \dfrac{v_1}{v_1 - v_2} \\ & \dfrac{-1}{v_1 - v_2} & \end{array} \right\rangle \\ &= V^{-1}{}_{R\{2;1,2\}}. \end{aligned}$$

Therefore,

$$V_{R\{2;1,2\}} = R_3^{-1} L_3^{-1}.$$

∎

Theorem 4.3.

Let $V_{R\{2;1,4\}}$ be a 5-dimensional special Vandermonde rhotrix. Then,

$$V_{R\{2;1,4\}}{}^{-1} = \left(R_5^{(2)}\right)^{-1} \left(L_5^{(2)}\right)^{-1} \left(R_5^{(1)}\right)^{-1} \left(L_5^{(1)}\right)^{-1},$$

where entries of $\left(R_5^{(1)}\right)^{-1}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i \ge j, j = 1,3, i = 3,4,5;$$
$$a_{ij} = 1, i \ge j, j = 1,2,3;$$
$$a_{ij} = \frac{v_2}{v_2 - v_1}, i = j, j = 2,$$

entries of $\left(L_5^{(1)}\right)^{-1}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 1, i \ge j, j = 1,2,3,4;$$
$$a_{ij} = \frac{v_2}{v_1 - v_2}, i > j, i = 3,5, j = 1;$$
$$a_{ij} = -\frac{v_2}{v_1 - v_2}, i \ge j, j = 2, i = 3,5;$$
$$a_{ij} = -1, j = 3, i = j + 2,$$

entries of $\left(R_5^{(2)}\right)^{-1}$ are

$$a_{ij} = 1, i = j, i, j = 1;$$
$$a_{ij} = 0, i > j, j = 1, i = 3,4,5;$$
$$a_{ij} = 1, j = 1, i = j + 1;$$
$$a_{ij} = 0; j = 2, i = j + 3;$$
$$a_{ij} = \frac{v_1}{v_1 - v_2}, i = 1, j = i + 1;$$
$$a_{ij} = \frac{v_2 v_1^2 - 2 v_1 v_2^2}{v_1 - v_2}, i = 1, j = i + 2;$$
$$a_{ij} = -1; j = i, i = 2;$$
$$a_{ij} = \frac{-1}{v_1 - v_2}, j = 2, i = j + 1;$$
$$a_{ij} = \frac{1}{v_2}, i = 2 + j, j = 2;$$
$$a_{ij} = 1, j = 3, i = j + 2,$$

and entries of $\left(L_5^{(2)}\right)^{-1}$ are

$$a_{ij} = 0, i \le j, i, j \ne 1;$$
$$a_{ij} = 0, i > j, j = 1, i = 3,4,5;$$
$$a_{ij} = 1, i \ge j, j = 1,2;$$
$$a_{ij} = 1, i \le j, j = 2,3;$$
$$a_{ij} = 0, j = 2, i = j + 3;$$
$$a_{ij} = \frac{v_1 - v_2}{v_1^2 v_2 - 8 v_1 v_2^2 + 6 v_2^3}, j = 3, i = j + 2.$$

Proof:

Let $V_{R\{2;1,4\}}$ be a special type of Vandermonde rhotrix as defined in (3.2) and its inverse is given in (4.1). Now,

$$\left(R_5^{(1)}\right)^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & \dfrac{v_2 - v_1}{v_2} & 0 & 0 \\ & 0 & 1 & 0 & \\ & & 1 & & \end{array} \right\rangle. \tag{4.7}$$

$$\left(L_5^{(1)}\right)^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & \dfrac{v_2}{v_1 - v_2} & 1 & 0 & \\ \dfrac{v_2}{v_1 - v_2} & 0 & \dfrac{-v_2}{v_1 - v_2} & 0 & 0 \\ & \dfrac{-v_2}{v_1 - v_2} & 1 & 0 & \\ & & -1 & & \end{array} \right\rangle. \tag{4.8}$$

$$\left(R_5^{(2)}\right)^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & \dfrac{v_1}{v_1 - v_2} & \\ 0 & 0 & \dfrac{-1}{v_1 - v_2} & -1 & \dfrac{v_1{}^2 v_2 - 2 v_1 v_2{}^2}{v_1 - v_2} \\ & 0 & \dfrac{1}{v_2} & \dfrac{-\left(v_1{}^2 - 2 v_2{}^2\right)}{v_1 - v_2} & \\ & & 1 & & \end{array} \right\rangle. \tag{4.9}$$

and

$$\left(L_5^{(2)}\right)^{-1} = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 & \\ & & \dfrac{v_1 - v_2}{v_1{}^2 v_2 - 8 v_1 v_2{}^2 + 6 v_2{}^3} & & \end{array} \right\rangle. \tag{4.10}$$

