Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 7, Issue 1 (June 2012), pp. 300 – 333

**Applications and Applied Mathematics:**
An International Journal
**(AAM)**

# A Mathematical Study on the Dynamics of an Eco-Epidemiological Model in the Presence of Delay

**T. K. Kar**
Department of Mathematics
Bengal Engineering and Science University
Shibpur, Howrah-711103, India
tkar1117@gmail.com

**Prasanta Kumar Mondal**
Department of Mathematics
Shibpur Sri Ramkrishna Vidyalaya
Howrah-711102, India
2prasant@gmail.com

Received: May 25, 2011; Accepted: August 16, 2011

## Abstract

In the present work a mathematical model of the prey-predator system with disease in the prey is proposed. The basic model is then modified by the introduction of time delay. The stability of the boundary and endemic equilibria are discussed. The stability and bifurcation analysis of the resulting delay differential equation model is studied and ranges of the delay inducing stability as well as the instability for the system are found. Using the normal form theory and center manifold argument, we derive the methodical formulae for determining the bifurcation direction and the stability of the bifurcating periodic solution. Some numerical simulations are carried out to explain our theoretical analysis.

**Keywords:** Eco-epidemiological model, Stability, Delay, Direction of Hopf-bifurcation

**MSC 2010:** 37N25, 92D25

## 1. Introduction

Both mathematical ecology and mathematical epidemiology are distinct major fields of study in biology. But there are some commonalities between them. A branch of ecology which considers the effect of transmissible diseases is called eco-epidemiology. On the other hand interaction between predators and their prey is a complex phenomenon in ecology. This complexity is further increased when one considers the presence of various infectious diseases prevalent in their populations. In the present paper, we consider a prey-predator model with disease in the prey. A biologically relevant example for this model can be found in the Salton Sea ecosystem. We could cite in particular the example of Tilapia (prey) and the Pelicans (predator). When the Tilapia become infected and struggle in their death, they tend to come closer to the surface of the sea and become more vulnerable as well as attractive to fish eating birds, like Pelicans (see Slack 1997). Chattopadhyay and Bairagi (2001) proposed an eco-epidemiological model of the Salton sea consisting of susceptible and infected tilapia fish populations and their predators, the pelican bird population, where it is assumed that predation is only on infected fish population. Hadeler and Freedman (1989) have discussed a predator-prey model where the prey population is infected by a parasite and in turn infects the predator with the parasite. Haque et al. (2008), proposed an eco-epidemiological predator-prey model with standard disease incidence.

The incidence rate, i.e., the rate of new infection plays an important role in the context of epidemiological modeling. Generally, the incidence rate is assumed to be bilinear in the infected fraction $I$ and the susceptible fraction $S$. However, there are many factors that emphasize the need for a modification of the standard bilinear form. It has been suggested by several authors that the disease transmission process may follow the saturation incidence (see Kar and Mondal, 2011; Cai *et al.*, 2009; Esteva and Matias, 2001). We have considered the incidence rate as $\beta SI/(\alpha+S)$, where $\beta$ is the transmission rate and $\alpha$ is a saturation factor, which is more realistic than the bilinear one, as it includes the behavioral change and crowding effect of the susceptible individual and also prevents unboundeness of the contact rate. The main aim of the paper is the study of the mutual relations occurring in an ecosystem where an epidemic runs through a prey population and the predator population being unaffected by the disease. We study some basic questions, among which whether and how the presence of the disease in the prey species affects the behavior of the model, but also whether the introduction of a sound predator can affect the dynamics of the disease in the prey.

Model with delay is much more realistic, as time delay occurs in almost every biological situation. For example, parasite is passed from one infected prey to another susceptible prey. So, the infection process cannot be done instantaneous. Therefore, the effect of time delay can't be ignored. Xiao and Chen (2001) claimed that they were the first to formulate and analyze an eco-epidemiological model with time delay. Bhattacharyya and Mukhopadhyay (2010) studied an analysis of periodic solutions in an eco-epidemiological model with saturation incidence and latency delay. They have incorporated the time required by the susceptible individuals to become infective after their interaction with the infectious individuals as a discrete time delay. They also elucidated the role of differential predation on disease dynamics and the role of latency delay in infection propagation. They have found an interval of the time delay parameter, in terms of different system parameter, that imparts stability around the infected equilibrium point and also deduced a threshold delay, which indicates the change of stability of the endemic equilibrium point. In this paper, we consider the discrete time delay in the disease transmission term and also consider an average information delay that measures the influence of the past disease. The main aim of this paper is to study the dynamics of the system around the biologically feasible equilibria.

We have two populations: (a) the prey, whose population is denoted by $N$ and (b) the predator, whose population is denoted by $\overline{P}$.

The following assumptions are made for formulating the basic mathematical model:

(i) In the absence of infection, the prey population grows according to the logistic law of growth with carrying capacity $K(>0)$ and intrinsic growth rate $r(>0)$.

(ii) In the presence of infection the total prey population $N$ is divided into two classes, susceptible population $\overline{S}$ and infected population $\overline{I}$. Therefore, at any time $t$,

$$N(t) = \overline{S}(t) + \overline{I}(t). \tag{1.1}$$

(iii) Susceptible prey becomes infected when it comes to the contact with the infected prey.

(iv) Infected individuals fail to contribute in the reproduction process and the growth dynamics of the susceptible.

(v) We assume that the predators' growth depend on past quantities of prey. Since prey populations are infected by a disease, so infected preys are weakened and become easier to catch.

Also, we assume that predator catches very small quantities of susceptible prey. Consider a continuous weight (or density) function $f_1$, whose role is to weight moments of the past and satisfies the following conditions:

$$f_1(s) \geq 0, \;\; s \in (0, +\infty), \qquad \int_0^{\infty} f_1(s)ds = 1. \tag{1.2}$$

Assume two weighted average over the past

$$\overline{Q_1}(t) = \int_{-\infty}^{t} \overline{I}(s) f_1(t-s)ds, \qquad \overline{Q_2}(t) = \int_{-\infty}^{t} \overline{S}(s) g_1(t-s)ds. \tag{1.3}$$

From the above assumptions we obtain the following model:

$$\begin{cases} \dfrac{d\overline{S}}{dt} = r\overline{S}(1-\dfrac{\overline{S}}{K}) - \dfrac{\beta\overline{S}\overline{I}(t-\overline{\tau})}{\alpha+\overline{S}} - p_1\overline{S}\,\overline{P}, \\ \dfrac{d\overline{I}}{dt} = \dfrac{\beta\overline{S}\overline{I}(t-\overline{\tau})}{\alpha+\overline{S}} - \mu\overline{I} - p_2\overline{I}\,\overline{P}, \\ \dfrac{d\overline{P}}{dt} = \overline{P}(-\varepsilon + h_1\displaystyle\int_{-\infty}^{t}\overline{I}(s)f_1(t-s)ds + h_2\int_{-\infty}^{t}\overline{S}(s)g_1(t-s)ds), \end{cases} \tag{1.4}$$

where $\overline{S}(t), \overline{I}(t)$ and $\overline{P}(t)$ denote the quantities of sound prey, infected prey and predator, respectively. $p_1, p_2(>0)$ are the capturing rates, $h_1, h_2(>0)$ are the product of the per-capita rate of predation and the rate of conversing prey into predation, $\mu, \varepsilon(>0)$ are the death rate of infective prey and predator, respectively. $\overline{\tau}$ is a time, during which the infectious prey develop in the fish population and only after that time the infected prey becomes itself infectious.

The predator species feeds on both the susceptible and infected prey, but as the infected prey becomes more vulnerable than the susceptible prey, the rate of predation on infected prey is much more than the susceptible prey. As the predation on susceptible prey is negligible so for the simplicity we do not consider the predation term on susceptible prey.

Therefore, we will discuss the integro-differential system as follows:

$$\begin{cases} \dfrac{d\overline{S}}{dt} = r\overline{S}(1-\dfrac{\overline{S}}{K}) - \dfrac{\beta\overline{S}\overline{I}(t-\overline{\tau})}{\alpha+\overline{S}}, \\ \dfrac{d\overline{I}}{dt} = \dfrac{\beta\overline{S}\overline{I}(t-\overline{\tau})}{\alpha+\overline{S}} - \mu\overline{I} - p_2\overline{I}\,\overline{P}, \\ \dfrac{d\overline{P}}{dt} = \overline{P}(-\varepsilon + h_1\displaystyle\int_{-\infty}^{t}\overline{I}(s)f_1(t-s)ds). \end{cases} \tag{1.5}$$

If we choose the density function $f_1(s) = \delta e^{-\delta s}$, where $\delta > 0$ is the average delay of the collected information on the disease, as well as the average length of the historical memory concerning the disease in study, then $f_1$ satisfies the condition (1.2). Then $\overline{Q_1}(t) = \int_{-\infty}^{t}\overline{I}(s)\delta e^{-\delta(t-s)}ds$ is the weighted average over the past values of disease and the choice of $f_1$ lays down exponentially fading memory (see Cushing, 1977; MacDonald, 1977; Farkas, 2001). Since $f_1$ is the probability density of an exponentially distributed random variable, the probabilistic interpretation is obvious. The smaller $\delta > 0$ is longer is the time interval in the past in which the values $\overline{I}(t)$ are taken into account, i.e., $1/\delta$ is the "measure of the influence of the past." Therefore, the system (1.5) can be transformed into the system of differential equations on the interval $[0, +\infty)$ as follows:

$$\begin{cases} \dfrac{d\bar{S}}{dt} = r\bar{S}(1-\dfrac{\bar{S}}{K}) - \dfrac{\beta\bar{S}\bar{I}(t-\bar{\tau})}{\alpha+\bar{S}}, \\ \dfrac{d\bar{I}}{dt} = \dfrac{\beta\bar{S}\bar{I}(t-\bar{\tau})}{\alpha+\bar{S}} - \mu\bar{I} - p_2\bar{I}\,\bar{P}, \\ \dfrac{d\bar{P}}{dt} = -\varepsilon\bar{P} + h_1\bar{P}\overline{Q_1}, \\ \dfrac{d\overline{Q_1}}{dt} = \delta(\bar{I} - \overline{Q_1}). \end{cases} \tag{1.6}$$

We set $\bar{S}/\alpha = S,\ \bar{I}/\alpha = I,\ \overline{Q_1} = \alpha Q,\ \bar{P} = P$ and use dimensionless time scale $\bar{t} = rt$. For simplicity, we replace the notation $\bar{t}$ by $t$. Then the system (1.6) can be written as follows:

$$\begin{cases} \dfrac{dS}{dt} = S(1-aS) - \dfrac{bSI(t-\tau)}{1+S}, \\ \dfrac{dI}{dt} = \dfrac{bSI(t-\tau)}{1+S} - d_1 I - pIP, \\ \dfrac{dP}{dt} = -d_2 P + hPQ, \\ \dfrac{dQ}{dt} = c(I-Q), \end{cases} \tag{1.7}$$

where $a = \alpha/K,\ b = \beta/r,\ d_1 = \mu/r,\ p = p_2/r,\ d_2 = \varepsilon/r,\ h = h_1\alpha/r,\ c = \delta/r,\ \tau = r\bar{\tau}.$

The initial conditions for the system (1.7) are

$$\begin{aligned} &(\phi(\theta),\psi_1(\theta),\psi_2(\theta),\psi_3(\theta)) \in C_+ = C([-\tau,0],\Re^4_+), \\ &\phi(0) > 0,\ \ \psi_i(0) > 0,\ \ i = 1,2,3, \end{aligned} \tag{1.8}$$

where

$$\Re^4_+ = \{(S,I,P,Q) \in \Re^4 : S,I,P,Q \ge 0\}.$$

We observe that the right hand side of Equation (1.7) is a smooth function of the variables $(S,I,P,Q)$ and the parameters, as long as these quantities are non-negative, so local existence and uniqueness properties hold in the positive octant.

