



## Solving Singularly Perturbed Differential Difference Equations via Fitted Method

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### Abstract

In this paper, we presented a fitted approach to solve singularly perturbed differential difference equations of second order with boundary at one end (left or right) of the interval. In this approach, with the help of Taylor series expansion, we approximated the terms containing negative and positive shifts and modified the singularly perturbed differential difference equation to singularly perturbed differential equation. A fitting parameter in the coefficient of the highest order derivative of the new equation is introduced and determined its value from the theory of singular perturbation. Finally, we obtained a three term recurrence relation which is solved using Thomas Algorithm. The applicability of the method is tested with six linear problems.

**Keywords:** Fitted, Differential-Difference, Singular Perturbation, Boundary Layer

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### 1. Introduction

The problems in which the highest order derivative term is multiplied by a small parameter are known to be perturbed problems and the parameter is known as the perturbation parameter. A singularly perturbed differential-difference equation is an ordinary differential equation in which

the highest derivative is multiplied by a small parameter and involving at least one delay or advance term. Recently by constructing a special type of mesh, so that the term containing delay lies on nodal points after discretization, Rao and Chakravarthy (2011) presented a fourth order finite difference method for solving singularly perturbed differential difference equations. Prasad and Reddy (2012) considered differential quadrature method for finding the numerical solution of boundary-value problems for a singularly perturbed differential-difference equation of mixed type. In the recent studies conducted by Kadalbajoo and Sharma (2004, 2008), Kadalbajoo and Ramesh (2007) and Kadalbajoo and Kumar (2008, 2010), the terms negative or left shift and positive or right shift have been used for delay and advance respectively.

The differential-difference equation plays an important role in the mathematical modeling of various practical phenomena in the biosciences and control theory. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense information and then react to it. For a detailed discussion on differential-difference equation, one may refer to Bellen and Zennaro (2003), Driver (1977), Bellman and Cooke (1963) books and high level monographs.

In El'sgolt's (1964), similar boundary value problems with solutions that exhibit rapid oscillations are studied. Kadalbajoo and Sharma (2004, 2008), Kadalbajoo and Ramesh (2007) and Kadalbajoo and Kumar (2008, 2010) initiated an extensive numerical work for solving singularly perturbed delay differential equations based on finite difference scheme, fitted mesh and B-spline technique, piecewise uniform mesh.

It is well known that the classical methods fail to provide reliable numerical results for such problems (in the sense that the parameter and the mesh size cannot vary independently). Lange and Miura (1985, 1994) gave asymptotic approaches in the study of class of boundary value problems for linear second order differential difference equations in which the highest order derivative is multiplied by small parameter. They have also discussed the effect of small shifts on the oscillatory solution of the problem.

In this paper, we presented a fitted approach to solve singularly perturbed differential difference equations of second order with boundary at end (left or right) of the interval. In this approach, the terms containing negative and positive shifts are approximated using Taylor series expansion and the singularly perturbed differential difference equation is modified get singularly perturbed differential equation. A fitting parameter in the coefficient of the highest order derivative of the new equation is introduced and determined its value from the theory of singular perturbation. Finally, we obtained a three term recurrence relation which is solved using Thomas Algorithm. The applicability of the method is tested by considering six linear problems (three for left layer and three for right layer).

## 2. Description of the Method

To describe the method, we first consider a linear singularly perturbed differential difference two-point boundary value problem of the form:

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x - \delta) + r(x)y(x) + s(x)y(x + \gamma) = f(x), 0 < x < 1. \quad (1)$$

Under the interval and boundary conditions

$$y(x) = \phi(x), \text{ on } -\delta \leq x \leq 0, \quad (2a)$$

and

$$y(x) = \varphi(x), \text{ on } 1 \leq x \leq 1 + \gamma, \quad (2b)$$

where,  $p(x), q(x), r(x), s(x), \phi(x), \varphi(x)$  and  $f(x)$  are sufficiently smooth functions, the singular perturbation parameter  $\varepsilon$  is a small positive parameter ( $0 < \varepsilon \ll 1$ ), and  $0 < \delta = o(\varepsilon), 0 < \gamma = o(\varepsilon)$  are respectively, the delay (negative shift) and the advance (positive shift) parameters. The solution of (1) and (2) exhibits, layer at the left end of the interval if  $p(x) - \delta q(x) + \gamma s(x) > 0$  and layer at the right end of the interval if  $p(x) - \delta q(x) + \gamma s(x) < 0$ . If  $p(x) = 0$ , then one may have oscillatory solution or two layers (one at each end) depending upon the cases whether  $q(x) + r(x) + s(x)$  is positive or negative.

Approximating  $y(x - \delta)$  and  $y(x + \gamma)$  by the Taylor series expansion, we have

$$y(x - \delta) \approx y(x) - \delta y'(x), \quad (3a)$$

$$y(x + \gamma) \approx y(x) + \gamma y'(x). \quad (3b)$$

Substituting equation (3) in to equation (1), we get

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \quad (4)$$

$$a(x) = p(x) - \delta q(x) + \gamma s(x), \quad (5)$$

$$b(x) = q(x) + r(x) + s(x). \quad (6)$$

From the theory of singular perturbations it is known that the solution of (4) and (2) is of the form (O' Malley (1974), pp. 22-26)

$$y(x) = y_0(x) + \frac{a(0)}{a(x)} (\phi(0) - y_0(0)) e^{-\int_0^x \left( \frac{a(x)}{\varepsilon} - \frac{b(x)}{a(x)} \right) dx} + O(\varepsilon), \quad (7)$$

where  $y_0(x)$  is the solution of  $a(x)y_0'(x) + b(x)y_0(x) = f(x)$ ,  $y_0(1) = \varphi(1)$ .

