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On the analytic solution for the steady drainage of magnetohydrodynamic (MHD) Sisko fluid film down a vertical belt

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Abstract

This paper presents an analytic study for the steady drainage of magnetohydrodynamic (MHD) Sisko fluid film down a vertical belt. The fluid film is assumed to be electrically conducting in the presence of a uniform transverse magnetic field. An analytic solution for the resulting non linear ordinary differential equation is obtained using the Adomian decomposition method. The effects of various available parameters especially the Hartmann number are observed on the velocity profile, shear stress and vorticity vector to get a physical insight of the problem. Furthermore, the shear thinning and shear thickening characteristics of the Sisko fluid are discussed. The physical quantities discussed for the Sisko fluid film have also been discussed for the Newtonian fluid film and comparison between them made.

Keywords: MHD; Thin film flow**;** Sisko fluid model; drainage; Adomian decomposition method

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1. Introduction

The phenomenon of thin film flow is involved in many natural and industrial problems, some of which on a rigid surface are driven by gravity. Along with the understanding and developments in the fundamental issues relating to such flows, a wide variety of industrial, biological and medical applications have benefited from scientific research. The recurring feature in this phenomenon is that when a fluid is disposed to a vertical rigid object, it adheres to it and drains down the object under conditions such that the gravitational and viscous forces dominate the inertial forces due to which the fluid film formed in contact with the free surface drain down under the action of gravity only. This type of drainage is termed as “free drainage” and it consists of an extention of fluid bounded by an object and with a free surface (usually air). The fluid film thickness δ, is much shorter than the length of the object so that the flow takes place predominantly in the longer dimensions under the action of gravity only. The flow velocity perpendicular to the plate is much smaller than the main flow velocity. Hence, we can consider it as a one dimensional flow, Jeffreys (1930), Green (1936), Denson (1970), Raghuraman (1971), Munson and Young (1994), O’ Brien and Schwartz (2002), Mayers (2005).

The study of thin film flow of an electrically conducting fluid in the presence of a transverse applied magnetic field has become the basis of numerous scientific and engineering applications. In recent times, great interest has been shown by many researchers towards the study of magnetohydrodynamic (MHD) thin film flows due to the effect of magnetic fields on the fluid film thickness and on the performance of many systems involving the phenomenon of thin liquid films of electrically conducting fluids, Dutta (1973), Hameed and Ellahi (2011), Alam et al. (2012). We will consider the particular example of the motion of MHD Sisko fluid film down an infinity, long vertical belt which can be extended to any frame of coatings and lubrication process.

In recent years, the flow of non-Newtonian fluids have gained considerable importance because of its promising applications in various fields of engineering, technology, biosciences, particularly in material processing, geophysics, chemical and nuclear industry, food industry and polymer processing. Drilling mud, tooth paste, greases, polymer melts, cement slurries, paints, blood, clay coatings etc., are some examples of non-Newtonian fluid. It is difficult to suggest a model for such broad and complex class of fluids that can single handily describe all the properties of non-Newtonian fluids. Therefore, several constitutive equations have been proposed to characterize and predict the physical structure and behaviour of such fluids for different materials, Mayers (2005).

The flow characteristics of the above mentioned fluids include shear-thinning, shear-thickening, viscoplasticity, viscoelasticity etc., which are usually analyzed with the help of the power law fluid model. Some of these flow characteristics were not fully described by the power law model. In view of this situation, Sisko proposed a constitutive equation (named after him “Sisko fluid model”) which includes the Newtonian and the power law fluids as special case to characterize these types of flows. Sisko fluids are capable of describing shear thinning and shear thickening phenomena, and have many important and well known industrial applications. It is the most appropriate model for describing the flow of greases having high viscosities at low shear rates and low viscosities at high shear rates. Waterborne coatings and metallic automotive basecoating where polymeric suspensions are used, cement slurries, lubricating greases, most psueodoplastic fluids and drilling fluids are some of its applications in industry, Sisko (1958), Siddiqiui et al. (2007), (2009), (2013), Mekheimer and El Kot (2012).