Rest of this paper is organized as follows: In section 2, we discussed the existence of equilibria and the stability of non-negative equilibria. The dynamical behavior of endemic equilibrium point and the existence of Hopf-bifurcation around the endemic equilibrium point are also presented in this section. In section 3, we have discussed the direction of Hopf-bifurcation and the stability of bifurcating periodic solutions by using normal form theory and the center manifold theorem due to Hassard et al. (1981). Some numerical simulations are given to satisfy our theoretical results in section 4. ◘

## 2. Equilibria and Stability Criteria

We now investigate non-negative equilibria for system (1.7). $E_0 = (0,0,0,0)$ is the trivial equilibrium, $E_1 = (1/a,0,0,0)$ is the axial equilibrium and $E_2 = (d_1/(b-d_1),(b-d_1-ad_1)/(b-d_1)^2,0,(b-d_1-ad_1)/(b-d_1)^2)$ is the boundary equilibrium. The interior equilibrium is $E^* = (S^*, I^*, P^*, Q^*)$, where $S^* = (h(1-a)+\sqrt{m})/2ah$, $I^* = Q^* = d_2/h$, $P^* = ((b-d_1)(h+\sqrt{m}-ah)-2ahd_1)/p(h+\sqrt{m}+ah)$ and $m = h^2(1-a)^2 + 4ah(h-bd_2)$.

We see that equilibria $E_0$, $E_1$ always exist. The boundary equilibria $E_2$ exists if $b > d_1(1+a)$. Now if $b < d_1(1+a)$, i.e. if the maximal renewal rate of infected prey is less than their mortality rate, then both infected prey and predator tends to zero. So, we note that the equilibrium $E_2$ arises from $E_1$ for $b = d_1(1+a)$ and persists for $b < d_1(1+a)$. The existence condition for the interior equilibrium $E^*$ is

$$(H_1): b < h(1+a)^2/4ad_2,\ (b-d_1)(h+\sqrt{m}-ah)-2ahd_1 > 0.$$

From the system (1.7), we observe that $(dI/dt)_{P=0} < 0$ if $S(0) < d_1/(b-d_1)$, since $S(t) \le S(0)$ at any time $t$. In this case $S(t) \le S(0) < d_1/(b-d_1)$. This is usually known as threshold phenomenon. If the initial susceptible prey population is less than the ratio of the death rate of infected prey to the maximal renewal rate of infected prey decreased by the death rate of infected prey, the epidemic can not at all spread. The more susceptible population in the system, the greater chance of becoming infective in the diseased system. ◘

Let $\overline{E} = (\overline{S}, \overline{I}, \overline{P}, \overline{Q})$ be any arbitrary equilibrium. Then the Jacobian matrix evaluated at $\overline{E}$ leads to the characteristic equation as follows:

$$\begin{vmatrix} \lambda - \{(1-a\overline{S}) - \frac{b\overline{I}}{1+\overline{S}}\} - \overline{S}\{-a + \frac{b\overline{I}}{(1+\overline{S})^2}\} & \frac{b\overline{S}}{1+\overline{S}} e^{-\lambda\tau} & 0 & 0 \\ -\frac{b\overline{I}}{(1+\overline{S})^2} & \lambda + d_1 + p\overline{P} - \frac{b\overline{S}}{1+\overline{S}} e^{-\lambda\tau} & p\overline{I} & 0 \\ 0 & 0 & \lambda + d_2 - h\overline{Q} & -h\overline{P} \\ 0 & -c & 0 & \lambda + c \end{vmatrix} = 0. \quad (2.1)$$

**Theorem 2.1:**

The trivial equilibrium $E_0$ is unstable.

***Proof:***

The characteristic equation (2.1) at the trivial equilibrium $E_0$ is

$$(\lambda - 1)(\lambda + d_1)(\lambda + d_2)(\lambda + c) = 0.$$

So, the trivial equilibrium is unstable (saddle).

**Theorem 2.2:**

The disease free equilibrium $E_1 = (1/a, 0, 0, 0)$ is

(i) Asymptotically stable when $(b - d_1)/ad_1 < 1,$
(ii) Linearly neutrally stable for $d_1 = b/(a+1)$ and
(iii) Unstable when $(b - d_1)/ad_1 > 1.$

***Proof:***

The characteristic equation at the disease-free equilibrium $E_1$ is

$$(\lambda + 1)(\lambda + d_2)(\lambda + c)(\lambda + d_1 - be^{-\lambda\tau}/(1+a)) = 0. \tag{2.2}$$

Thus the stability of the disease-free equilibrium depends on $\Gamma(\lambda) = \lambda + d_1 - be^{-\lambda\tau}/(1+a).$

(i) $\Gamma(\lambda) = \lambda + d_1 - be^{-\lambda\tau}/(1+a) = 0,$ $\lambda_{\max} = (be^{-\lambda\tau}/(1+a) - d_1)|_{\tau=0} = b/(1+a) - d_1 < 0$ if $(b - d_1)/ad_1 < 1.$ In fact, the root of $\Gamma(\lambda) = 0$ has negative real part for $\tau \geq 0.$ Thus, if $(b - d_1)/ad_1 < 1,$ the disease-free equilibrium is asymptotically stable for all $\tau \geq 0.$

(ii) If $d_1 = b/(1+a),$ we see that $\lambda = 0$ is a simple root of $\Gamma(\lambda) = 0.$ If $\Gamma(\lambda)$ has a characteristic root as $\lambda = \gamma + i\omega,$ then we have

$$\gamma + i\omega + d_1 - d_1 e^{-\gamma\tau}(\cos\omega\tau - i\sin\omega\tau) = 0.$$

This implies that

$$(\gamma + d_1)^2 + \omega^2 = d_1^{\,2} e^{-2\gamma\tau},$$

which holds only when $\gamma < 0$. If $\gamma > 0$, we will reach a contradiction. Thus, when $d_1 = b/(1+a)$, the disease-free equilibrium is linearly neutrally stable.

(iii) When $(b-d_1)/ad_1 > 1$, $\Gamma(0) < 0$ and $\Gamma(+\infty) > 0$. Thus, the characteristic equation (2.2) has at least one positive root. So the disease-free equilibrium is unstable and the equilibrium point $E_2$ exists. ◘

To discuss the stability of the equilibrium point $E_2$, we state the following theorem, which is set up in Kar, (2003).

**Theorem 2.3:**

A set of necessary and sufficient conditions for $(x^*, y^*)$ to be asymptotically stable for all $\tau \geq 0$ is the following**:**

1. The real parts of all the roots of $\Delta(\lambda, 0) = 0$ are negative.
2. For all real $v$ and $\tau \geq 0$, $\Delta(iv, \tau) \neq 0$, where $i = \sqrt{-1}$.

**Theorem 2.4:**

The boundary equilibrium $E_2 = (d_1/(b-d_1), (b-d_1-ad_1)/(b-d_1)^2, 0, (b-d_1-ad_1)/(b-d_1)^2)$ is

(i) Locally asymptotically stable for all $\tau \geq 0$, if $b-d_1-ab < ad_1$,
(ii) Unstable for $\tau = 0$, if $b-d_1-ab > ad_1$.

***Proof:***

The characteristic equation at the equilibrium point $E_2$ is

$$\begin{aligned}(-c-\lambda)\{-d_2 + h(b-d_1-ad_1)/(b-d_1)^2 - \lambda\}\{&(x-\lambda)(d_1 e^{-\lambda\tau} - d_1 - \lambda) \\ &+ d_1(b-d_1-ad_1)e^{-\lambda\tau}/b\} = 0,\end{aligned} \tag{2.3}$$

where

$$x = d_1(b-ab-d_1-ad_1)/b(b-d_1).$$

As $(H_1)$ holds, so

$$\lambda = -d_2 + h(b-d_1-ad_1)/(b-d_1)^2 = (-d_2 d_1^2 + (2bd_2 - h - ah)d_1 + bh - b^2 d_2)/(b-d_1)^2 < 0.$$

Thus, the stability of the equilibrium point $E_2$ depends on the equation

$$\Delta(\lambda,\tau) \equiv (x-\lambda)(d_1 e^{-\lambda\tau} - d_1 - \lambda) + d_1(b-d_1-ad_1)e^{-\lambda\tau}/b = 0. \tag{2.4}$$

$$\therefore \Delta(\lambda,\tau) = \lambda^2 + (d_1-x)\lambda - xd_1 - e^{-\lambda\tau}[d_1\lambda - xd_1 - d_1(b-d_1-ad_1)/b].$$

For $\tau = 0,$ the equation (2.4) can be written as

$$\lambda^2 - x\lambda + d_1(b-d_1-ad_1)/b = 0. \tag{2.5}$$

If $x<0$ i.e. $b-d_1-ab<ad_1$ and since $b-d_1-ad_1>0,$ all roots of the Equation (2.5) are real and negative or complex conjugate with negative real parts. Therefore, the equilibrium point $E_2$ is locally asymptotically stable for $\tau = 0.$

For $\tau \neq 0,$ if $\lambda = i\omega$ is a root of the Equation (2.4), then we have

$$\begin{aligned}(-\omega^2 - xd_1) + i(d_1-x)\omega = {} & [d_1\omega\sin\omega\tau - \{xd_1 + d_1(b-d_1-ad_1)/b\}\cos\omega\tau] \\ & + i[d_1\omega\cos\omega\tau + \{xd_1 + d_1(b-d_1-ad_1)/b\}\sin\omega\tau].\end{aligned} \tag{2.6}$$

Separating real and imaginary parts, we get

$$(-\omega^2 - xd_1) = d_1\omega\sin\omega\tau - \{xd_1 + d_1(b-d_1-ad_1)/b\}\cos\omega\tau,$$

$$(d_1-x)\omega = d_1\omega\cos\omega\tau + \{xd_1 + d_1(b-d_1-ad_1)/b\}\sin\omega\tau.$$

Squaring and adding the above two equations, we have

$$\omega^4 + x^2\omega^2 - d_1(b-d_1-ad_1)\{2d_1x + d_1(b-d_1-ad_1)/b\}/b = 0. \tag{2.7}$$

Now, $2d_1x + d_1(b-d_1-ad_1)/b = d_1[-(1+a)d_1^2 - 3abd_1 + b^2]/b(b-d_1) < 0.$ So, the equation (2.7) does not have any real solutions. Hence, by the theorem 2.3., the theorem is proved. ◘

Now we consider the following assumptions:

$(H_2): \{S^{*2}(a-f)^2 - hpI^*P^*\} - 4fgS^{*2}(a-f)^2 < 0,$

$(H_3): 4fgS^*(a-f)\{d_2fgpP^*S^*(a-f) + f^2g^2S^*(a-f) - d_2pP^*S^{*3}(a-f)^3\}$
$- \{S^{*2}(fg-d_2pP^*)(a-f)^2 - d_2pP^*(fg+d_2pP^*)\}^2 > 0,$

$(H_4): c < 2d_2gpP^*S^{*2}(a-f)^2/[d_2^2p^2P^{*2} - g^2\{S^*(a-f)-f\}^2 - 2d_2pP^*S^*(a-f)\{g+S^*(a-f)\}],$

where $f = bI^*/(1+S^*)^2,\ g = bS^*/1+S^*.$

Explicit biological interpretations of the conditions $H_2 - H_4$ seem to be difficult. These may simply be regarded as some conditions to be satisfied by the biological and technical parameters for the stability of the endemic equilibrium.

