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ON EXTENSION OF MITTAG-LEFFLER FUNCTION

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Abstract

In this paper, we study the extended Mittag -Leffler function by using generalized beta function and obtain various differential properties, integral representations. Further, we discuss Mellin transform of these functions in terms of generalized Wright hyper geometric function and evaluate Laplace transform, and Whittaker transform in terms of extended beta function. Finally, several interesting special cases of extended Mittag -Leffler functions have also be given.

Keywords: Beta Functions; Mellin Transform; Laplace Transform; Whittaker Transform; Extended Riemann Liouville Fractional derivative operator

MSC 2010 No. : 33E12, 44A10

1. Introduction

The Mittag-Leffler function occurs naturally in the solution of fractional order and integral equation. The importance of such functions in physics and engineering is steadily increasing. Some application of the Mittag-Leffler is carried out in the Study of Kinetic Equation, Study of Lorenz System, Random Walk, Levy Flights and Complex System and also in applied problems such as fluid flow, electric network, probability and statistical distribution theory. Haubold et al. (2011) studied various properties of Mittag-Leffler function.

In 1903, the Swedish mathematician Gosta Mittag-Leffler introduced the function.

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{1}$$

where z is a complex variable, Γ is a Gamma function, and $\alpha \geq 0$. It is a direct generalization of the exponential function for $\alpha=1$, and for $0<\alpha<1$, it interpolates between exponential and hypergeometric function 1/(1-z). The generalization of $E_\alpha(z)$ was further studied by Wiman (1905) as:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta\in\mathbb{C},\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0), \tag{2}$$

Prabhakar (1971) introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in as:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},\ (\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0), \tag{3}$$

where $(Y)_n$ is the Pochhamer symbol by Rainville (1960) such that

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},\ (\gamma)_0=1,$$
$$(\gamma)_n=\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1),\ \ for\ n\geq 1.$$

Shukla and Prajapati (2007) introduced the extended Mittag-Leffler function $E^{\gamma,q}_{\alpha,\beta}(z)$ which is defined as follows:

$$E^{\gamma,q}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},$$
$$\{\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ q\in(0,1)\cup\mathbb{N}\}, \tag{4}$$

$E_{\alpha,\beta}^{\gamma,q}(z)$ Converge absolutely $\forall\ |z|<1$, if $q=\mathrm{Re}(\alpha)+1$, $\forall z$ and for Beta function defined as:

$$B(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du=\frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)},\quad \mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0. \tag{5}$$

Laplace transform of the function $f(z)$ is defined as:

$$L(f(z);s)=\int_0^\infty e^{-sz}f(z)\,dz. \tag{6}$$

Mellin-transform and its inverse Transform of the function $f(z)$ is defined as:

$$M\left[f(z);s\right]=f^*(s)=\int_0^\infty z^{s-1}f(z)dz,\quad \mathrm{Re}(s)>0, \tag{7}$$

$$f(z)=M^{-1}\left[f^*(s);z\right]=\frac{1}{2\pi i}\int_L f^*(s)z^{-s}ds. \tag{8}$$

Whittaker transforms (Whittaker and Watson (1996)) is defined as:

$$\int_0^\infty e^{-t/2}\,t^{v-1}\,W_{\lambda,\mu}(t)\,dt=\frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)}, \tag{9}$$

where $Re(\mu\pm v)>-\frac{1}{2}$ and $w_{\lambda,\mu}(t)$ is the Whittaker confluent hyper geometric function.

Wright generalized hypergeometric function is defined as:

$$ {}_p\psi_q\left[(a_1,A_1),\dots,(a_p,A_p);(b_1,B_1),\dots,(b_q,B_q);z\right] =\sum_{n=0}^{\infty}\frac{\prod_{i=0}^{P}\Gamma(a_i+A_i\,n)}{\prod_{j=0}^{q}\Gamma(b_j+B_j\,n)}.\frac{z^n}{n!}. \tag{10}$$

The classical Riemann-Liouville fractional derivative of order μ is usually defined by

$$D_z^{\mu}\left[f(z)\right]=\frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}dt,\quad (\mathrm{Re}(\mu)<0),$$

where the integration path is a line from 0 to z in the complex t-plane. For the case $(m-1) < Re(\mu) < m\ (m = 1,2,3,\ldots)$, it is defined by

$$D_z^{\mu}\left[f(z)\right] = \frac{d^m}{dz^m} D_z^{\mu-m}\left\{f(z)\right\}$$
$$= \frac{d^m}{dz^m}\left[\frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}\,dt\right].$$