The Adomian Decomposition method has proven to be a valuable alternative analytic tool for solving linear, nonlinear ordinary as well as partial differential equations, which occur in engineering and applied sciences, Adomian (1987). This method does not require any small parameter, linearization, perturbation and other similar restrictions rather it provides a direct scheme. An advantage of this method is that it provides a solution in the form of an infinite convergent series in which each component can easily be determined by recursion. Hosseini and Nasabzadeh (2006) have discussed the rapid convergence of the series solution obtained by ADM. The first several computed terms of the series solution obtained, usually provides a good approximation with a high degree of accuracy when compared with other techniques, Alam et al. (2012). Wazwaz (1999), (2001), (2009) and Siddiqiui et al. (2010), (2012) have applied this method for different types of linear and nonlinear differential equations. As in our case, for the drainage of magnetohydrodynamic (MHD) Sisko fluid film down a vertical belt, an exact solution seems to be difficult, so a truncated number of terms will be used for the solution purpose.

The purpose of the present paper is to analytically study the steady drainage of magnetohydrodynamic (MHD) Sisko fluid film down a vertical belt. We extend the work of Siddiqiui et al. (2013) to observe the effects of the uniform transverse magnetic field. We shall solve this problem for the first time by ADM. We shall find the physical expressions like the velocity profile, the volume flow rate, the average film velocity, the shear stress, the force exerted by the fluid film on the belt surface and the vorticity vector. We shall also discuss the influences of the fluid behaviour index, The sisko fluid parameter, Stokes number and the Hartmann number on the velocity profile, the shear stress, the flow rate and the vorticity vector via tables and on velocity profile via graphs.

The rest of the paper is organized as follows: the basic governing equations and the constitutive equation for the incompressible Sisko fluid model are given in section 2, in section 3 the problem is formulated, section 4 contains the solution of the problem using the ADM and includes the volume flow rate, the average film velocity, the shear stress, the force exerted by the fluid film on the belt surface and the vorticity vector, in section 5 the influences of the available parameters and dimensionless numbers are discussed through tables and graphs and finally concluding remarks are given in section 6.

2. Basic Equations

The continuous flow behaviour of an electrically conducting Sisko fluid film is governed by the Maxwells' equations and Ohm's law together with the equation of continuity, momentum balance and the constitutive equation for the incompressible Sisko fluid model:
The Maxwells' equations in simplified form are

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \quad \nabla . \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla . \mathbf{E} = \frac{\breve{\rho}}{\varepsilon}, \tag{1}$$

where $\mathbf{B}$ is the total magnetic field, μ_0 is the magnetic permeability, $\mathbf{J}$ is the electric current density, $\mathbf{E}$ is the electric field, ε is the permittivity of free space (an electric constant) and $\breve{\rho}$ is the charge density. An electrically conducting Sisko fluid film moving with velocity $\mathbf{V}$ in the presence of an external magnetic field $\mathbf{B}$, in addition to an electric field $\mathbf{E}$, is subjected to an extra term $(\mathbf{V}\times\mathbf{B})$, that accounts for the current induced by the Lorentz force on the charge carriers. Ohm's law professes that the current density is proportional to the total electric field, mathematically in generalized form, it can be written as

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}), \tag{2}$$

where σ is the electric conductivity. The magnetic induction equation for the magnetic flux $\mathbf{B}$ can easily be derived from Maxwells' equations (1) and Ohm's law (2). It suggests that an applied magnetic field be induces a magnetic field $\mathbf{b}$ in the medium and total field $\mathbf{B}$ is the sum of the applied and induced magnetic fields, i.e., $\mathbf{B} = B_0 + \mathbf{b}$. If the magnetic Reynolds number and the effect of the polarization of the ionized fluid, are negligibly small. Then the so that induced magnetic field $\mathbf{b}$ and electric field $\mathbf{E}$ respectively are assumed to be zero.