**Theorem 2.5:**

If $(H_2)$ and $(H_3)$ hold, then the endemic equilibrium $E^*$ is locally asymptotically stable for $\tau = 0.$

***Proof:***

The characteristic equation at $E^* = (S^*, I^*, P^*, Q^*)$ is

$$\begin{vmatrix} \lambda - S^*\{-a + \dfrac{bI^*}{(1+S^*)^2}\} & \dfrac{bS^*}{1+S^*}e^{-\lambda\tau} & 0 & 0 \\ -\dfrac{bI^*}{(1+S^*)^2} & \lambda + \dfrac{bS^*}{1+S^*} - \dfrac{bS^*}{1+S^*}e^{-\lambda\tau} & pI^* & 0 \\ 0 & 0 & \lambda & -hP^* \\ 0 & -c & 0 & \lambda + c \end{vmatrix} = 0. \tag{2.8}$$

When $\tau = 0,$ the characteristic equation (2.8) yields

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{2.9}$$

where

$$\begin{aligned} &a_1 = S^*(a-f) + c,\ a_2 = cS^*(a-f) + fg,\ a_3 = cd_2(g-d_1) + cfg, \\ &a_4 = cd_2S^*(g-d_1)(a-f). \end{aligned} \tag{2.10}$$

Since $a - f = \sqrt{m}/h(1+S^*) > 0$ *and* $g - d_1 = pP^* > 0,$ then $a_i > 0$ for $i = 1,2,3,4.$ By the conditions $(H_2)$ and $(H_3)$ the following conditions hold trivially.

$$\begin{aligned} a_1a_2 - a_3 &= cS^{*2}(a-f)^2 + S^*(a-f)(fg + c^2) - cd_2(g-d_1) \\ &= S^*(a-f)c^2 + \{S^{*2}(a-f)^2 - hpI^*P^*\}c + fgS^*(a-f) \\ &> 0, \end{aligned}$$

$$a_3(a_1a_2 - a_3) - a_4a_1^2 = c(d_2pP^* + fg)[S^*(a-f)c^2 + \{S^{*2}(a-f)^2 - hpI^*P^*\}c$$
$$+ fgS^*(a-f)] - cd_2pP^*S^*(a-f)\{S^*(a-f)+c\}^2$$
$$= Lc^3 + Mc^2 + Nc$$
$$> 0,$$

where

$$L = fgS^*(a-f) > 0,$$
$$M = S^{*2}(a-f)^2(fg - d_2pP^*) - d_2pP^*(fg + d_2pP^*),$$
$$N = fgS^*(a-f)(fg + d_2pP^*) - d_2pP^*S^{*3}(a-f)^3.$$

Hence, the Routh-Hurwitz criterion is satisfied. Thus, it follows that the endemic equilibrium $E^*$ is locally asymptotically stable for $\tau = 0$. ◘
We now give a definition, which can be found in Beretta (2002).

**Definition 2.1:**

The equilibrium $E^*$ is absolutely stable if it is asymptotically stable for all delays $\tau \geq 0$ and is conditionally stable if it is asymptotically stable for $\tau$ in some finite interval. ◘

Next we will investigate the distribution of roots of the following equation

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} = 0, \tag{2.11}$$

where $m_i,\ n_i \in \Re\,(i = 0, 1, 2, 3)$ *and* $\sum_{i=0}^{3} n_i^2 \neq 0.$

When $\tau = 0,$ the equation (2.11) reduces to

$$\lambda^4 + (m_3 + n_3)\lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0. \tag{2.12}$$

Obviously, $i\omega(\omega > 0)$ is a root of equation (2.11) if and only if $\omega$ satisfies

$$\omega^4 - m_3\omega^3 i - m_2\omega^2 + m_1\omega i + m_0 + (\cos\omega\tau - i\sin\omega\tau)(-n_3\omega^3 i - n_2\omega^2 + n_1\omega i + n_0) = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} \omega^4 - m_2\omega^2 + m_0 = (-n_2\omega^2 + n_0)\cos\omega\tau + (-n_3\omega^3 + n_1\omega)\sin\omega\tau, \\ -m_3\omega^3 + m_1\omega = (-n_3\omega^3 + n_1\omega)\cos\omega\tau - (-n_2\omega^2 + n_0)\sin\omega\tau, \end{cases} \tag{2.13}$$

which implies

$$w^4 + p'w^3 + q'w^2 + r'w + s' = 0, \tag{2.14}$$

where

$$w = \omega^2,\ p' = -2m_2 + m_3^{\ 2} - n_3^{\ 2},\ q' = m_2^{\ 2} + 2m_0 - 2m_1m_3 - n_2^{\ 2} + 2n_1n_3,$$
$$r' = -2m_0m_2 + m_1^{\ 2} + 2n_0n_2 - n_1^{\ 2},\ s' = m_0^{\ 2} - n_0^{\ 2}.$$

Let us denote

$$l_1 = \frac{1}{2}q' - \frac{3}{16}p'^2,\ l_2 = \frac{1}{32}p'^3 - \frac{1}{8}p'q' + r',\ \Sigma = \left(\frac{l_2}{2}\right)^3 + \left(\frac{l_1}{3}\right)^3,\ \zeta = \frac{-1+\sqrt{3}i}{2},$$
$$y_1 = \sqrt[3]{-\frac{l_2}{2} + \sqrt{\Sigma}} + \sqrt[3]{-\frac{l_2}{2} - \sqrt{\Sigma}},\ y_2 = \zeta\sqrt[3]{-\frac{l_2}{2} + \sqrt{\Sigma}} + \zeta^2\sqrt[3]{-\frac{l_2}{2} - \sqrt{\Sigma}},$$
$$y_3 = \zeta^3\sqrt[3]{-\frac{l_2}{2} + \sqrt{\Sigma}} + \zeta\sqrt[3]{-\frac{l_2}{2} - \sqrt{\Sigma}},\ w_i = y_i - \frac{3p'}{4},\quad i = 1,2,3.$$

Li et al. (2005) obtained the following results on the distribution of roots of Equation (2.14).

**Lemma 2.1:**

For the polynomial equation (2.14)

(i) If $s' < 0,$ then Eq. (2.14) has at least one positive root;
(ii) If $s' \geq 0$ *and* $\Sigma \geq 0,$ then Eq. (2.14) has positive roots if and only if $w_1 > 0$ and $\rho(w_1) < 0;$
(iii) If $s' \geq 0$ and $\Sigma < 0,$ then Eq. (2.14) has positive roots if and only if there exists at least one $w^* \in \{w_1, w_2, w_3\}$ such that $w^* > 0$ and $\rho(w^*) \leq 0,$ where $\rho(w) = w^4 + p'w^3 + q'w^2 + r'w + s'.$

**Lemma 2.2:**

(i) The positive equilibrium $E^*$ of system (1.7) is absolutely stable if and only if the equilibrium $E^*$ of the corresponding ordinary differential equation system is asymptotically stable and the characteristic equation (2.8) has no purely imaginary roots for any $\tau > 0;$

(ii) The positive equilibrium $E^*$ of system (1.7) is conditionally stable if and only if all roots of the characteristic equation (2.8) have negative real parts at $\tau = 0$ and there exist some positive values $\tau$ such that the characteristic equation (2.8) has a pair of purely imaginary roots $\pm i\omega_0.$ ◘

**Theorem 2.6:**

If $(H_2)$, $(H_3)$ and $(H_4)$ hold, then the endemic equilibrium point $E^*$ is conditionally stable.

***Proof:***

For $\tau > 0$, the characteristic equation (2.8) can be expanded as

$$\lambda^4 + A'\lambda^3 + B'\lambda^2 + C'\lambda + D' = e^{-\lambda\tau}[g\lambda^3 + E'\lambda^2 + F'\lambda], \tag{2.15}$$

where

$$\begin{aligned} &A' = S^*(a-f) + g + c,\ B' = S^*(a-f)(c+g) + cg,\ C' = cd_2(g-d_1), \\ &D' = cd_2 S^*(g-d_1)(a-f),\ E' = g(c-f) + gS^*(a-f),\ F' = cg\{S^*(a-f) - f\}. \end{aligned} \tag{2.16}$$

Assume that for some $\tau > 0$, $i\omega(\omega > 0)$ is a root of (2.15), then we have

$$\omega^4 - A'\omega^3 i - B'\omega^2 + C'\omega i + D' = (\cos\omega\tau - i\sin\omega\tau)(-g\omega^3 i - E'\omega^2 + F'\omega i). \tag{2.17}$$

Separating real and imaginary parts, we have

$$\begin{cases} \omega^4 - B'\omega^2 + D' = -E'\omega^2\cos\omega\tau + (-g\omega^3 + F'\omega)\sin\omega\tau, \\ -A'\omega^3 + C'\omega = (-g\omega^3 + F'\omega)\cos\omega\tau + E'\omega^2\sin\omega\tau, \end{cases} \tag{2.18}$$

which implies that

$$\omega^8 + x_1\omega^6 + x_2\omega^4 + x_3\omega^2 + x_4 = 0, \tag{2.19}$$

where

$$\begin{aligned} &x_1 = A'^2 - 2B' - g^2,\ x_2 = B'^2 + 2D' - 2A'C' - E'^2 + 2gF', \\ &x_3 = C'^2 - 2B'D' - F'^2,\ x_4 = D'^2. \end{aligned} \tag{2.20}$$