The extended Riemann-Liouville fractional derivative operator was defined by Özarslan and Özergin (2010) as follows:

$$D_z^{\mu,p}\left[f(z)\right] = \frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1} \exp\left(\frac{-pz^2}{t(z-t)}\right)dt, \tag{11}$$

$$\mathrm{Re}(\mu) < 0, \mathrm{Re}(p) > 0 \text{ and for } (m-1) < \mathrm{Re}(\mu) < m, (m = 1,2,3,\ldots).$$

$$D_z^{\mu,p}\left[f(z)\right] = \frac{d^m}{dz^m} D_z^{\mu-m}\left[f(z)\right]$$
$$= \frac{d^m}{dz^m}\left[\frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1} \exp\left(\frac{-pz^2}{t(z-t)}\right)dt\right], \tag{12}$$

where the path of integration is a line from 0 to z in the complex t-plane. For the case $p = 0$, we obtain the classical Reimann-Liouville fraction derivative operator.

2. Main Result

The extended Mittag-Leffler function can be given as:

$$E_{\alpha,p}^{(\gamma,c);q}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}}{\Gamma(\alpha n+\beta)} \cdot \frac{z^n}{n!}, \tag{13}$$
$$p \geq 0, \mathrm{Re}(c) > 0, \mathrm{Re}(\gamma) > 0,\ q < \mathrm{Re}(\alpha)+1,$$

where

$$B_p(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} \exp\left[\frac{-p}{u(1-u)}\right]du, \tag{14}$$
$$\mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0.$$

The above Mittag-Leffler function can be derived by using the following relations which are given by Chaudhary et al. (2004), Chaudhary and Zubair (2001):

$$\frac{(\gamma)_{nq}}{(c)_{nq}} = \frac{B(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)},$$

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \cdot \frac{(c)_{nq}}{(c)_{nq}} \frac{z^n}{n!}. \tag{15}$$

Now, we state some theorems on this function

3. Fractional Derivative of Extended Mittag-Leffler function

Theorem 1.

Let $p \geq 0$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, where $q < \mathrm{Re}(\alpha) + 1$, then

$$D_z^{\lambda-\mu,p}\left[z^{\lambda-1} E_{\alpha,\beta}^{(\mu);q}(z^q;p)\right] = \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda,\mu-\lambda) E_{\alpha,\beta}^{(\lambda,\mu);q}(z^q;p), \tag{16}$$

Proof:

Taking L.H.S. and using the extended Riemann-Liouville fractional derivative

$$\mathrm{D}_z^{\lambda-\mu,p}\left[z^{\lambda-1} E_{\alpha,\beta}^{(\mu);q}(z^q;p)\right] = \frac{1}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1} E_{\alpha,\beta}^{(\mu);q}(t^q;p)(z-t)^{-\lambda+\mu-1} \exp\left[\frac{-p\, z^2}{t(z-t)}\right] dt,$$

on putting $t = zu$ and using Equation (4) , we obtain

$$\frac{z^{u-1}}{\Gamma(\mu-\lambda)} \int_0^1 u^{\lambda-1}(1-u)^{-\lambda+\mu-1} \exp\left[\frac{-p}{u(1-u)}\right] \sum_{k=0}^{\infty} \frac{(\mu)_{kq}}{\Gamma(\alpha k+\beta)} \frac{z^{qk} u^{qk}}{k!} du.$$

After changing the order of summation and integration, we get the desired result.

Theorem 2.

$$\frac{d^n}{dz^n}\left\{E_{\alpha,\beta}^{(\gamma,c);q}(z;p)\right\} = (\gamma)_{qn} E_{\alpha,\beta+nq}^{\gamma+nq,c+nq}(z;p). \tag{17}$$

Proof:

Taking first derivative with respect to *z* of Equation (14), we get

$$\frac{d}{dz}\left[E_{\alpha,\beta}^{(\gamma,c);q}(z;p)\right]=\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+(n+1)q,c-\gamma\right)}{B(\gamma,c-\gamma)}\frac{(c)_{(n+1)q}}{\Gamma\alpha(n+1)+\beta}.\frac{z^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{B_p\left(\gamma+q+nq,c-\gamma\right)}{B(\gamma,c-\gamma)}\frac{(c)_q\,(c+q)_{nq}}{\Gamma\left(\alpha(n+1)+\beta\right)}.\frac{z^n}{n!}$$

$$=(\gamma)_q\,E_{\alpha,\beta+\alpha}^{(\gamma+q;\,c+q);q}(z;p).$$

Further, applying the process *n* times, we get the result.