Finally, it is assumed tacitly that a magnetic field with a constant magnetic flux density B_0 is applied perpendicular to the velocity field. Hence, the final form of the MHD body force caused by the external magnetic field is

$$\mathbf{J}\times\mathbf{B} = -\sigma B_0^2\mathbf{V}. \tag{3}$$

The equation of continuity and momentum balance respectively are

$$div\mathbf{V} = 0, \tag{4}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + div\mathbf{S} + \mathbf{J}\times\mathbf{B}, \tag{5}$$

where $\mathbf{V}$ is the velocity vector, ρ is the constant density, $\mathbf{f}$ is the body force per unit mass, p is the dynamic pressure, $\mathbf{S}$ is the extra stress tensor and $\frac{D}{Dt}$ is the material time derivative. The constitutive equation for incompressible Sisko fluid model, Sisko (1958), Siddiqiui et al. (2009), (2013), Mekheimer and El Kot (2012) is given by

$$\mathbf{S} = \left[a + b\left(\sqrt{\frac{1}{2}tr\mathbf{A}_1^2}\right)^{n-1}\right]\mathbf{A}_1, \tag{6}$$

where a, b are material constants and n is the fluid behaviour index. If $a = 0$ the equations for the power law fluid model and if $b=0$ for Newtonian fluid are obtained and A_1 is the Rivlin-Ericksen tensor:

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \qquad \mathbf{L} = \nabla \mathbf{V}, \tag{7}$$

superscript T denotes the transpose and $\nabla \mathbf{V}$ is the velocity gradient.

3. Problem Formulation

We consider a steady, laminar and parallel flow of an electrically conducting incompressible Sisko fluid flowing down an infinite vertical belt. As a result, a thin uniform fluid film of thickness δ is formed in contact with the stationary air. We choose an xz-coordinate system such that x-axis is normal to the belt and z-axis along the belt in downward direction as shown in the Fig. 1. We neglect the thermal effects and assume that the fluid completely wets the belt, the belt extends to infinity in the y-direction so that $\frac{d}{dy}=0$, there is no applied pressure driving the flow and fluid film fall under the action of gravity only. Therefore, the only velocity component is in z-direction. Accordingly we assume that

$$\mathbf{V} = [0, 0, w(x)], \qquad \mathbf{S} = \mathbf{S}(x). \tag{8}$$

A uniform transverse magnetic field with a constant magnetic flux density B_0 is applied perpendicular to the belt in the direction of positive x-axis. Therefore, the electromagnetic force per unit volume becomes

$$\mathbf{J} \times \mathbf{B} = [0, 0, -\sigma B_0^2 w(x)]. \tag{9}$$

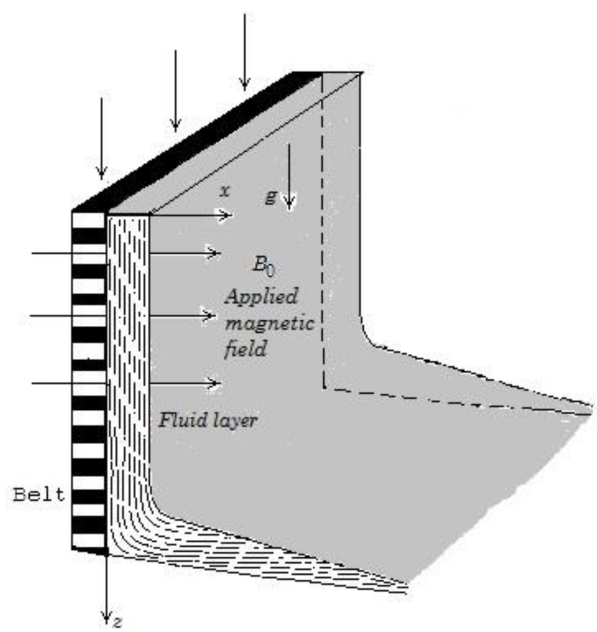

Figure 1. The geometry of the problem

Profile (8) identically satisfies the equation of continuity (4). Equation (6) upon using Equation (7) and profile (8) yields the following non zero components of extra stress tensor:

$$S_{xz} = \left[a + b\left(\frac{dw}{dx}\right)^{n-1} \right] \frac{dw}{dx} = S_{zx}. \tag{10}$$