Since $a - f > 0$, we have

$$\begin{aligned} &x_1 = S^{*2}(a-f)^2 + c^2 > 0, \\ &x_2 = S^{*2}(a-f)^2(c+g)^2 - f^2g^2 + 2cgS^*(c+g)(a-f) - 2cd_2(c+g)(g-d_1), \\ &x_3 = Uc^2 + Vc < 0, \end{aligned}$$

where

$$U = {d_2}^2 (g-d_1)^2 - g^2 (S^*(a-f)-f)^2 - 2gd_2S^*(g-d_1)(a-f) - 2d_2{S^*}^2(a-f)^2(g-d_1),$$
$$V = -2d_2g{S^*}^2(a-f)^2(g-d_1).$$

Let $\omega_0$ be one of the positive root of the Equation (2.19). Then the characteristic Equation (2.15) has a pair of purely imaginary roots of the form $\pm i\omega_0$. From the Equation (2.18), we get the corresponding $\tau_k > 0$ such that the characteristic Equation (2.15) has a pair of purely imaginary roots.

$$\tau_k = \frac{1}{\omega_0}\cos^{-1}[\{(C'\omega_0 - A'{\omega_0}^3)(F'\omega_0 - g{\omega_0}^3) - E'{\omega_0}^2({\omega_0}^4 - B'{\omega_0}^2 + D')\} / \{(F'\omega_0 - g{\omega_0}^3) + E'^2{\omega_0}^4\}] + \frac{2n\pi}{\omega_0}, \qquad n = 0, 1, 2, 3, \ldots$$

Hence, by Lemma (2.2) and Theorem (2.5), the endemic equilibrium $E^*$ is conditionally stable. ◘

Now we will show that $\left.\dfrac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_k} > 0.$

This implies that there exists at least one eigen value with positive real part for $\tau > \tau_k$. Differentiating (2.15) with respect to $\tau$, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^4 + 2A'\lambda^3 + B'\lambda^2 - D'}{-\lambda^2(\lambda^4 + A'\lambda^3 + B'\lambda^2 + C'\lambda + D')} + \frac{2g\lambda^2 + E'\lambda}{\lambda^2(g\lambda^2 + E'\lambda + F')} - \frac{\tau}{\lambda}.$$

Therefore,

$$sign\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\tau=\tau_k} = sign\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0}$$
$$= sign\left\{\frac{4\omega_0^6 + 3x_1\omega_0^4 + 2x_2\omega_0^2 + x_3}{\omega_0^2[(-g\omega_0^2 + F')^2 + E'^2\omega_0^2]}\right\}.$$

Thus, we have

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_k} > 0.$$

This shows that the root of characteristic Equation (2.8) crosses the imaginary axis from the left to the right as $\tau$ increases through $\tau_k$ and the conditions for Hopf bifurcation are then satisfied at $\tau = \tau_k$. ◘

As we know, Hopf-bifurcation is a very important dynamic phenomenon in epidemiology. It can be used to interpret the periodic behavior for some infectious diseases. For more details we refer the reader to see Greehalgh et al. (2004); Hethcote et al. (1999). It has been recognized that delay may have very complicated impact on the dynamic behavior of a system. It can cause the loss of stability and can bifurcate various periodic solutions. Some recent literatures on this subject are Yan and Zhang (2008), Ruan and Wang (2003), and Yan and Li (2006).

## 3. Stability and Direction of Hopf Bifurcations

In this section, we shall study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using the normal theory and the center manifold theorem due to Hassard et al. (1981).

Let $u_1 = S - S^*$, $u_2 = I - I^*$, $u_3 = P - P^*$, $u_4 = Q - Q^*$, $\bar{u}_i(t) = u_i(\tau t)$, $\tau = \nu + \tau_k$ and dropping the bars for simplification of notations, the system (1.7) becomes a functional differential equation in $C_1 = C_1([-1, 0], R^4)$ as

$$\dot{u}(t) = L_\nu(u_t) + \bar{f}(\nu, u_t), \tag{3.1}$$

where

$u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in R^4$ and $L_\nu : C_1 \to R^4$, $\bar{f} : R \times C_1 \to R^4$ are given by

$$L_\nu(\phi) = (\tau_k + \nu)\begin{bmatrix} -aS^* + fS^* & 0 & 0 & 0 \\ f & -g & -pI^* & 0 \\ 0 & 0 & 0 & hP^* \\ 0 & c & 0 & -c \end{bmatrix}\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \\ \phi_4(0) \end{bmatrix} + (\tau_k + \nu)\begin{bmatrix} 0 & -g & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \\ \phi_4(-1) \end{bmatrix}, \tag{3.2}$$

and

$$\bar{f}(\nu, \phi) = (\tau_k + \nu)\begin{bmatrix} -a\phi_1^2(0) - bM_1 \\ bM_1 - p\phi_2(0)\phi_3(0) \\ h\phi_3(0)\phi_4(0) \\ 0 \end{bmatrix}, \tag{3.3}$$

where

$$M_1 = \frac{1}{(1+S^*)^2}\phi_1(0)\phi_2(-1) - \frac{1}{(1+S^*)^3}\phi_1^2(0)\phi_2(-1) + \frac{1}{(1+S^*)^4}\phi_1^3(0)\phi_2(-1) + \ldots$$

By the Riesz representation theorem (Hale and Verduyn, 1993), there exists a function $\eta(\theta, \nu)$ of bounded variation for $\theta \in [-1, 0]$ such that

$$L_\nu(\phi) = \int_{-1}^{0} d\eta(\theta, \nu)\,\phi(\theta), \text{ for } \phi \in C_1. \tag{3.4}$$

In fact, if we choose

$$\eta(\theta, \nu) = (\tau_k + \nu)\begin{bmatrix} -(a-f)S^* & 0 & 0 & 0 \\ f & -g & -pI^* & 0 \\ 0 & 0 & 0 & hP^* \\ 0 & c & 0 & -c \end{bmatrix}\delta_1(\theta) - (\tau_k + \nu)\begin{bmatrix} 0 & -g & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\delta_1(\theta+1), \tag{3.5}$$

where $\delta_1$ is the Dirac delta function, then (3.4) is satisfied.

For $\phi \in C_1([-1, 0], R^4)$, define

$$A(\nu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0) \\ \displaystyle\int_{-1}^{0} d\eta(s,\nu)\,\phi(s), & \theta = 0 \end{cases} \tag{3.6}$$

and

$$R(\nu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ \bar{f}(\nu, \varphi), & \theta = 0. \end{cases} \tag{3.7}$$

Then the system (3.1) is equivalent to

$$\dot{u}_t = A(\nu)u_t + R(\nu)u_t, \tag{3.8}$$

where $u_t(\theta) = u(t+\theta)$, for $\theta \in [-1, 0]$.

For $\psi \in C_1^1([0, 1], (R^4)^*)$, define

$$A^* \psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, 1] \\ \int\limits_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0 \end{cases} \tag{3.9}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int\limits_{-1}^{0} \int\limits_{\xi=0}^{\theta} \overline{\psi}(\xi-\theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi, \tag{3.10}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and $A^*$ are adjoint operators. By the discussion in section 2, we know that $\pm i\omega_0\tau_k$ are eigenvalues of $A(0)$. Hence, they are also eigenvalues of $A^*$. We first need to compute the eigenvectors of $A(0)$ and $A^*$ corresponding to $i\omega_0\tau_k$ and $-i\omega_0\tau_k$, respectively.

Suppose $q(\theta) = (1, q_1, q_2, q_3)^T e^{i\omega_0\tau_k\theta}$ is the eigenvectors of $A(0)$ corresponding to $i\omega_0\tau_k$. So,

$A(0)q(\theta) = i\omega_0\tau_k q(\theta).$ Then from the definition of $A(0)$ and (3.2), (3.4) and (3.5), we get

$$\tau_k \begin{bmatrix} -(a-f)S^* & 0 & 0 & 0 \\ f & -g & -pI^* & 0 \\ 0 & 0 & 0 & hP^* \\ 0 & c & 0 & -c \end{bmatrix} q(0) + \tau_k \begin{bmatrix} 0 & -g & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} q(-1) = i\omega_0\tau_k q(0).$$

Since $q(-1) = q(0)e^{-i\omega_0\tau_k}$, then we have

$$q_1 = \frac{-i\omega_0 - (a-f)S^*}{ge^{-i\omega_0\tau_k}},\ q_2 = \frac{hP^*}{i\omega_0} q_3,\ q_3 = \frac{c}{c + i\omega_0} q_1.$$

Similarly, let $q^*(s) = D(1, q_1^*, q_2^*, q_3^*)e^{i\omega_0\tau_k s}$ be the eigenvectors of $A^*$ corresponding to $-i\omega_0\tau_k$. Then, by the definition of $A^*$ and (3.2), (3.4) and (3.5), we have

$$q_1^* = \frac{(a-f)S^* - i\omega_0}{f},\ q_2^* = \frac{pI^*}{i\omega_0} q_1^*,\ q_3^* = \frac{hP^*}{c - i\omega_0} q_2^*.$$

In order to assure $\langle q^*(s), q(\theta) \rangle = 1,$ we need to determine the value of $D.$ From (3.10), we have

$$\begin{aligned}
\langle q^*(s),\, q(\theta)\rangle &= \overline{q^*}(0)q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{q^*}(\xi-\theta)\, d\eta(\theta,0)q(\xi)\, d\xi \\
&= \bar{D}(1,\, \overline{q_1^*},\, \overline{q_2^*},\, \overline{q_3^*})(1,\, q_1,\, q_2,\, q_3)^T \\
&\qquad - \int_{-1}^{0}\int_{0}^{\theta} \bar{D}\,(1,\, \overline{q_1^*},\, \overline{q_2^*},\, \overline{q_3^*})e^{-i\omega_0\tau_k(\xi-\theta)}\, d\eta(\theta)(1,\, q_1,\, q_2,\, q_3)^T e^{i\omega_0\tau_k\xi} d\xi \\
&= \bar{D}\{1 + q_1\overline{q_1^*} + q_2\overline{q_2^*} + q_3\overline{q_3^*} - \int_{-1}^{0}(1,\, \overline{q_1^*},\, \overline{q_2^*},\, \overline{q_3^*})\theta\, e^{i\omega_0\tau_k\theta}\, d\eta(\theta)(1,\, q_1,\, q_2,\, q_3)^T\} \\
&= \bar{D}\{1 + q_1\overline{q_1^*} + q_2\overline{q_2^*} + q_3\overline{q_3^*} + \tau_k g q_1(-1+\overline{q_1^*})e^{-i\omega_0\tau_k}\}.
\end{aligned}$$

Therefore, we can choose $D$ as

$$D = \frac{1}{1 + \bar{q}_1 q_1^* + \bar{q}_2 q_2^* + \bar{q}_3 q_3^* + \tau_k g \bar{q}_1(-1+q_1^*)e^{i\omega_0\tau_k}}.$$