On multiplying (4.7)-(4.10), we get

$$\left(R_5^{(2)}\right)^{-1}\left(L_5^{(2)}\right)^{-1}\left(R_5^{(1)}\right)^{-1}\left(L_5^{(1)}\right)^{-1}=$$

$$=\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad \frac{v_1}{v_1-v_2} \\ 0 \quad 0 \quad \frac{-1}{v_1-v_2} \quad -1 \quad \frac{v_1^2v_2-2v_1v_2^2}{v_1-v_2} \\ 0 \quad \frac{1}{v_2} \quad \frac{-\left(v_1^2-2v_2^2\right)}{v_1-v_2} \\ 1 \end{array}\right\rangle \cdot \left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 1 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ \frac{v_1-v_2}{v_1^2v_2-8v_1v_2^2+6v_2^3} \end{array}\right\rangle$$

$$\cdot\left\langle \begin{array}{c} 1 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad \frac{v_2-v_1}{v_2} \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 1 \end{array}\right\rangle \cdot \left\langle \begin{array}{c} 1 \\ \frac{v_2}{v_1-v_2} \quad 1 \quad 0 \\ \frac{v_2}{v_1-v_2} \quad 0 \quad \frac{-v_2}{v_1-v_2} \quad 0 \quad 0 \\ \frac{-v_2}{v_1-v_2} \quad 1 \quad 0 \\ -1 \end{array}\right\rangle.$$

$$=\left\langle \begin{array}{c} \frac{6v_2^2}{v_1^2-8v_1v_2+6v_2^2} \\ -8\frac{v_2}{v_1^2-8v_1v_2+6v_2^2} \quad 1 \quad \frac{v_1^2-8v_1v_2}{v_1^2-8v_1v_2+6v_2^2} \\ \frac{1}{v_1^2-8v_1v_2+6v_2^2} \quad 0 \quad \frac{8v_2}{v_1^2-8v_1v_2+6v_2^2} \quad -1 \quad -\frac{v_1^2-2v_1v_2}{v_1^2-8v_1v_2+6v_2^2} \\ \frac{-1}{v_1^2-8v_1v_2+6v_2^2} \quad \frac{1}{v_2} \quad \frac{1}{v_2}\frac{v_1^2-2v_2^2}{v_1^2-8v_1v_2+6v_2^2} \\ -\frac{1}{v_2}\frac{v_1-v_2}{v_1^2-8v_1v_2+6v_2^2} \end{array}\right\rangle$$

$$=V_{R\{2;1,4\}}^{-1}.$$

Hence,

$$V_{R\{2;1,4\}}^{-1}=\left(R_5^{(2)}\right)^{-1}\left(L_5^{(2)}\right)^{-1}\left(R_5^{(1)}\right)^{-1}\left(L_5^{(1)}\right)^{-1}.$$

∎

5. CONCLUSION

In the present era of information technology it has become essential to provide security to the data which is at rest (for example, Hardware); that is, in hardware, and secondly to the data which is in transit (for example, in the wire). Cryptography provides the security to the data and makes e-commerce, e-mail, transaction from ATM, internet banking safe to the users. Strong algorithms and structures of mathematics are behind the cryptography which are responsible for providing security to the users who are interacting with different computers at distant places for safe communications or financial transactions. In this context some researchers used Vandermonde matrices for the encryption and decryption of messages which travel over the insecure channels and the financial transaction over the internet. With the appearance of a new mathematical object known as rhotrix in the literature, the Vandermonde rhotrices played an important role in cryptography.

In the present paper we introduced a special type of Vandermonde rhotrix. Since every rhotrix of dimension $m = 2n + 1$ can be represented as coupled matrices of order $(n + 1)$ and n respectively. The total entries in the rhotrix are always $\frac{m^2+1}{2}$. The coupled matrices of this rhotrix will have the elements $(n + 1)^2$ and n^2, respectively. The importance of matrices in cryptography is discussed initially by Lester S. Hill. Upper triangular matrices and lower triangular matrices have much importance for fast calculation in addition and multiplication operations of matrices. Therefore, such type of matrices would help to increase the efficiency of the computer hardware and software for fast encryption and decryption.

Keeping in view the utilities of upper triangular and lower triangular matrices in cryptography, we use the special type of Vandermonde rhotrix, which is represented by coupled matrices. Initially we factorize the special Vandermonde rhotrix into two rhotrices known as left triangular rhotrix and right triangular rhotrix. These left and right triangular rhotrices are then converted into the coupled matrices. The structure of rhotrices is the representation of coupled matrices which provides double security to the cryptosystems because the use of rhotrix means use of two matrices which becomes tedious to know for the adversary. If the original message will be confused and diffused by making use of two matrices then the security of the data which travel over the insecure channels will be doubled, and to retrieve the original data without knowing the key will be very difficult. In this way the use of Vandermonde rhotrices in the field of cryptography will be more important. The obtained coupled matrices are decomposed in upper and lower triangular matrices. Then we composed the coupled lower and upper triangular matrices into the structure of rhotrices and further we obtained the left and right triangular rhotrices.

In cryptography, obtaining the original message from the encrypted messages is known as inverse. Therefore, in communication, e-commerce, transaction from ATM, internet banking, e-mail etc., the inverse is important for the decryption of the data. In order to meet this requirement we used the factorization of special Vandermonde rhotrix to obtain the inverse of rhotrix. The inverse of a rhotrix is obtained with the help of inverses of its left triangular rhotrix and right triangular rhotrix. For the process to obtain inverses of left triangular rhotrices and right triangular rhotrices, we have written these rhotrices in the form of coupled matrices and obtained their inverses. Again, composing the coupled matrices we obtained the respective left and right triangular rhotrices. Further compositions of left and right triangular rhotrices again have become the special Vandermonde rhotrix.

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