By taking the Taylor's series expansion for  $a(x)$  and  $b(x)$  about the point '0' and restricting to their first terms, (7) becomes,

$$y(x) = y_0(x) + \frac{a(0)}{a(x)} (\phi(0) - y_0(0)) e^{-\left( \frac{a(0)}{\varepsilon} - \frac{b(0)}{a(0)} \right) x} + O(\varepsilon). \quad (8)$$

Now, we divide the interval  $[0, 1]$  into  $N$  equal parts with constant mesh length  $h$ . Let  $0 = x_0, x_1, \dots, x_N = 1$  be the mesh points. Then we have  $x_i = ih; i = 0, 1, \dots, N$ .

From (8) we have

$$y(ih) = y_0(ih) + \frac{a(0)}{a(ih)}(\phi(0) - y_0(0))e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{\varepsilon a(0)}\right)ih}, \tag{9}$$

and

$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\phi(0) - y_0(0))e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)i\rho}, \quad \rho = \frac{h}{\varepsilon}. \tag{10}$$

### 2.1. Derivation of the Special Second Order Scheme for (4)

Consider a typical pivotal point in the mesh, at  $x = x_i + gh$ . The following expression can be written for  $y$ ,  $y'$  and  $y''$ :

$$y_g = y(x_i + gh) = E^g y(x_i), \tag{11a}$$

$$y_g' = Dy_g, \tag{11b}$$

$$y_g'' = D^2 y_g. \tag{11c}$$

The shift operator

$$E = e^{hD} \tag{11d}$$

can be related to the central difference operators  $\mu, \delta$  by using the following expressions

$$hD = \mu\delta - \frac{1}{6}\mu\delta^3 + \frac{1}{30}\mu\delta^5 + \dots, \tag{11e}$$

$$h^2 D^2 = \delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 + \dots, \tag{11f}$$

$$h^3 D^3 = \mu\delta^3 - \frac{1}{4}\mu\delta^5 + \dots, \tag{11g}$$

$$h^4 D^4 = \delta^4 - \frac{1}{6}\delta^6 + \dots, \tag{11h}$$

where,  $\mu$  and  $\delta$  denote the usual central difference operators and  $E$  and  $D$  denote the shift (displacement) and the differential operators, respectively. By substituting (11d)-(11h) into (11a)-(11c), we get

$$y_g = [1 + g\mu\delta + \frac{g^2}{2}\delta^2 + \frac{1}{6}g(g^2-1)\mu\delta^3 + \frac{g^2(g^2-1)}{24}\delta^4 + \dots]y_i, \quad (12)$$

$$y_g' = \frac{1}{h}[\mu\delta + g\delta^2 + \frac{(3g^2-1)}{6}\mu\delta^3 + g(2g^2-1)\frac{\delta^4}{12} + \dots]y_i, \quad (13)$$

$$y_g'' = \frac{1}{h^2}[\delta^2 + g\mu\delta^3 + \frac{6g^2-1}{12}\delta^4 + g(2g^2-3)\frac{\mu\delta^5}{6} + \dots]y_i. \quad (14)$$

By substituting equations (12-14) with  $g = 1/2$  into equation (4) we get:

$$\frac{\varepsilon}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_{i+1/2}}{h}(y_{i+1} - y_i) + b_{i+1/2}\left(\frac{3y_{i+1} + 6y_i - y_{i-1}}{8}\right) = f_{i+1/2}, \quad (15)$$

$$i = 1, 2, \dots, N-1$$

For further discussion on the special second order scheme (one can see (2)). Now, we introduce a fitting factor  $\sigma(\rho)$  in the above scheme (15)

$$\frac{\sigma(\rho)\varepsilon}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{a_{i+1/2}}{h}(y_{i+1} - y_i) + b_{i+1/2}\left(\frac{3y_{i+1} + 6y_i - y_{i-1}}{8}\right) = f_{i+1/2}, \quad (16)$$

$$i = 1, 2, \dots, N-1$$

The fitting factor  $\sigma(\rho)$  is to be determined in such a way that the solution of (16) converges uniformly to the solution of (1)-(2). Multiplying (16) by  $h$  and taking the limit as,  $h \rightarrow 0$ , we get

$$\lim_{h \rightarrow 0} \left[ \frac{\sigma}{\rho} (y(ih+h) - 2y(ih) + y(ih-h)) + a(ih/2)(y(ih+h) - y(ih)) \right] = 0. \quad (17)$$

By substituting (10) in (17) and simplifying, we get the constant fitting factor

$$\sigma = \frac{\rho a(0)}{4} \frac{[1 - e^{-\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)\rho}]}{[\sinh\left(\left(\frac{a^2(0) - \varepsilon b(0)}{a(0)}\right)\rho/2\right)]^2}. \quad (18)$$