Next we will compute the co-ordinate to describe the center manifold $C_0$ at $\nu = 0$. Let $u_t$ be the solution of (3.8) when $\nu = 0$. Define

$$z(t) = \langle q^*, u_t\rangle, \quad W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \tag{3.11}$$

On the center manifold $C_0$, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots, \tag{3.12}$$

$z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\overline{q^*}$. Note that $W$ is real if $u_t$ is real. We only consider real solutions. For solution $u_t \in C_0$ of (3.8), since $\nu = 0$, we have

$$\begin{aligned}
\dot{z}(t) &= \langle q^*, \dot{u}_t\rangle \\
&= i\omega_0\tau_k z + \overline{q^*}(0)\bar{f}(0, W(z,\bar{z},0) + 2\,\mathrm{Re}\{zq(\theta)\}) \\
&\cong i\omega_0\tau_k z + \overline{q^*}(0)\bar{f}_0(z,\bar{z}).
\end{aligned}$$

We rewrite this equation as

$$\dot{z}(t) = i\omega_0\tau_k z(t) + g'(z,\bar{z}),$$

where

$$g'(z,\bar{z}) = \overline{q^*}(0)\bar{f}_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \dots \tag{3.13}$$

It follows from (3.11) and (3.12) that

$$\begin{aligned} u_t(\theta) &= W(t,\theta) + 2\,\mathrm{Re}\{z(t)q(\theta)\} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + zq + \bar{z}\,\bar{q} + \dots \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1,q_1,q_2,q_3)^T e^{i\omega_0\tau_k\theta}z + (1,\bar{q}_1,\bar{q}_2,\bar{q}_3)^T e^{-i\omega_0\tau_k\theta}\bar{z} + \dots \end{aligned} \tag{3.14}$$

It follows together with (3.3) that

$$\begin{aligned} g'(z,\bar{z}) &= \overline{q^*}(0)\bar{f}_0(z,\bar{z}) \\ &= \tau_k\overline{D}\,(1,\overline{q_1^*},\overline{q_2^*},\overline{q_3^*})\begin{bmatrix} -au_{1t}^2(0) \\ -pu_{2t}(0)u_{3t}(0) \\ hu_{3t}(0)u_{4t}(0) \\ 0 \end{bmatrix} + \tau_k\overline{D}\,(1,\overline{q_1^*},\overline{q_2^*},\overline{q_3^*})\begin{bmatrix} -bM_1 \\ bM_1 \\ 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{3.15}$$

where

$$M_1 = \frac{1}{(1+S^*)^2}u_{1t}(0)u_{2t}(-1) - \frac{1}{(1+S^*)^3}u_{1t}^2(0)u_{2t}(-1) + \frac{1}{(1+S^*)^4}u_{1t}^3(0)u_{2t}(-1) + \dots$$

$$\begin{aligned} &(1,\overline{q_1^*},\overline{q_2^*},\overline{q_3^*})\begin{bmatrix} -au_{1t}^2(0) \\ -pu_{2t}(0)u_{3t}(0) \\ hu_{3t}(0)u_{4t}(0) \\ 0 \end{bmatrix} \\ &= -a\{z+\bar{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+O(|(z,\bar{z})|^3)\}^2 \\ &\quad - p\overline{q_1^*}\{zq_1+\bar{z}\bar{q}_1+W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+O(|(z,\bar{z})|^3)\} \\ &\quad \times\{zq_2+\bar{z}\bar{q}_2+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\bar{z}+W_{02}^{(3)}(0)\frac{\bar{z}^2}{2}+O(|(z,\bar{z})|^3)\} \\ &\quad + h\overline{q_2^*}\{zq_2+\bar{z}\bar{q}_2+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\bar{z}+W_{02}^{(3)}(0)\frac{\bar{z}^2}{2}+O(|(z,\bar{z})|^3)\} \\ &\quad \times\{zq_3+\bar{z}\bar{q}_3+W_{20}^{(4)}(0)\frac{z^2}{2}+W_{11}^{(4)}(0)z\bar{z}+W_{02}^{(4)}(0)\frac{\bar{z}^2}{2}+O(|(z,\bar{z})|^3)\} \\ &= a_{11}z^2 + a_{12}z\bar{z} + a_{13}\bar{z}^2 + a_{14}z^2\bar{z} + \dots, \end{aligned}$$

where

$$a_{11} = -a - pq_1q_2\overline{q_1^*} + hq_2q_3\overline{q_2^*}, \quad a_{12} = -2a - p\overline{q_1^*}(q_1\overline{q}_2 + \overline{q}_1q_2) + h\overline{q_2^*}(q_2\overline{q}_3 + \overline{q}_2q_3),$$

$$a_{13} = -a - p\overline{q_1^*}\,\overline{q}_1\overline{q}_2 + h\overline{q_2^*}\,\overline{q}_2\overline{q}_3,$$

$$a_{14} = -a\{2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\} - p\overline{q_1^*}\{\overline{q}_1W_{20}^{(3)}(0)/2 + q_1W_{11}^{(3)}(0) + \overline{q}_2W_{20}^{(2)}(0)/2 + q_2W_{11}^{(2)}(0)\}$$
$$+ h\overline{q_2^*}\{q_2W_{11}^{(4)}(0) + \overline{q}_2W_{20}^{(4)}(0)/2 + \overline{q}_3W_{20}^{(3)}(0)/2 + q_3W_{11}^{(3)}(0)\}.$$

and

$$M_1 = b_{11}z^2 + b_{12}z\overline{z} + b_{13}\overline{z}^2 + b_{14}z^2\overline{z} + \ldots,$$

where

$$b_{11} = \frac{1}{(1+S^*)^2}q_1e^{-i\omega_0\tau_k},\ b_{12} = \frac{1}{(1+S^*)^2}(\overline{q}_1e^{i\omega_0\tau_k} + q_1e^{-i\omega_0\tau_k}),\ b_{13} = \frac{1}{(1+S^*)^2}\overline{q}_1e^{i\omega_0\tau_k},$$

$$b_{14} = \frac{1}{(1+S^*)^2}\{W_{11}^{(2)}(-1) + W_{20}^{(2)}(-1)/2 + \overline{q}_1W_{20}^{(1)}(0)e^{i\omega_0\tau_k}/2 + q_1W_{11}^{(1)}(0)e^{-i\omega_0\tau_k}\}$$
$$-\frac{1}{(1+S^*)^3}\{2q_1e^{-i\omega_0\tau_k} + \overline{q}_1e^{i\omega_0\tau_k}\}.$$

From (3.15), we have

$$g'(z,\overline{z}) = \tau_k\overline{D}\{(a_{11}z^2 + a_{12}z\overline{z} + a_{13}\overline{z}^2 + a_{14}z^2\overline{z} + \ldots) + b(-1+\overline{q_1^*})(b_{11}z^2 + b_{12}z\overline{z} + b_{13}\overline{z}^2 + b_{14}z^2\overline{z} + \ldots)\}. \tag{3.16}$$

Comparing the coefficients of (3.13) and (3.16), we get

$$\begin{cases} g_{20} = 2\tau_k\overline{D}\{a_{11} + b(\overline{q_1^*} - 1)b_{11}\}, \\ g_{11} = \tau_k\overline{D}\{a_{12} + b(\overline{q_1^*} - 1)b_{12}\}, \\ g_{02} = 2\tau_k\overline{D}\{a_{13} + b(\overline{q_1^*} - 1)b_{13}\}, \\ g_{21} = 2\tau_k\overline{D}\{a_{14} + b(\overline{q_1^*} - 1)b_{14}\}. \end{cases} \tag{3.17}$$

Since $W_{20}(\theta)$ and $W_{11}(\theta)$ are in $g_{21}$, we still need to compute them. From (3.8) and (3.11), we have

$$\begin{aligned}\dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\,\bar{q} \\ &= AW - 2\operatorname{Re}\{\overline{q^*}(0)\bar{f}_0 q(\theta)\} + Ru_t \\ &= \begin{cases} AW - 2\operatorname{Re}\{\overline{q^*}(0)\bar{f}_0 q(\theta)\}, & \textit{if } -1 \le \theta \le 0, \\ AW - 2\operatorname{Re}\{\overline{q^*}(0)\bar{f}_0 q(\theta)\} + \bar{f}_0, & \textit{if } \theta = 0, \end{cases} \\ &\equiv A(0)W + H(z,\bar{z},\theta), \end{aligned} \tag{3.18}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots \tag{3.19}$$

We know

$$W = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots$$

$$\begin{aligned}\therefore \dot{W} &= W_z\dot{z}(t) + W_{\bar{z}}\dot{\bar{z}}(t) \\ &= (W_{20}(\theta)z + W_{11}(\theta)\bar{z} + \ldots)(i\omega_0\tau_k z(t) + g'(z,\bar{z})) \\ &\qquad + (W_{11}(\theta)z + W_{02}(\theta)\bar{z} + \ldots)(-i\omega_0\tau_k\bar{z}(t) + \overline{g}'(z,\bar{z})). \end{aligned} \tag{3.20}$$

From (3.18),

$$\begin{aligned}\dot{W} &= A(0)(W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots) + H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots \\ &= (A(0)W_{20}(\theta) + H_{20}(\theta))\frac{z^2}{2} + (A(0)W_{11}(\theta) + H_{11}(\theta))z\bar{z} + (A(0)W_{02}(\theta) + H_{02}(\theta))\frac{\bar{z}^2}{2} + \ldots \end{aligned} \tag{3.21}$$

Comparing the coefficients of $z^2$ and $z\bar{z}$ from (3.20) and (3.21), we get

$$(A(0) - 2i\omega_0\tau_k I)W_{20}(\theta) = -H_{20}(\theta), \qquad A(0)W_{11}(\theta) = -H_{11}(\theta). \tag{3.22}$$

For $\theta \in [-1,0]$, we have from (3.18) and (3.13)

$$\begin{aligned}H(z,\bar{z},\theta) &= -\overline{q^*}(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\ &= -g'(z,\bar{z})q(\theta) - \overline{g}'(z,\bar{z})\bar{q}(\theta) \\ &= -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + \ldots)q(\theta) - (\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \ldots)\bar{q}(\theta). \end{aligned} \tag{3.23}$$

Again, comparing the coefficients of $z^2$ and $z\bar{z}$ between (3.19) and (3.23), we get

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{20}\bar{q}(\theta), \tag{3.24}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.25}$$

From the definition of $A$ and (3.22) and (3.24), we get

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Since $q(\theta) = (1,\, q_1,\, q_2,\, q_3)^T e^{i\omega_0\tau_k\theta}$, so we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_1' e^{2i\omega_0\tau_k\theta}, \tag{3.26}$$

where $E_1' = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)})^T$ is a constant vector.

Similarly, from (3.22) and (3.25), we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_k}q(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{q}(0)e^{-i\omega_0\tau_k\theta} + E_2', \tag{3.27}$$

where $E_2' = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}, E_2^{(4)})^T$ is a constant vector.