Corollary 1.

For the extended Mittag-Leffler function, the following differentiation formula holds:

$$\frac{d^n}{dz^n}\left[z^{\beta-1}E_{\alpha,\beta}^{(\gamma;c);q}\left(\lambda z^{\alpha};p\right)\right]=z^{\beta-n-1}E_{\alpha,\beta-n}^{(\gamma;c);q}\left(\lambda z^{\alpha};p\right),$$

in Equation (17). We put $z=\lambda z^{\alpha}$ and multiply by $z^{\beta-1}$. Then taking *z* derivative *n* times, we get required result.

Similarly, another interesting derivative formula for the extended Mittag-Leffler function is

$$E_{\alpha,\beta}^{(\gamma,c);q}(z;p)=\beta\,E_{\alpha,\beta+1}^{(\gamma,c);q}(z;p)+\alpha\,z\,\frac{d}{dz}E_{\alpha,\beta+1}^{(\gamma,c);q}(z;p), \tag{18}$$

4. Integral Representation of Extended Mittag-Leffler function

Theorem 3.

For Extended MittagLeffler function, we have

$$E_{\alpha,\beta}^{(\Upsilon,C);q}\ (Z;p)=\frac{1}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}e^{-\frac{p}{t(1-t)}}E_{\alpha,\beta}^{c;q}\left(t^q z\right)dt, \tag{19}$$

where $p\geq 0, \mathrm{Re}(c)>0, \mathrm{Re}(\gamma)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, q<\mathrm{Re}(\alpha)+1.$

Proof:

Using Equation (14) in Equation (13), we get

$$E_{\alpha,\beta}^{(\gamma,c);q} = \sum_{k=0}^{\infty} \int_0^1 t^{\gamma+kq-1}(1-t)^{c-\gamma-1} e^{-p/t(1-t)} \frac{(c)_{kq}}{B(\gamma,c-\gamma)} . \frac{1}{\Gamma(\alpha k+\beta)} . \frac{z^k}{k!} dt.$$

On interchanging order of summation and integration in the above equation, we get

$$\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} e^{-\frac{p}{t(1-t)}} \sum_{k=0}^{\infty} \frac{(c)_{kq}}{B(\gamma,c-\gamma)} . \frac{(t^q z)^k}{\Gamma(\alpha k+\beta)k!} dt, \tag{20}$$

and using Equation (4) in Equation (20), we get the desired result.

Corollary 2.

$$E_{\alpha,\beta}^{(\gamma,c);q}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \int_0^{\infty} \frac{u^{\gamma-1}}{(u+1)^c} exp.\left(\frac{-p(1+u)^2}{u}\right) E_{\alpha,\beta}^{c;q}\left[\left(\frac{u}{1+u}\right)^q z;p\right] du. \tag{21}$$

The above result can be prove by putting $t = \frac{u}{1+u}$ in Theorem 3.

Corollary 3.

$$E_{\alpha,\beta}^{(\gamma,c);q}(z;p) = \frac{1}{B(\gamma,c-\gamma)}\left[2\int_0^{\pi/2} sin^{2\gamma-1}\theta \, cos^{2(c-\gamma)-1}\theta \; exp.\left(\frac{-p}{sin^2\theta cos^2\theta}\right)\right] . E_{\alpha,\beta}^{c;q}\left(sin^{2q}\theta z\right) d\theta. \tag{22}$$

Taking $t = sin^2\theta$ in theorem 3, we get above trigonometric form of Extended Beta function.

5. Integral transform of extended Mittag-Leffler function

Theorem 4.

The Mellin transform of the extended Mittag-Leffler function is given by

$$M\left[E_{\alpha,\beta}^{(\gamma,c);q}(z;p);s\right] = \frac{\Gamma s \Gamma(c+s-\gamma)}{\Gamma\gamma\Gamma(c-\gamma)} {}_2\Psi_2\left[\begin{matrix}(c,q),(\gamma+s,q)\\(\beta,\alpha),(c+2s,q)\end{matrix};z\right]. \tag{23}$$

Proof:

We start with

$$M\left[E_{\alpha,\beta}^{(\gamma,c);q}(z;p);s\right]=\int_0^\infty p^{s-1}E_{\alpha,\beta}^{(\gamma,c);q}(z;p)\,dp, \tag{24}$$

and using Equation (19) in Equation (24), we get

$$M\left[E_{\alpha,\beta}^{(\gamma,c);q}(z;p);s\right]$$
$$=\frac{1}{B(\gamma,c-\gamma)}\int_0^\infty p^{s-1}\left[\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}e^{-\frac{p}{t(1-t)}}\right]E_{\alpha,\beta}^{c;q}(t^q z)\,dt\,dp.$$