The equivalent three-term recurrence relation of equation (16) is given by:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, i = 1, 2, \dots, N-1, \quad (19)$$

where

$$E_i = \frac{\sigma\varepsilon}{h^2} - \frac{b_{i+1/2}}{8}, F_i = \frac{2\sigma\varepsilon}{h^2} + \frac{a_{i+1/2}}{h} - \frac{6b_{i+1/2}}{8}, G_i = \frac{\sigma\varepsilon}{h^2} + \frac{a_{i+1/2}}{h} + \frac{3b_{i+1/2}}{8}, H_i = f_{i+1/2}.$$

This gives us the tri diagonal system which can be solved easily by Thomas Algorithm.

## 2.2. Thomas Algorithm

A brief discussion on solving the three term recurrence relation using Thomas Algorithm, which also called *Discrete Invariant Imbedding* (Bellman and Cooke (1963)), is presented as follows:

Consider the scheme given in (19) i.e.,  $E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, i = 1, 2, \dots, N-1$  subject to the boundary conditions

$$y_0 = y(0) = \alpha, y_N = y(1) = \beta. \quad (20a)$$

We set

$$y_i = W_i y_{i+1} + T_i, i = N-1, N-2, \dots, 2, 1, \quad (20b)$$

where,  $W_i = W(x_i)$  and  $T_i = T(x_i)$  which are to be determined.

From (20b), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \quad (20c)$$

By substituting (20c) in (19), we get

$$y_i = \left( \frac{G_i}{F_i - E_i W_{i-1}} \right) y_{i+1} + \left( \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right). \quad (20d)$$

By comparing (20d) and (20b), we get the recurrence relations

$$W_i = \left( \frac{G_i}{F_i - E_i W_{i-1}} \right), \quad (20e)$$

and

$$T_i = \left( \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right). \quad (20f)$$

To solve these recurrence relations for  $i = 0, 1, \dots, N-1$ , we need the initial conditions for  $W_0$  and  $T_0$ . For this we take  $y_0 = \alpha = W_0 y_1 + T_0$ . We choose  $W_0 = 0$  so that the value of  $T_0 = \alpha$ . With these initial values, we compute  $W_i$  and  $T_i$  for  $i = 0, 1, \dots, N-1$ , from (20e) and (20f) in forward process, and then obtain  $y_i$  in the backward process from (20b) and (20a). The conditions for the discrete invariant embedding algorithm to be stable are (Andargie and Reddy (2007) and Bellman and Cooke (1963)):

$$E_i > 0, F_i > 0, F_i \geq E_i + G_i \text{ and } |E_i| \leq |G_i|. \quad (20g)$$

In this method, if the assumptions  $p(x) - \delta q(x) + \gamma s(x) > 0$ ,  $q(x) + r(x) + s(x) < 0$  and  $\varepsilon > 0$  hold, one can easily show that the conditions given in (16g) hold and, thus, the invariant imbedding algorithm are stable.

### 3. Right Layer Problems

For problems with layer at the right end of the interval, from the theory of singular perturbations it is known that the solution of (4) and (2) is of the form (O' Malley (1974))

$$y(x) = y_0(x) + \frac{a(1)}{a(x)} (\varphi(1) - y_0(1)) e^{\int_1^x \left( \frac{a(x) - b(x)}{\varepsilon - a(x)} \right) dx} + O(\varepsilon), \quad (21)$$

where  $y_0(x)$  is the solution of  $a(x)y_0'(x) + b(x)y_0(x) = f(x)$ ,  $y_0(0) = \phi(0)$ . By applying Taylor's series expansion on  $a(x)$  and  $b(x)$  about the point '1' and restricting to their first terms, (21) becomes,

$$y(x) = y_0(x) + (\varphi(1) - y_0(1)) e^{\left( \frac{a(1) - b(1)}{\varepsilon - a(1)} \right) (1-x)} + O(\varepsilon). \quad (22)$$

Now, we divide the interval  $[0, 1]$  into  $N$  equal parts with constant mesh length  $h$ . Let  $0 = x_0, x_1, \dots, x_N = 1$  be the mesh points. Then, we have  $x_i = ih; i = 0, 1, \dots, N$ . From (22) we have

$$y(ih) = y_0(ih) + (\varphi(1) - y_0(1)) e^{\left( \frac{a(1) - b(1)}{\varepsilon - a(1)} \right) (1-ih)} + O(\varepsilon), \quad (23)$$

and

$$\lim_{h \rightarrow 0} y(ih) = y_0(0) + (\varphi(1) - y_0(1)) e^{\left( \frac{a^2(1) - \varepsilon b(1)}{a(1)} \right) \left( \frac{1-ih}{\varepsilon} \right)}, \quad \rho = \frac{h}{\varepsilon}. \quad (24)$$