The momentum balance (5) with the help of Equations (8)-(10) and assumptions we made, will lead to

$$\frac{d^2 w}{dx^2} + \frac{b}{a}\frac{d}{dx}\left(\frac{dw}{dx}\right)^n - \frac{\sigma B_0^2}{a} w = -\frac{\rho g}{a}. \tag{11}$$

The boundary conditions associated with Equation (11) are

$$w = 0 \qquad at \qquad x = 0, \qquad (no\ slip), \tag{12}$$

$$S_{xy} = 0 \qquad at \qquad x = \delta, \qquad (free\ surface). \tag{13}$$

The free surface condition (13) after making use of Equation (10) takes the form

$$\frac{dw}{dx} = 0 \qquad at \qquad x = \delta. \tag{14}$$

Considering non-dimensional parameters

$$w^* = \frac{w}{\sqrt{g\delta}}, \qquad x^* = \frac{x}{\delta},$$

into Equation (11) and boundary conditions (12) and (14), after omitting \`$*$', we obtain

$$\frac{d^2 w}{dx^2} + \beta \frac{d}{dx}\left(\frac{dw}{dx}\right)^n - M^2 w = -S_t, \tag{15}$$

$$w = 0 \qquad at \qquad x = 0, \tag{16}$$

$$\frac{dw}{dx} = 0 \qquad at \qquad x = 1, \tag{17}$$

where

$$S_t = \frac{\rho g \delta^2}{a\sqrt{g\delta}}$$

is the Stokes number,

$$\beta = \frac{b}{a\left(\frac{\delta}{\sqrt{g\delta}}\right)^{n-1}}$$

is the Sisko fluid parameter and

$$M^2 = \frac{\sigma B_0^2 \delta^2}{a}$$

is the Hartmann number. Equation (15) is a second order non-linear and inhomogeneous ordinary differential equation. It seems to be difficult to have exact solution for (15). In the next section, we will apply ADM to solve (15) subject to boundary conditions (16) and (17).

4. Solution of the Problem

Keeping in mind the main steps of ADM (for reference see Adomian (1987), Wazwaz (2009)), we rewrite (15) in operator form as

$$L_{xx}(w) = -S_t - \beta L_x N(w) + M^2 w, \tag{18}$$

where

$$L_{xx} = \frac{d^2}{dx^2} \text{ and } L_x = \frac{d}{dx},$$

respectively, are two and one fold linear operators and

$$N(w) = \left(\frac{dw}{dx}\right)^n$$

is a nonlinear term. Since L_{xx} is invertible, then the two folds inverse operator L_{xx}^{-1} is defined as

$$L_{xx}^{-1}(*) = \iint (*)dxdx.$$

Operating L_{xx}^{-1} on both sides of Equation (18), we get

$$w(x) = A + Bx - \frac{S_t}{2}x^2 - \beta L_{xx}^{-1}\left[L_x N(w)\right] + M^2 L_{xx}^{-1}\left[w(x)\right], \tag{19}$$

where A and B are constants to be determined. We present the decomposition of unknown

$w(x)$ by the decomposition series

$$w(x)=\sum_{k=0}^{\infty} w_k(x), \tag{20}$$

and the expansion of non linear term $N(w)$ by an infinite series of Adomian polynomials

$$N(w)=\sum_{k=0}^{\infty} A_k, \tag{21}$$

where the components $w_k(x), k\geq 0$ and the Adomian polynomials $A_k, k\geq 0$ can easily be computed. Substituting the decomposition series (20) and infinite series of Adomian polynomials (21) into (19), we have

$$\sum_{k=0}^{\infty} w_k(x)=A+Bx-\frac{S_t}{2}x^2-\beta L_{xx}^{-1}\left[L_x\sum_{k=0}^{\infty}A_k\right]+M^2L_{xx}^{-1}\left[\sum_{k=0}^{\infty}w_k(x)\right]. \tag{22}$$