We find the values of $E_1'$ and $E_2'$. From the definition of $A(0)$ and (3.22), we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0), \tag{3.28}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{3.29}$$

where $\eta(\theta) = \eta(0,\theta)$.

By (3.18), we know when $\theta = 0$,

$$\begin{aligned} H(z,\bar{z},0) &= -2\,\mathrm{Re}\{\overline{q^*}(0)f_0 q(0)\} + \bar{f}_0 \\ &= -g'(z,\bar{z})q(0) - \bar{g}'(z,\bar{z})\bar{q}(0) + \bar{f}_0. \end{aligned}$$

i.e. $$H_{20}(0)\frac{z^2}{2} + H_{11}(0)z\bar{z} + H_{02}(0)\frac{\bar{z}^2}{2} + \ldots = -(g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \ldots)q(0)$$
$$- (\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \ldots)\bar{q}(0) + \bar{f}_0.$$

(3.30)

By (3.3), we have

$$\bar{f}_0 = \tau_k \begin{bmatrix} -au_{1t}^2(0) - bM_1 \\ bM_1 - pu_{2t}(0)u_{3t}(0) \\ hu_{3t}(0)u_{4t}(0) \\ 0 \end{bmatrix}.$$

By (3.11), we obtain

$$\begin{aligned} u_t(\theta) &= W(t,\theta) + 2\,\mathrm{Re}\{z(t)q(\theta)\} \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta) + \ldots \end{aligned}$$

Then, we have

$$\bar{f}_0 = \tau_k \begin{bmatrix} -a - \dfrac{b}{(1+S^*)^2} q_1 e^{-i\omega_0\tau_k} \\ \dfrac{b}{(1+S^*)^2} q_1 e^{-i\omega_0\tau_k} - pq_1q_2 \\ hq_2q_3 \\ 0 \end{bmatrix} z^2 + \tau_k \begin{bmatrix} -2a - \dfrac{2b}{(1+S^*)^2}\mathrm{Re}\{q_1 e^{-i\omega_0\tau_k}\} \\ \dfrac{2b}{(1+S^*)^2}\mathrm{Re}\{q_1 e^{-i\omega_0\tau_k}\} - 2p\,\mathrm{Re}\{q_1\bar{q}_2\} \\ 2h\,\mathrm{Re}\{q_2\bar{q}_3\} \\ 0 \end{bmatrix} z\bar{z} + \ldots \quad (3.31)$$

From (3.30) and (3.31), we get

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_k \begin{bmatrix} -a - bb_{11} \\ bb_{11} - pq_1q_2 \\ hq_2q_3 \\ 0 \end{bmatrix}, \quad (3.32)$$

and

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_k \begin{bmatrix} n_1' \\ n_2' \\ n_3' \\ 0 \end{bmatrix}, \quad (3.33)$$

where

$$n_1' = -a - \frac{b}{(1+S^*)^2}\operatorname{Re}\{q_1 e^{-i\omega_0\tau_k}\},\ n_2' = \frac{b}{(1+S^*)^2}\operatorname{Re}\{q_1 e^{-i\omega_0\tau_k}\} - p\operatorname{Re}\{q_1\overline{q}_2\},\ n_3' = h\operatorname{Re}\{q_2\overline{q}_3\}.$$

Since $i\omega_0\tau_k$ is the eigenvalue of $A(0)$ and $q(0)$ is the corresponding eigenvector, we obtain

$$(i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta} d\eta(\theta))q(0) = 0,$$

and

$$(-i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta} d\eta(\theta))\overline{q}(0) = 0.$$

Therefore, substituting (3.26) and (3.32) into (3.28), we get

$$(2i\omega_0\tau_k I - \int_{-1}^{0} e^{2i\omega_0\tau_k\theta} d\eta(\theta))E_1' = 2\tau_k \begin{bmatrix} -a - bb_{11} \\ bb_{11} - pq_1q_2 \\ hq_2q_3 \\ 0 \end{bmatrix}.$$

That is,

$$\begin{bmatrix} 2i\omega_0 + (a-f)S^* & ge^{-2i\omega_0\tau_k} & 0 & 0 \\ -f & 2i\omega_0 + g - ge^{-2i\omega_0\tau_k} & pI^* & 0 \\ 0 & 0 & 2i\omega_0 & -hP^* \\ 0 & -c & 0 & 2i\omega_0 + c \end{bmatrix} E_1' = 2\begin{bmatrix} -a - bb_{11} \\ bb_{11} - pq_1q_2 \\ hq_2q_3 \\ 0 \end{bmatrix}.$$

This implies that

$$E_1^{(1)} = \frac{2}{\Lambda_1}\begin{vmatrix} -a - bb_{11} & ge^{-2i\omega_0\tau_k} & 0 & 0 \\ bb_{11} - pq_1q_2 & 2i\omega_0 + g - ge^{-2i\omega_0\tau_k} & pI^* & 0 \\ hq_2q_3 & 0 & 2i\omega_0 & -hP^* \\ 0 & -c & 0 & 2i\omega_0 + c \end{vmatrix},$$

$$E_1^{(2)} = \frac{2}{\Lambda_1}\begin{vmatrix} 2i\omega_0 + (a-f)S^* & -a-bb_{11} & 0 & 0 \\ -f & bb_{11} - pq_1q_2 & pI^* & 0 \\ 0 & hq_2q_3 & 2i\omega_0 & -hP^* \\ 0 & 0 & 0 & 2i\omega_0 + c \end{vmatrix},$$

$$E_1^{(3)} = \frac{2}{\Lambda_1}\begin{vmatrix} 2i\omega_0 + (a-f)S^* & ge^{-2i\omega_0\tau_k} & -a-bb_{11} & 0 \\ -f & 2i\omega_0 + g - ge^{-2i\omega_0\tau_k} & bb_{11} - pq_1q_2 & 0 \\ 0 & 0 & hq_2q_3 & -hP^* \\ 0 & -c & 0 & 2i\omega_0 + c \end{vmatrix},$$

$$E_1^{(4)} = \frac{2}{\Lambda_1}\begin{vmatrix} 2i\omega_0 + (a-f)S^* & ge^{-2i\omega_0\tau_k} & 0 & -a-bb_{11} \\ -f & 2i\omega_0 + g - ge^{-2i\omega_0\tau_k} & pI^* & bb_{11} - pq_1q_2 \\ 0 & 0 & 2i\omega_0 & hq_2q_3 \\ 0 & -c & 0 & 0 \end{vmatrix},$$

where

$$\Lambda_1 = \begin{vmatrix} 2i\omega_0 + (a-f)S^* & ge^{-2i\omega_0\tau_k} & 0 & 0 \\ -f & 2i\omega_0 + g - ge^{-2i\omega_0\tau_k} & pI^* & 0 \\ 0 & 0 & 2i\omega_0 & -hP^* \\ 0 & -c & 0 & 2i\omega_0 + c \end{vmatrix}.$$

Similarly, substituting (3.27) and (3.33) into (3.29), we get

$$\begin{bmatrix} (a-f)S^* & g & 0 & 0 \\ -f & 0 & pI^* & 0 \\ 0 & 0 & 0 & -hP^* \\ 0 & -c & 0 & c \end{bmatrix} E_2' = 2\begin{bmatrix} n_1' \\ n_2' \\ n_3' \\ 0 \end{bmatrix}.$$

Hence, we obtain

$$E_2^{(1)} = \frac{2}{\Lambda_2}\begin{vmatrix} n_1' & g & 0 & 0 \\ n_2' & 0 & pI^* & 0 \\ n_3' & 0 & 0 & -hP^* \\ 0 & -c & 0 & c \end{vmatrix}, \quad E_2^{(2)} = \frac{2}{\Lambda_2}\begin{vmatrix} (a-f)S^* & n_1' & 0 & 0 \\ -f & n_2' & pI^* & 0 \\ 0 & n_3' & 0 & -hP^* \\ 0 & 0 & 0 & c \end{vmatrix},$$

$$E_2^{(3)} = \frac{2}{\Lambda_2}\begin{vmatrix} (a-f)S^* & g & n_1' & 0 \\ -f & 0 & n_2' & 0 \\ 0 & 0 & n_3' & -hP^* \\ 0 & -c & 0 & c \end{vmatrix}, \quad E_2^{(4)} = \frac{2}{\Lambda_2}\begin{vmatrix} (a-f)S^* & g & 0 & n_1' \\ -f & 0 & pI^* & n_2' \\ 0 & 0 & 0 & n_3' \\ 0 & -c & 0 & c \end{vmatrix},$$

where

$$\Lambda_2 = \begin{vmatrix} (a-f)S^* & g & 0 & 0 \\ -f & 0 & pI^* & 0 \\ 0 & 0 & 0 & -hP^* \\ 0 & -c & 0 & c \end{vmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (3.26) and (3.27). Furthermore, we can compute $g_{21}$ by (3.17). Hence, we can compute the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_k}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\left\{\frac{d\lambda(\tau_k)}{d\tau}\right\}}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\left\{\frac{d\lambda(\tau_k)}{d\tau}\right\}}{\omega_0\tau_k}, \quad k = 0,1,2,... \end{aligned} \tag{3.34}$$

By the result of Hassard et al. (1981), we have the following theorem:

**Theorem 3.1:**

In (3.34), the sign of $\mu_2$ determined the direction of the Hopf bifurcation: if $\mu_2 > 0\,(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solution exist for $\tau > \tau_k\,(\tau < \tau_k)$; $\beta_2$ determines the stability of the bifurcating periodic solution: the bifurcating periodic solution is stable (unstable) if $\beta_2 < 0\,(\beta_2 > 0)$ and $T_2$ determines the period of the bifurcating periodic solution: the period increases (decreases) if $T_2 > 0\,(T_2 < 0)$.