By substituting equations (12)-(14) with  $g = -1/2$  into equation (4) and introducing the fitting factor  $\sigma(\rho)$ , we get:

$$\frac{\sigma \varepsilon}{h^2} (y_{i+1} - 2y_i + y_{i-1}) + \frac{a_{i-1/2}}{h} (y_i - y_{i-1}) + b_{i-1/2} \left( \frac{-y_{i+1} + 6y_i + 3y_{i-1}}{8} \right) = f_{i-1/2}, \quad (25)$$

$$i = 1, 2, \dots, N-1$$

Multiplying (25) by  $h$  and taking the limit as  $h \rightarrow 0$ , we get

$$\lim_{h \rightarrow 0} \left[ \frac{\sigma(\rho)}{\rho} (y(ih+h) - 2y(ih) + y(ih-h)) + a(ih-h/2)(y(ih) - y((i-1)h)) \right] = 0. \quad (26)$$

Using (24) in (26) and simplifying, we get

$$\sigma = \frac{\rho}{4} a(0) \frac{[e^{\frac{(a^2(1)-\varepsilon b(1))\rho}{a(1)}} - 1]}{[\sinh(\frac{(a(1)^2 - \varepsilon b(1))\rho}{2a(1)})]^2}. \quad (27)$$

An equivalent three-term recurrence relation for equation (25) is:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, i = 1, 2, \dots, N-1, \quad (28)$$

where

$$E_i = \frac{\varepsilon\sigma}{h^2} - \frac{a_{i-1/2}}{h} + \frac{3b_{i-1/2}}{8}, F_i = \frac{2\varepsilon\sigma}{h^2} - \frac{a_{i-1/2}}{h} - \frac{6b_{i-1/2}}{8}, G_i = \frac{\varepsilon\sigma}{h^2} - \frac{b_{i-1/2}}{8}, H_i = f_{i-1/2}.$$

#### 4. Numerical Examples

To demonstrate the applicability of the method, we considered six linear problems (three with left layer and three with right layer). We presented the tables of the numerical results along with the exact solution. Now let us consider the singularly perturbed differential difference equation

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x-\delta) + r(x)y(x) + s(x)y(x+\gamma) = f(x), x \in (0,1),$$

subject to the interval and boundary conditions

$$y(x) = \phi(x), -\delta \leq x \leq 0; \quad y(x) = \varphi(x), 1 \leq x \leq 1+\gamma.$$

The exact solution of such boundary value problems having constant coefficients is given by:

$$y(x) = c_1 \exp(m_1 x) + c_2 \exp(m_2 x) + \frac{f}{c},$$



$$\begin{aligned}
c &= q + r + s, \\
m_1 &= \frac{-(p - \delta q + \gamma s) + \sqrt{(p - \delta q + \gamma s)^2 - 4\epsilon c}}{2\epsilon}, \\
m_2 &= \frac{-(p - \delta q + \gamma s) - \sqrt{(p - \delta q + \gamma s)^2 - 4\epsilon c}}{2\epsilon}, \\
c_1 &= \frac{-f + \beta c + \exp(m_2)(f - \alpha c)}{(\exp(m_1) - \exp(m_2))c}, \\
c_2 &= \frac{f - \beta c + \exp(m_1)(-f + \alpha c)}{(\exp(m_1) - \exp(m_2))c}, \alpha = y(0), \beta = y(1)
\end{aligned}$$

**Example 1:** Consider the singularly perturbed differential difference equation with layer at the left end

$$\begin{aligned}
\epsilon y''(x) + y'(x) + 2y(x - \delta) - 3y(x) &= 0, \text{ and} \\
y(x) &= 1, \quad -\delta \leq x \leq 0, \quad y(x) = 1, \quad 1 \leq x \leq 1 + \gamma.
\end{aligned}$$

**Example 2:** Consider the singularly perturbed differential difference equation with layer at the left end

$$\begin{aligned}
\epsilon y''(x) + y'(x) - 3y(x) + 2y(x + \gamma) &= 0, \text{ and} \\
y(x) &= 1, \quad -\delta \leq x \leq 0, \quad y(x) = 1, \quad 1 \leq x \leq 1 + \gamma.
\end{aligned}$$

**Example 3:** Consider the singularly perturbed differential difference equation with layer at the left end

$$\begin{aligned}
\epsilon y''(x) + y'(x) - 2y(x - \delta) - 5y(x) + y(x + \gamma) &= 0, \text{ and} \\
y(x) &= 1, \quad -\delta \leq x \leq 0, \quad y(x) = 1, \quad 1 \leq x \leq 1 + \gamma.
\end{aligned}$$

**Example 4:** Consider the singularly perturbed differential difference equation with right layer

$$\begin{aligned}
\epsilon y''(x) - y'(x) - 2y(x - \delta) + y(x) &= 0, \text{ and} \\
y(x) &= 1, \quad -\delta \leq x \leq 0, \quad y(x) = -1, \quad 1 \leq x \leq 1 + \gamma.
\end{aligned}$$

**Example 5:** Consider the singularly perturbed differential difference equation with right layer

$$\begin{aligned}
\epsilon y''(x) - y'(x) + y(x) - 2y(x + \gamma) &= 0, \text{ and} \\
y(x) &= 1, \quad -\delta \leq x \leq 0, \quad y(x) = -1, \quad 1 \leq x \leq 1 + \gamma.
\end{aligned}$$

**Example 6:** Consider the singularly perturbed differential difference equation with right layer

$$\varepsilon y''(x) - y'(x) - 2y(x - \delta) + y(x) - 2y(x + \gamma) = 0, \text{ and}$$

$$y(x) = 1, \quad -\delta \leq x \leq 0, \quad y(x) = -1, \quad 1 \leq x \leq 1 + \gamma.$$

### 5. Discussion and Conclusions

We presented a fitted approach to solve singularly perturbed differential equations of second order with boundary at one (left or right) end of the interval. In this approach, the singularly perturbed differential difference equation is modified to a singularly perturbed differential equation by approximating the term containing small shifts (negative and positive) using Taylor series expansion. In the new singularly perturbed problem, we introduced a fitting parameter and determined its value using the theory of singular Perturbation. Finally, we get a three term recurrence relation which is solved using Thomas algorithm.