Boundary conditions (16) and (17), after making use of decomposition series (20), yield

$$\begin{cases} w_0(0)=w_1(0)=w_2(0)=\cdots=0, \\ w_0'(1)=w_1'(1)=w_2'(2)=\cdots=0. \end{cases} \tag{23}$$

Equation (22) follows with the following recursive relation

$$w_0(x)=A+Bx-\frac{S_t}{2}x^2, \tag{24}$$

$$w_{k+1}(x)=-\beta L_{xx}^{-1}\left[L_xA_k\right]+M^2L_{xx}^{-1}\left[w_k(x)\right], \qquad k\geq 0. \tag{25}$$

Equation (24) with the help of boundary conditions (23) leads to

$$w_0(x)=\frac{S_t}{2}\left[1-(1-x)^2\right]. \tag{26}$$

Substituting the decomposition series (20) into the infinite series of Adomian polynomials (21), then using binomial expansion, we acquire

$$A_0=\left(w_0'(x)\right)^n, \tag{27}$$

$$A_1=n\left(w_0'(x)\right)^{n-1}w_1'(x), \tag{28}$$

$$\vdots$$

where `dash' over w reprents the derivative with respect to 'x'. By making use of boundary conditions (23), Equation (26) and Adomian polynomials A_k from Equations (27) and (28); the recursive relation (25) gives the following components of $w(x)$:

$$w_1(x) = -\frac{\beta S_t^n}{n+1}\left[1-(1-x)^{n+1}\right] + \frac{M^2 S_t}{24}\left[6x^2 - 12x + 1 - (1-x)^4\right], \tag{29}$$

$$\begin{aligned} w_2(x) = {} & \frac{\beta^2 S_t^{2n-1}}{2}\left[1-(1-x)^{2n}\right] + \frac{n\beta M^2 S_t^n}{6}\left[\frac{3\left(1-(1-x)^{n+1}\right)}{n+1}\right. \\ & \left. - \frac{1-(1-x)^{n+3}}{n+3}\right] - \frac{\beta M^2 S_t^n}{n+1}\left[\frac{x^2}{2} - x + \frac{1-(1-x)^{n+3}}{(n+2)(n+3)}\right] \\ & + \frac{M^4 S_t}{720}\left[15x^4 - 60x^3 + 15x^2 + 90x + 1 - (1-x)^6\right], \end{aligned} \tag{30}$$

$$\vdots$$

Inserting components $w_0(x), w_1(x)$ and $w_2(x)$ from Equations (26), (29) and (30) into the decomposition series (20), we get the solution of Equation (15) of the following form:

$$\begin{aligned} w(x) = {} & \frac{S_t}{2}\left[1-(1-x)^2\right] - \frac{\beta S_t^n}{n+1}\left[1-(1-x)^{n+1}\right] + \frac{\beta^2 S_t^{2n-1}}{2}\left[1-(1-x)^{2n}\right] \\ & + M^2\left[\frac{S_t}{24}\left\{6x^2 - 12x + 1 - (1-x)^4\right\} + \frac{n\beta S_t^n}{6}\left\{\frac{3\left(1-(1-x)^{n+1}\right)}{n+1} - \frac{1-(1-x)^{n+3}}{n+3}\right\}\right. \\ & + \frac{n\beta S_t^n}{6}\left\{\frac{3\left(1-(1-x)^{n+1}\right)}{n+1} - \frac{1-(1-x)^{n+3}}{n+3}\right\} - \frac{\beta S_t^n}{n+1}\left\{\frac{x^2}{2} - x + \frac{1-(1-x)^{n+3}}{(n+2)(n+3)}\right\} \\ & \left. + \frac{M^2 S_t}{720}\left\{15x^4 - 60x^3 + 15x^2 + 90x + 1 - (1-x)^6\right\} + \cdots\right], \end{aligned} \tag{31}$$

which is the velocity profile for the steady drainage of magnetohydrodynamic (MHD) Sisko fluid film downs a vertical belt.