## 4. Numerical Simulations and Discussion

In this section, we have studied the existence of the Hopf bifurcation of the system (1.7) by choosing a set of parameter values. Consider the following system:

$$\begin{cases} \dfrac{dS}{dt} = S(1-0.11S) - \dfrac{0.7SI(t-\tau)}{1+S}, \\ \dfrac{dI}{dt} = \dfrac{0.7SI(t-\tau)}{1+S} - 0.42I - 0.02IP, \\ \dfrac{dP}{dt} = -0.3P + 0.4PQ, \\ \dfrac{dQ}{dt} = 3(I-Q). \end{cases} \tag{4.1}$$

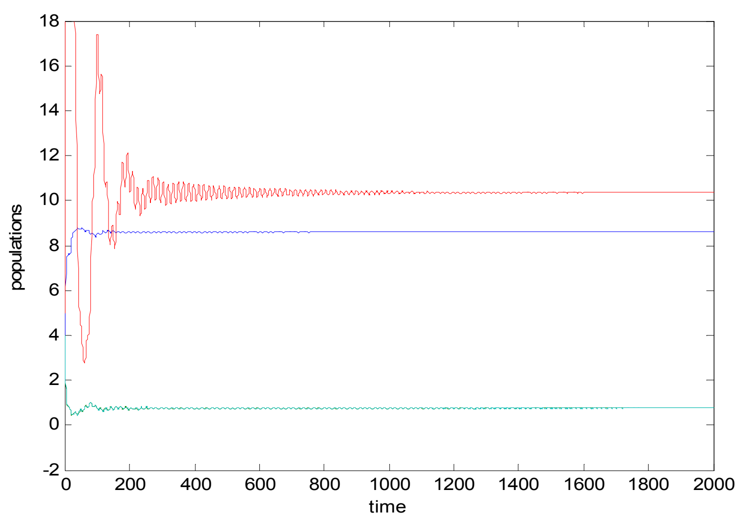


**Figure 1.** The endemic equilibrium $E^* = (8.59341, 0.75, 10.3517, 0.75)$ of the system (4.1) is asymptotically stable for $\tau = 14.73 < \tau_0$. All the trajectories of the state variables converge to their respective equilibrium values.

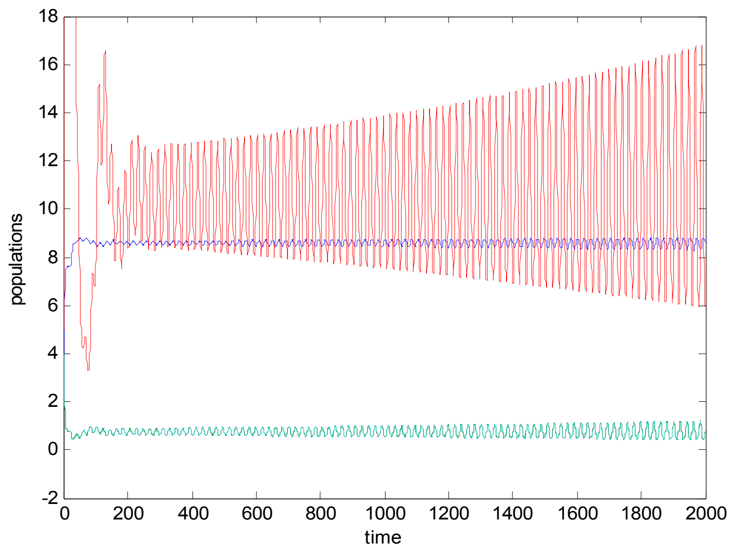


**Figure 2.** The bifurcating periodic solutions for $\tau = 20.11 > \tau_0$.
The figure shows that the equilibrium point
$E^* = (8.59341, 0.75, 10.3517, 0.75)$ is unstable.

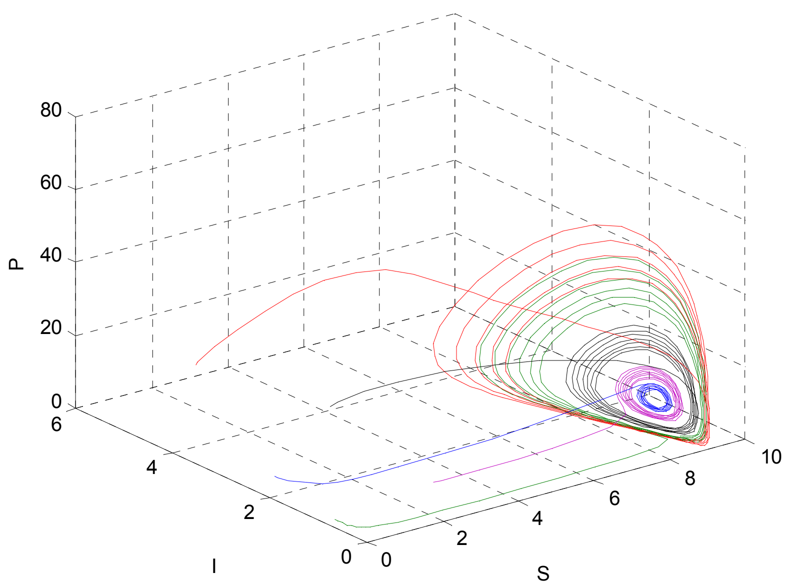


**Figure 3.** Bifurcation diagram of the system (4.1) in $(S, I, P)$-space for $\tau = 0$.

The system (4.1) has an endemic equilibrium $E^* = (8.59341, 0.75, 10.3517, 0.75)$ and for $\tau = 0$, the endemic equilibrium $E^*$ is locally asymptotically stable. The values of $\omega_0 = 0.0800723$, $\tau_0 = 19.901$ and from the formulae (3.34), we obtain $c_1(0) = 22.9545 - 19.7103i$, $\mu_2 = 11290 > 0$, $\beta_2 = 45.909 > 0$ and $T_2 = 29.345 > 0$. Thus, the endemic equilibrium $E^*$ is asymptotically stable when $\tau = 14.73 < \tau_0$ and unstable when

$\tau = 20.11 > \tau_0$, as shown in the Figures 1 and 2 respectively. In Figure 2, it is found that all the trajectories of susceptible, infected prey and predator bifurcate periodically around their equilibrium point. The equilibrium point $(8.59341, 0.75, 10.3517)$ is periodically stable when no delay parameter is introduced in the model system (4.1) and is numerically investigated in figure 3.

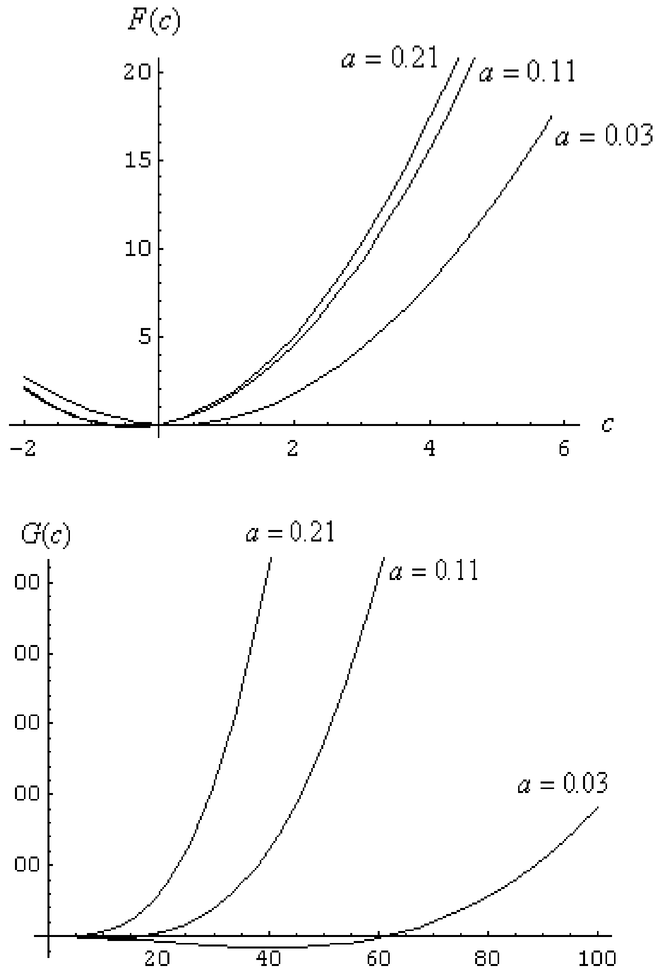


**Figure 4.** Local stability properties of endemic equilibrium $E^*$ through the functions $F(c) = a_1 a_2 - a_3$ and $G(c) = Lc^3 + Mc^2 + Nc$ for $\tau = 0$. $F(c), G(c) > 0$ ensure the local asymptotic stability of $E^*$. The functions $F(c)$ and $G(c)$ are plotted for different values of $a$ i.e. for different values of carrying capacity and the saturation constant.

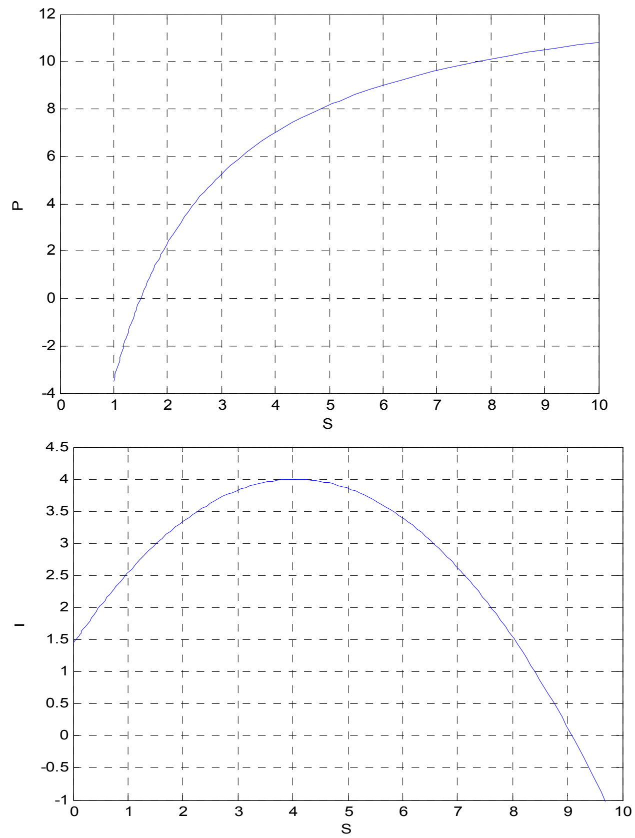


**Figure 5.** The figures indicate variation of both the predator and infectious prey populations with the susceptible prey population. The plot functions imply the growth of predator depends on the prey population.

In Figure 4, it is observed that for different values of $a$, i.e., for different values of saturation factor and carrying capacity, the endemic equilibrium $E^*$ is locally asymptotically stable for $\tau = 0$. Also, when the susceptible prey gradually increases, the predator population increases but infected population decreases and extinct entirely for $S \geq 9.1$, which is shown in Figure 5.

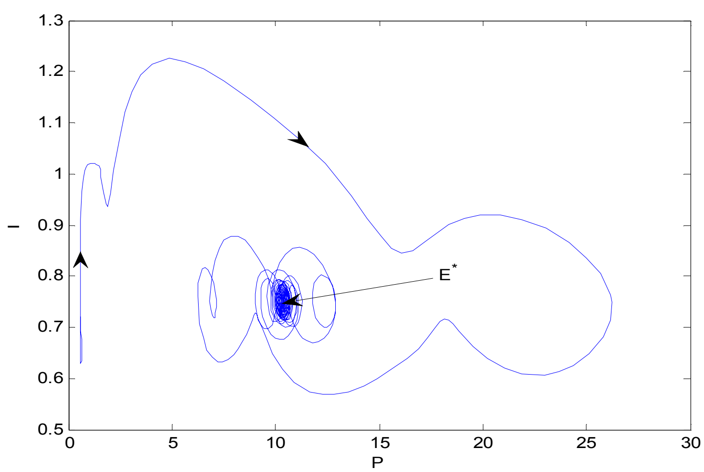


**Figure 6.** The endemic equilibrium $(10.3517, 0.75)$ is locally asymptotically stable for some initial parameter values in $(P, I)$ plane when $\tau = 14.73 < \tau_0$.