Many methods have been presented producing good approximation in the outer region of the singularly perturbed problems. Here, to test the applicability of this method, we considered six linear problems (three for left layer and three for right layer). We presented tables of values and the point wise error for meshes in the inner region of the problems by taking different values of the perturbation parameter  $\varepsilon$ , the delay parameter  $\delta$  and the advance parameter  $\gamma$ . The results demonstrate that the present method produced good approximation to the exact solution.

**Table 1. (a)** Absolute Maximum Error for Example 1 with  $n=100, \varepsilon=0.01, \gamma=0.5*\varepsilon$

x(i)	$\delta=0.00$			$\delta=0.005$			$\delta=0.009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.01	0.601899	0.604134	2.23E-03	0.601836	0.604121	2.28E-03	0.601803	0.604129	2.33E-03
0.02	0.460076	0.462325	2.25E-03	0.458557	0.460859	2.30E-03	0.457355	0.459703	2.35E-03
0.03	0.411116	0.413081	1.96E-03	0.408546	0.410554	2.01E-03	0.406492	0.408536	2.04E-03
0.04	0.395823	0.397578	1.76E-03	0.392681	0.394471	1.79E-03	0.390158	0.391975	1.82E-03
0.05	0.392750	0.394390	1.64E-03	0.389332	0.391000	1.67E-03	0.386580	0.388271	1.69E-03
0.06	0.394128	0.395713	1.58E-03	0.390587	0.392195	1.61E-03	0.387731	0.389358	1.63E-03
0.07	0.397145	0.398703	1.56E-03	0.393550	0.395130	1.58E-03	0.390649	0.392247	1.60E-03
0.08	0.400779	0.402325	1.55E-03	0.397164	0.398731	1.57E-03	0.394246	0.395830	1.58E-03
0.09	0.404663	0.406203	1.54E-03	0.401042	0.402602	1.56E-03	0.398118	0.399695	1.58E-03
1.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00

**Table 1. (b)** Absolute Maximum Error for Example 1 with  $n=100$ ,  $\epsilon=0.005$ ,  $0.1*\epsilon$

x(i)	$\delta = 0.000$			$\delta = 0.0005$			$\delta = 0.0009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.01	0.456876	0.457859	9.83E-04	0.456726	0.457712	9.86E-04	0.456606	0.457594	9.88E-04
0.02	0.387246	0.388453	1.21E-03	0.386931	0.388141	1.21E-03	0.386679	0.387890	1.21E-03
0.03	0.381175	0.382424	1.25E-03	0.380818	0.382070	1.25E-03	0.380533	0.381786	1.25E-03
0.04	0.383664	0.384920	1.26E-03	0.383299	0.384558	1.26E-03	0.383008	0.384268	1.26E-03
0.05	0.387334	0.388591	1.26E-03	0.386969	0.388228	1.26E-03	0.386676	0.387937	1.26E-03
0.06	0.391197	0.392454	1.26E-03	0.390832	0.392090	1.26E-03	0.390539	0.391799	1.26E-03
0.07	0.395119	0.396375	1.26E-03	0.394754	0.396012	1.26E-03	0.394462	0.395721	1.26E-03
0.08	0.399084	0.400338	1.25E-03	0.398719	0.399975	1.26E-03	0.398427	0.399685	1.26E-03
0.09	0.403088	0.404342	1.25E-03	0.402724	0.403979	1.26E-03	0.402432	0.403689	1.26E-03
1.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00

**Table 2. (a)** Absolute Maximum Error for Example 2 with  $n=100$ ,  $\epsilon=0.01$ ,  $\delta=0.5*\epsilon$

x(i)	$\gamma = 0.00$			$\gamma = 0.005$			$\gamma = 0.009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.01	0.601899	0.604134	2.23E-03	0.601986	0.604172	2.19E-03	0.602073	0.604220	2.15E-03
0.02	0.460076	0.462325	2.25E-03	0.461613	0.463810	2.20E-03	0.462856	0.465012	2.16E-03
0.03	0.411116	0.413081	1.96E-03	0.413687	0.415610	1.92E-03	0.415745	0.417634	1.89E-03
0.04	0.395823	0.397578	1.76E-03	0.398950	0.400672	1.72E-03	0.401441	0.403138	1.70E-03
0.05	0.392750	0.394390	1.64E-03	0.396142	0.397756	1.61E-03	0.398838	0.400431	1.59E-03
0.06	0.394128	0.395713	1.58E-03	0.397639	0.399200	1.56E-03	0.400425	0.401968	1.54E-03
0.07	0.397145	0.398703	1.56E-03	0.400704	0.402241	1.54E-03	0.403527	0.405047	1.52E-03
0.08	0.400779	0.402325	1.55E-03	0.404357	0.405882	1.53E-03	0.407194	0.408703	1.51E-03
0.09	0.404663	0.406203	1.54E-03	0.408247	0.409766	1.52E-03	0.411087	0.412590	1.50E-03
1.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00