Remark:

We recover the solutions for (i) the Sisko fluid film [Siddiqui et al. (2013)] when $M \to 0$, (ii) the MHD Newtonian fluid film when $\beta \to 0$ and (iii) the Newtonian fluid film [Van Rossum (1958)] when both M and $\beta \to 0$ in (31).

The non-dimensional volume flow rate Q and average film velocity $\overline{w}$ are defined by

$$Q = \overline{w} = \int_0^1 w(x)dx, \tag{32}$$

which upon using (31) leads to

$$Q = \overline{w} = \frac{S_t}{3} - \frac{\beta S_t^n}{n+2} - \frac{2M^2 S_t}{15} + \frac{n\beta^2 S_t^{2n-1}}{2n+1} + \beta M^2 S_t^n \left[\frac{n^2+6n+5}{3(n^2+6n+8)} \right] + \frac{17M^4 S_t}{315} + \cdots. \tag{33}$$

The shear stress (10) exerted by the belt on fluid film in dimensionless form is given by

$$S_{XZ} = \frac{dw}{dx} + \beta \left(\frac{dw}{dx} \right)^n, \tag{34}$$

where

$$S_{XZ} = \frac{S_{xz}}{a\sqrt{\frac{g}{\delta}}}.$$

$$\begin{aligned} S_{XZ} &= S_t(1-x) - \beta S_t^n (1-x)^n - \frac{M^2 S_t}{6}\left\{3(1-x) - (1-x)^3\right\} + n\beta^2 S_t^{2n-1}(1-x)^{2n-1} \\ &+ \frac{n\beta M^2 S_t^n}{6}\left\{3(1-x)^n - (1-x)^{n+2}\right\} + \frac{\beta M^2 S_t^n}{n+1}\left\{(1-x) - \frac{(1-x)^{n+2}}{n+2}\right\} \\ &+ \frac{M^4 S_t}{120}\left\{10x^3 - 30x^2 + 5x + 15 + (1-x)^5\right\} + \beta\Big[S_t(1-x) - \beta S_t^n (1-x)^n \\ &- \frac{M^2 S_t}{6}\left\{3(1-x) - (1-x)^3\right\} + n\beta^2 S_t^{2n-1}(1-x)^{2n-1} + \frac{n\beta M^2 S_t^n}{6}\left\{3(1-x)^n - (1-x)^{n+2}\right\} \\ &+ \frac{\beta M^2 S_t^n}{n+1}\left\{(1-x) - \frac{(1-x)^{n+2}}{n+2}\right\} + \frac{M^4 S_t}{120}\left\{10x^3 - 30x^2 + 5x + 15 + (1-x)^5\right\} \\ &+ \ldots\Big]^n + \cdots. \end{aligned} \tag{35}$$

Equation (35) at $x = 0$ gives the shear stress exerted by the belt on fluid film at the belt surface:

$$\begin{aligned} S_{XZ}\,|_{x=0} &= S_t - \beta S_t^n - \frac{M^2 S_t}{3} + n\beta^2 S_t^{2n-1} + \frac{n\beta M^2 S_t^n}{3} + \frac{\beta M^2 S_t^n}{n+2} + \frac{M^4 S_t}{120} \\ &+ \beta\left[S_t - \beta S_t^n - \frac{M^2 S_t}{3} + n\beta^2 S_t^{2n-1} + \frac{n\beta M^2 S_t^n}{3} + \frac{\beta M^2 S_t^n}{n+2} + \frac{M^4 S_t}{120} + \cdots \right]^n + \cdots. \end{aligned} \tag{36}$$