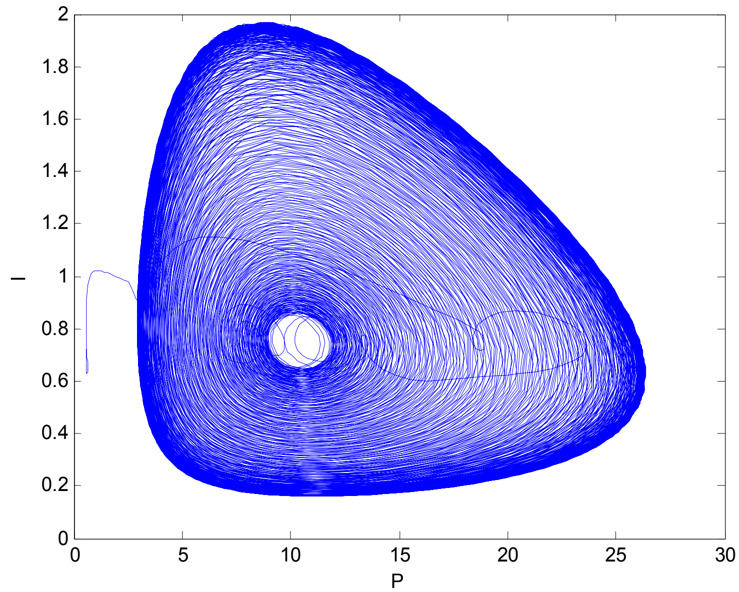


**Figure 7.** The figure indicates projection of the phase portrait of system (4.1) in $(P, I)$ plane for $\tau = 20.11 > \tau_0$.

The Figure 6 shows that when the value of delay parameter lies below the critical value, the infected population initially increases and when rate of predation increases the number of infected prey population drops off and the path approaches to their equilibrium values in finite time. But, when the value of delay parameter is beyond the critical value, Hopf-bifurcation occurs for the system (4.1) and there exist limit cycle near the equilibrium point $(10.3517, 0.75)$ as demonstrated in Figure 7.

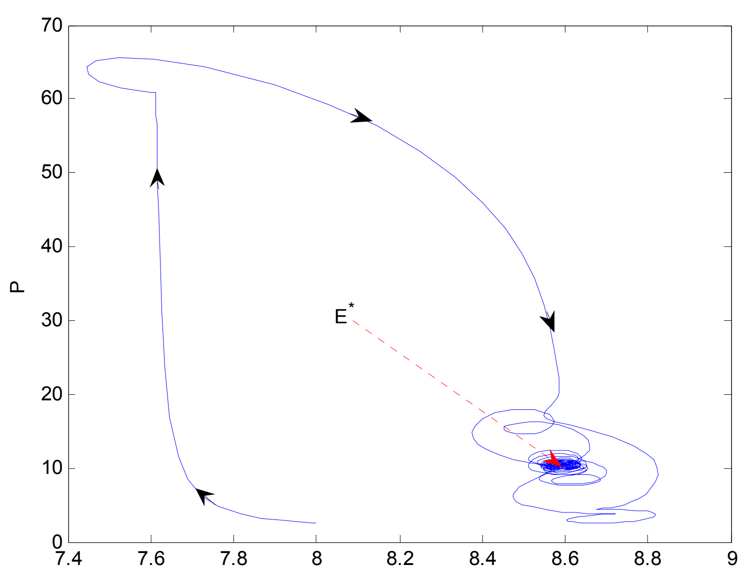


**Figure 8.** The positive equilibrium $(8.59341, 10.3517)$ is locally asymptotically stable for some hypothetical parameter values in $(S, P)$ plane when the delay parameter value below the critical delay value i.e. $\tau = 14.73 < \tau_0$.

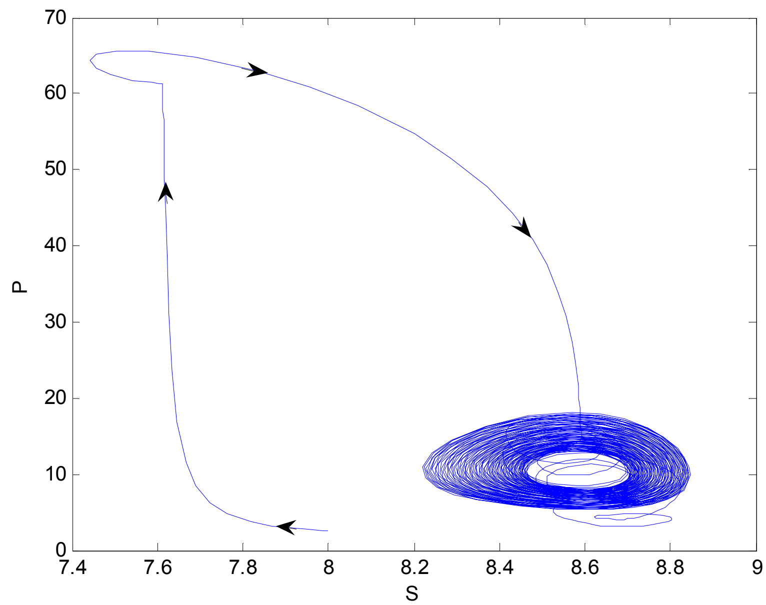


**Figure 9.** The figure shows projection of the phase portrait of system (4.1) in $(S, P)$ plane for $\tau = 20.11 > \tau_0$.

Again, assuming the value of the delay parameter $\tau = 14.73$ below the critical value $\tau_0 = 19.901$, we sketch the trajectory in $(S, P)$-plane as depicted in figure 8. Here we observe that initially the number of predator population increases when the susceptible population decreases and after some finite time the number of predator population decreases while the number of susceptible population increases and the path approaches to their equilibrium value $(8.59341, 10.3517)$ in finite time. But, when the value of delay parameter $\tau = 20.11 > \tau_0$, Hopf-bifurcation occurs for the system and there exist limit cycle nears the equilibrium point $(8.59341, 10.3517)$ as demonstrated in Figure 9.

## 5. Conclusions

In this paper, we propose a prey-predator model with the assumption that the disease is spreading only among the prey species and though the predator species feeds on both the susceptible and infected prey species, the rate of predation on infected prey is more than the susceptible prey as it becomes more vulnerable to predation. The dynamical behavior of the system is investigated from the point of view of stability analysis. The system is locally asymptotically stable in some region of the parametric space and exhibits periodic oscillations in some other region. Some conditions are obtained for small amplitude periodic solutions bifurcating from a positive interior equilibrium by applying both mathematical and numerical techniques. The stability as well as the direction of bifurcation is obtained by applying the algorithm due to Hassard et al. (1981) that depends on the centre manifold theorem. There is a minimum force of infection below which the disease does not spread out. Numerical simulations substantiate the analytical results.

## *Acknowledgements*

*The authors are thankful to the referees for their valuable comments and suggestions on the manuscripts*.

## REFERENCES

Beretta, E. and Kuang, Y. (2002). Geometric stability switch criteria in delay differential systems with delay dependent parameters, SIAM J. Math. Anal., 33, pp. 1144-1165.

Chattopadhyay, J. and Bairagi, N. (2001). Pelicans at risk in Salton sea-an ecoepidemiological study, Ecol. Model, 136, pp. 103-112.

Greenhalgh, D., Khan, Q.J.A. Lewis, F.I. (2004). Hopf-bifurcationin two SIRS density dependent epidemic models, Math. Comput. Modell., 39, pp. 1261-1283.

Hadeler, K.P. and Freedman, H.I. (1989). Predator-prey populations with parasitic infection, J. Math. Biol., 27, pp. 609-631.

Hale, J.K. and Verduyn. (1993). S.M., Introduction to Functional Differential Equations, Springer-Verlag, New York.

Hassard, B., Kazarinoffand, D. and Wan, Y. (1981). Theorey and Application of Hopf Bifurcation, Cambridge University Press, Cambridge.

Hethcote, H.W., L. Yi and Zhujun, J. (1999). Hopf-bifurcation in models for pertussis epidemiology, Math. Comput. Model, 30, pp. 29-45.

Kar, T.K. (2003). Selective Harvesting in a prey-predator fishery with time delay, Math. Comp. Model., 38, pp. 449-458.

Li, X., Wei, J. (2005). On the zeros of a fourth degree exponential polynomial with applications to a neural network model with delays, Chaos Solitions Fractals, 26, pp. 519-526.

Ruan, S.G. and Wang, W.D. (2003). Dynamical behaviour of an epidemic model with a nonlinear incidence rate, J. Differ. Equat., 188, pp. 135-163.

Slack, G. (1997). Salton sea fisheries, Pacific Discovery, Winter.

Xio, Y.N. and Chen, L.S. (2001). Modelling and analysis of a predator-prey model with disease in the prey, Math. Biosci., 171, pp. 59-82.

Yan, X. and Zhang, C. (2008). Hopf-bifurcation in a delayed Lotka-Volterra predator-prey system, Nonlinear Anal. RWA, 9, pp. 114-127.

Yan, X. and Li, W. (2006). Hopf-bifurcation and global periodic solutions in a delayed predator-prey system, Appl. Math. Comput., 177, pp. 427-445.

Cai, L., Guo, S., Li, X. and Ghosh, M. (2009). Global dynamics of a dengue epidemic mathematical model, Chaos Solitons Fractals, 42, pp. 2297–2304.

Esteva, L. and Matias, M. (2001). A model for vector transmitted diseases with saturation incidence, J. Biol. Syst., 9 (4), pp. 235–245.

Kar, T.K. and Mondal, P.K. (2011). Global dynamics and bifurcation in delayed SIR epidemic model, Nonlinear Analysis: Real World Applications, 12, pp. 2058–2068.

Bhattacharyya, R. and Mukhopadhyay, B. (2010). Analysis of periodic solutions in an eco-epidemiological model with saturation incidence and latency delay, Nonlinear Analysis: Hybrid Systems, 4, pp. 176-188.

Cushing, J.M. (1977). Intrgro-differential equations and delay models in populations' dynamics, Lecture Notes in Biomathematics, vol. 20, Springer, Berlin / New York.

Farkas, M. (2001). Dynamical models in biology, Academic press, New York.
MacDonald, N. (1977). Time delay in predator-prey model II, Math. Biosci., 33, pp. 227-234.
Haque, M. Zhen, J. and Venturino, E. (2008). An ecoepidemiological predator-prey model with standard disease incidence, Math. Meth. Appl. Sci., pp. 1-24.