**Table 2. (b)** Absolute Maximum Error for Example 2 with  $n=100$ ,  $\epsilon=0.005$ ,  $\delta=0.1*\epsilon$

x(i)	$\gamma = 0.000$			$\gamma = 0.0005$			$\gamma = 0.0009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.01	0.456876	0.457859	9.83E-04	0.457026	0.458006	9.81E-04	0.457146	0.458124	9.79E-04
0.02	0.387246	0.388453	1.21E-03	0.387560	0.388766	1.21E-03	0.387812	0.389016	1.20E-03
0.03	0.381175	0.382424	1.25E-03	0.381530	0.382778	1.25E-03	0.381815	0.383061	1.25E-03
0.04	0.383664	0.384920	1.26E-03	0.384027	0.385282	1.25E-03	0.384318	0.385571	1.25E-03
0.05	0.387334	0.388591	1.26E-03	0.387699	0.388955	1.26E-03	0.387991	0.389245	1.25E-03
0.06	0.391197	0.392454	1.26E-03	0.391562	0.392817	1.25E-03	0.391854	0.393107	1.25E-03
0.07	0.395119	0.396375	1.26E-03	0.395484	0.396738	1.25E-03	0.395776	0.397028	1.25E-03
0.08	0.399084	0.400338	1.25E-03	0.399448	0.400701	1.25E-03	0.399740	0.400991	1.25E-03
0.09	0.403088	0.404342	1.25E-03	0.403453	0.404704	1.25E-03	0.403744	0.404994	1.25E-03
1.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00

**Table 3. (a)** Absolute Maximum Error for Example 3 with  $n=100$ ,  $\epsilon=0.01$

x(i)	$\delta = 0.000, \gamma = 0.009$			$\delta = 0.005, \gamma = 0.005$			$\delta = 0.009, \gamma = 0.000$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.01	0.335996	0.347172	1.12E-02	0.334298	0.345297	1.10E-02	0.333451	0.344364	1.09E-02
0.02	0.114522	0.122344	7.82E-03	0.113442	0.121109	7.67E-03	0.112907	0.120498	7.59E-03
0.03	0.040747	0.045021	4.27E-03	0.040271	0.044450	4.18E-03	0.040036	0.044170	4.13E-03
0.04	0.016275	0.018540	2.26E-03	0.016134	0.018352	2.22E-03	0.016067	0.018262	2.20E-03
0.05	0.008267	0.009589	1.32E-03	0.008285	0.009589	1.30E-03	0.008296	0.009592	1.30E-03
0.06	0.005763	0.006690	9.27E-04	0.005853	0.006777	9.24E-04	0.005899	0.006822	9.23E-04
0.07	0.005106	0.005887	7.81E-04	0.005229	0.006014	7.86E-04	0.005291	0.006080	7.88E-04
0.08	0.005076	0.005818	7.42E-04	0.005216	0.005966	7.50E-04	0.005287	0.006041	7.54E-04
0.09	0.005266	0.006013	7.47E-04	0.005417	0.006174	7.57E-04	0.005494	0.006256	7.62E-04
1.00	1.000000	1.000000	1.11E-16	1.000000	1.000000	1.11E-16	1.000000	1.000000	1.11E-16

**Table 3. (b)** Absolute Maximum Error for Example 3 with  $n=100$ ,  $\epsilon=0.005$

x(i)	$\delta = 0.000, \gamma = 0.0009$			$\delta = 0.0005, \gamma = 0.0005$			$\delta = 0.0009, \gamma = 0.000$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.01	0.131020	0.130201	8.19E-04	0.130887	0.130062	8.26E-04	0.130821	0.129992	8.29E-04
0.02	0.019441	0.019512	7.11E-05	0.019415	0.019485	7.03E-05	0.019402	0.019472	6.99E-05
0.03	0.005254	0.005580	3.26E-04	0.005257	0.005584	3.27E-04	0.005259	0.005586	3.27E-04
0.04	0.003598	0.003990	3.92E-04	0.003608	0.004001	3.93E-04	0.003613	0.004006	3.93E-04
0.05	0.003565	0.003984	4.20E-04	0.003576	0.003997	4.21E-04	0.003582	0.004003	4.21E-04
0.06	0.003750	0.004192	4.42E-04	0.003762	0.004205	4.43E-04	0.003768	0.004211	4.43E-04
0.07	0.003976	0.004439	4.64E-04	0.003988	0.004453	4.65E-04	0.003995	0.004460	4.65E-04
0.08	0.004219	0.004705	4.87E-04	0.004232	0.004720	4.88E-04	0.004239	0.004727	4.88E-04
0.09	0.004477	0.004987	5.11E-04	0.004491	0.005002	5.11E-04	0.004498	0.005010	5.12E-04
1.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00