Now, changing the order of integration and putting $u=\dfrac{p}{t(1-t)}$, we get

$$M\left[E_{\alpha,\beta}^{(\gamma,c);q}(z;p);s\right]$$
$$=\frac{\Gamma s}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+s-1}(1-t)^{c+s-\gamma-1}.\sum_{k=0}^{\infty}\frac{(c)_{kq}}{\Gamma(\alpha k+\beta)}.\frac{(t^q z)^k}{k!}\,dt.$$

Finally, using Beta function, [Equation (5)] in the above case, we get the desired result.

Corollary 4.

Taking $s=1$ in Equation (23), we get

$$\int_0^\infty E_{\alpha,\beta}^{(\gamma,c);q}(z;p)\,dp=\frac{\Gamma(c+1-\gamma)}{\Gamma\gamma\,\Gamma(c-\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(c,q),(\gamma+1,q)\\(\beta,\alpha),(c+2,q)\end{matrix};z\right]. \tag{25}$$

Corollary 5.

Taking Inverse Mellin transform

$$E_{\alpha,\beta}^{(\gamma,c);q}(z;p)\,dp$$
$$=\frac{1}{2\pi i}\frac{1}{\Gamma\gamma\Gamma c-\gamma}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma s\,\Gamma(c+s-\gamma)\,{}_2\Psi_2\left[\begin{matrix}(c,q),(\gamma+s,q)\\(\beta,\alpha),(c+2s,q)\end{matrix};z\right].p^{-s}ds, \tag{26}$$

Similarly, Laplace and Whittaker transform [4] of Extended Mittag-Leffler function are as follows:

$$L\left[z^{b-1}E_{\alpha,\beta}^{(\gamma,c);q}\left(z^{q};p\right);s\right]=\int_{0}^{\infty}z^{b-1}e^{-sz}E_{\alpha,\beta}^{(\gamma,c);q}\left(z^{q};p\right)dz$$

$$=\frac{\Gamma b}{s^{b}}\sum_{k=0}^{\infty}\frac{B_{p}\left(\gamma+kq,c-\gamma\right)}{B\left(\gamma,c-\gamma\right)}\frac{(c)_{kq}(b)_{k}}{k!}\cdot\frac{1}{s^{k}},\tag{27}$$

and

$$\int_{0}^{\infty}e^{-ft/2}\,t^{\varepsilon-1}\,W_{\lambda,\psi}(ft)\,E_{\alpha,\beta}^{(\gamma,c);q}\left(w\,t^{\eta}\right)dt$$

$$=f^{-\varepsilon}\sum_{k=0}^{\infty}\frac{B_{p}\left(\gamma+kq,c-\gamma\right)}{B\left(\gamma,c-\gamma\right)}\frac{(c)_{kq}}{\Gamma\left(\alpha k+\beta\right)}\cdot\frac{1}{k!}\left(\frac{\omega}{f^{\eta}}\right)^{k}\frac{\Gamma\left(\varepsilon+\eta k-\psi+\frac{1}{2}\right)\Gamma\left(\varepsilon+\eta k+\psi+\frac{1}{2}\right)}{\Gamma\left(\varepsilon+\eta k-\lambda+1\right)}.\tag{28}$$

Using the definition of extended Mittag-Leffler function [Equation (14)] in middle term of Equation (27), and changing the order of integration and summation, and using Laplace Transform

$$\int_{0}^{\infty}e^{-st}t^{n-1}dt=\frac{\Gamma n}{s^{n}},$$

we get the required result. Taking L.H.S. of Equation (28) and put $ft=v$ and using definition of Extended Mittag-Leffler function changing order of summation and integration and using Equation (9) we get the desired result.

6. Special Cases

(i) If we put $q=1$, in above Theorems 1, 3, and 4, it reduces to the result given recently by Özarslan and Yilmaz (2014).

(ii) If we put $c=q=1$, and $p=0$, in the above Theorems 1, 3, and 4, it reduces to Prabhakar function (1971) .

7. Conclusion

In this investigation, we established and evaluated some fractional derivative formulas involving extended Mittag Leffler function by using extended beta function and also evaluated integral representation of this extended Mittage-leffler function. Other than this, we obtained some integral transforms in terms of generalized Wright function and extended beta function. The approach presented in this investigation is general but can be extended to establish other properties of special functions.

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