**Table 4. (a)** Absolute Maximum Error for Example 4 with  $n=100$ ,  $\epsilon=0.01, \delta=0.1*\epsilon, \gamma=0.5*\epsilon$

x(i)	$\delta = 0.00$			$\delta = 0.005$			$\delta = 0.009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.91	0.404448	0.405977	1.53E-03	0.400807	0.402355	1.55E-03	0.397866	0.399430	1.56E-03
0.92	0.400186	0.401705	1.52E-03	0.396523	0.398060	1.54E-03	0.393563	0.395115	1.55E-03
0.93	0.395507	0.397001	1.49E-03	0.391796	0.393306	1.51E-03	0.388797	0.390320	1.52E-03
0.94	0.389608	0.391041	1.43E-03	0.385792	0.387237	1.44E-03	0.382705	0.384159	1.45E-03
0.95	0.380271	0.381565	1.29E-03	0.376227	0.377524	1.30E-03	0.372951	0.374250	1.30E-03
0.96	0.361377	0.362369	9.93E-04	0.356858	0.357839	9.81E-04	0.353194	0.354165	9.71E-04
0.97	0.316031	0.316421	3.90E-04	0.310625	0.310980	3.56E-04	0.306241	0.306568	3.27E-04
0.98	0.197603	0.196962	6.41E-04	0.190889	0.190189	7.00E-04	0.185460	0.184710	7.50E-04
0.99	-0.12263	-0.12437	1.74E-03	-0.12983	-0.13163	1.81E-03	-0.13561	-0.13748	1.86E-03
1.00	-1.00000	-1.00000	0.00E+00	-1.00000	-1.00000	0.00E+00	-1.00000	-1.00000	0.00E+00

**Table 4. (b)** Absolute Maximum Error for Example 4 with  $n=100, \epsilon=0.005, \delta=0.1*\epsilon, \gamma=0.1*\epsilon$

x(i)	$\delta = 0.000$			$\delta = 0.0005$			$\delta = 0.0009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.91	0.403088	0.404342	1.25E-03	0.402724	0.403979	1.26E-03	0.402432	0.403689	1.26E-03
0.92	0.399084	0.400338	1.25E-03	0.398719	0.399975	1.26E-03	0.398426	0.399684	1.26E-03
0.93	0.395118	0.396373	1.26E-03	0.394752	0.396010	1.26E-03	0.394460	0.395719	1.26E-03
0.94	0.391185	0.392442	1.26E-03	0.390820	0.392078	1.26E-03	0.390527	0.391787	1.26E-03
0.95	0.387247	0.388505	1.26E-03	0.386881	0.388141	1.26E-03	0.386588	0.387849	1.26E-03
0.96	0.383016	0.384275	1.26E-03	0.382646	0.383908	1.26E-03	0.382351	0.383614	1.26E-03
0.97	0.376345	0.377613	1.27E-03	0.375960	0.377229	1.27E-03	0.375652	0.376923	1.27E-03
0.98	0.351245	0.352544	1.30E-03	0.350788	0.352088	1.30E-03	0.350422	0.351723	1.30E-03
0.99	0.188546	0.189868	1.32E-03	0.187866	0.189187	1.32E-03	0.187322	0.188642	1.32E-03
1.00	-1.000000	-1.000000	0.00E+00	-1.000000	-1.000000	0.00E+00	-1.000000	-1.000000	1.11E-16

**Table 5. (a)** Absolute Maximum Error for Example 5 with  $n=100, \epsilon=0.01, \delta=0.1*\epsilon, \gamma=0.5*\epsilon$

x(i)	$\gamma=0.00$			$\gamma=0.005$			$\gamma=0.009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.91	0.408050	0.409559	1.51E-03	0.408050	0.409559	1.51E-03	0.408050	0.409559	1.51E-03
0.92	0.403809	0.405310	1.50E-03	0.403809	0.405310	1.50E-03	0.403809	0.405310	1.50E-03
0.93	0.399175	0.400653	1.48E-03	0.399175	0.400653	1.48E-03	0.399175	0.400653	1.48E-03
0.94	0.393377	0.394798	1.42E-03	0.393377	0.394798	1.42E-03	0.393377	0.394798	1.42E-03
0.95	0.384261	0.385551	1.29E-03	0.384261	0.385551	1.29E-03	0.384261	0.385551	1.29E-03
0.96	0.365828	0.366831	1.00E-03	0.365828	0.366831	1.00E-03	0.365828	0.366831	1.00E-03
0.97	0.321358	0.321780	4.22E-04	0.321358	0.321780	4.22E-04	0.321358	0.321780	4.22E-04
0.98	0.204235	0.203652	5.84E-04	0.204235	0.203652	5.84E-04	0.204235	0.203652	5.84E-04
0.99	-0.11547	-0.11715	1.68E-03	-0.11547	-0.11715	1.68E-03	-0.11547	-0.11715	1.68E-03
1.00	-1.000000	-1.000000	0.00E+00	-1.000000	-1.000000	0.00E+00	-1.000000	-1.000000	0.00E+00

**Table 5. (b)** Absolute Maximum Error for Example 4 with  $n=100, \epsilon=0.005, \delta=0.1*\epsilon, \gamma=0.1*\epsilon$

x(i)	$\gamma=0.000$			$\gamma=0.0005$			$\gamma=0.0009$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.91	0.403088	0.404342	1.25E-03	0.403452	0.404704	1.25E-03	0.403744	0.404994	1.25E-03
0.92	0.399084	0.400338	1.25E-03	0.399448	0.400701	1.25E-03	0.399739	0.400991	1.25E-03
0.93	0.395118	0.396373	1.26E-03	0.395483	0.396736	1.25E-03	0.395774	0.397027	1.25E-03
0.94	0.391185	0.392442	1.26E-03	0.391551	0.392805	1.25E-03	0.391842	0.393096	1.25E-03
0.95	0.387247	0.388505	1.26E-03	0.387613	0.388869	1.26E-03	0.387906	0.389160	1.25E-03
0.96	0.383016	0.384275	1.26E-03	0.383384	0.384642	1.26E-03	0.383679	0.384936	1.26E-03
0.97	0.376345	0.377613	1.27E-03	0.376729	0.377995	1.27E-03	0.377036	0.378301	1.26E-03
0.98	0.351245	0.352544	1.30E-03	0.351701	0.352999	1.30E-03	0.352066	0.353362	1.30E-03
0.99	0.188546	0.189868	1.32E-03	0.189224	0.190548	1.32E-03	0.189766	0.191092	1.33E-03
1.00	-1.000000	-1.000000	0.00E+00	-1.000000	-1.000000	0.00E+00	-1.000000	-1.000000	0.00E+00

**Table 6. (a)** Absolute Maximum Error for Example 5 with  $n=100$ ,  $\epsilon=0.01$

x(i)	$\delta = 0.000, \gamma = 0.009$			$\delta = 0.005, \gamma = 0.005$			$\delta = 0.009, \gamma = 0.000$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.91	0.071229	0.073575	2.35E-03	0.068032	0.070363	2.33E-03	0.064873	0.067186	2.31E-03
0.92	0.069049	0.071335	2.29E-03	0.065891	0.068158	2.27E-03	0.062770	0.065015	2.24E-03
0.93	0.066661	0.068861	2.20E-03	0.063515	0.065688	2.17E-03	0.060401	0.062544	2.14E-03
0.94	0.063561	0.065611	2.05E-03	0.060360	0.062369	2.01E-03	0.057182	0.059144	1.96E-03
0.95	0.058288	0.060042	1.75E-03	0.054885	0.056568	1.68E-03	0.051487	0.053091	1.60E-03
0.96	0.046625	0.047764	1.14E-03	0.042707	0.043723	1.02E-03	0.038766	0.039646	8.80E-04
0.97	0.016367	0.016275	9.11E-05	0.011362	0.011065	2.96E-04	0.006298	0.005779	5.19E-04
0.98	-0.06779	-0.07002	2.23E-03	-0.07460	-0.07713	2.53E-03	-0.08149	-0.08434	2.85E-03
0.99	-0.30803	-0.31252	4.48E-03	-0.31607	-0.32087	4.80E-03	-0.32413	-0.32926	5.13E-03
1.00	-1.00000	-1.00000	0.00E+00	-1.00000	-1.00000	0.00E+00	-1.00000	-1.00000	0.00E+00

**Table 6. (b)** Absolute Maximum Error for Example 4 with  $n=100$ ,  $\epsilon=0.005, \delta=0.1*\epsilon, \gamma=0.1*\epsilon$

x(i)	$\delta = 0.000, \gamma = 0.0009$			$\delta = 0.0005, \gamma = 0.0005$			$\delta = 0.0009, \gamma = 0.000$		
	y(i)	Exact	Error	y(i)	Exact	Error	y(i)	Exact	Error
0.00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00	1.000000	1.000000	0.00E+00
0.91	0.066338	0.068184	1.85E-03	0.066019	0.067865	1.85E-03	0.065701	0.067546	1.85E-03
0.92	0.064389	0.066202	1.81E-03	0.064076	0.065888	1.81E-03	0.063764	0.065575	1.81E-03
0.93	0.062497	0.064276	1.78E-03	0.062190	0.063968	1.78E-03	0.061884	0.063661	1.78E-03
0.94	0.060657	0.062402	1.75E-03	0.060356	0.062100	1.74E-03	0.060055	0.061798	1.74E-03
0.95	0.058839	0.060552	1.71E-03	0.058543	0.060255	1.71E-03	0.058248	0.059958	1.71E-03
0.96	0.056837	0.058521	1.68E-03	0.056543	0.058226	1.68E-03	0.056250	0.057932	1.68E-03
0.97	0.053084	0.054758	1.67E-03	0.052776	0.054448	1.67E-03	0.052469	0.054139	1.67E-03
0.98	0.035682	0.037420	1.74E-03	0.035283	0.037017	1.73E-03	0.034883	0.036613	1.73E-03
0.99	-0.08587	-0.08392	1.95E-03	-0.08659	-0.08465	1.94E-03	-0.08730	-0.08538	1.92E-03
1.00	-1.00000	-1.00000	0.00E+00	-1.00000	-1.00000	0.00E+00	-1.00000	-1.00000	0.00E+00

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