The force exerted by the fluid film on the belt surface, in dimensionless form during drainage is defined by

$$F_z = -\int_0^1 S_{XZ}\,|_{x=0}\, dx, \tag{37}$$

which after making use of (36) yields

$$F_z = -S_t + \beta S_t^n + \frac{M^2 S_t}{3} - n\beta^2 S_t^{2n-1} - \frac{n\beta M^2 S_t^n}{3} - \frac{\beta M^2 S_t^n}{n+2} - \frac{M^4 S_t}{120}$$
$$-\beta\left[S_t - \beta S_t^n - \frac{M^2 S_t}{3} + n\beta^2 S_t^{2n-1} + \frac{n\beta M^2 S_t^n}{3} + \frac{\beta M^2 S_t^n}{n+2} + \frac{M^4 S_t}{120} + \cdots \right]^n - \cdots, \tag{38}$$

which provides the force exerted by the fluid film at the belt surface. In dimensionless form, the vorticity vector $\bar{\Omega}$ is calculated as

$$\bar{\Omega} = -\left[S_t(1-x) - \beta S_t^n (1-x)^n - \frac{M^2 S_t}{6}\left\{3(1-x) - (1-x)^3\right\} + n\beta^2 S_t^{2n-1}(1-x)^{2n-1} \right.$$
$$+ \frac{n\beta M^2 S_t^n}{6}\left\{3(1-x)^n - (1-x)^{n+2}\right\} + \frac{\beta M^2 S_t^n}{n+1}\left\{(1-x) - \frac{(1-x)^{n+2}}{n+2}\right\}$$
$$\left. + \frac{M^4 S_t}{120}\left\{10x^3 - 30x^2 + 5x + 15 + (1-x)^5\right\} + \cdots \right]\mathbf{j}, \tag{39}$$

where j is the unit vector in the y-direction.

5. Results and Discussion

In this section, we shall observe the quantitative effects of fluid behaviour index n, Sisko fluid parameter β, Stokes number S_t and Hartmann number M involved in the present analysis. For the numerical evaluations of the analytical results obtained in the previous section, we developed numerical codes in Mathematica and displayed some of the important results via tables (see Tables 1-12) and graphs (see Figures 2-6).

Tables 1-4 show the distribution of velocity profile of the MHD Sisko fluid film, draining from the vertical belt. From these tables, it is noted that as we move within the domain i.e., $\forall\, x \in [0,1]$ the velocity increases. We delineated the decrease in velocity with increase in the Hartman number M. This is because of the fact that the introduction of transverse magnetic field damps the thin film of Sisko fluid, flowing down the belt. Moreover, these tables disclosed the slower drainage of Sisko fluid film when compared with the Newtonian fluid film. The slower drainage of shear thinning fluid film as compared to the shear thickening fluid film has also been evident. This indicates that the rheology of the fluid has significant effects on the thin film flow.

Tables 5-8 are tabulated to observe the effect of Hartman number M on the shear stress. From these tables we observed that shear stress decreases as we move from belt surface to the free surface. With increase in the Hartman number M shear stress decreases. It is also observed that shear stress experienced by the Sisko fluid film is greater in amount than that of the Newtonian fluid film. Shear thickening fluid film bears more shear stress as compared to the shear thinning fluid film which decreases with increasing heartman number M.

Distribution of vorticity vector is shown in tables 9-12. From these tables, we infer that as we proceed within the domain $\forall\ x \in [0,1]$, the vorticity effect decreases, i.e., maximum near the belt and minimum at the free surface. Negative sign indicates that film has clockwise rotational effects. Decrease in vorticity effect with increase in the Hartman number M is seen. Furthermore, we also noted that the Sisko fluid film has lesser clockwise rotational effects than that of the Newtonian fluid film. Shear thickening fluid film have more rotational effects as comparing to the shear thinning fluid film which is again the evidence of the rehological effects of Sisko fluid.

Table 1. Velocity distribution for the thin film flow of MHD Sisko fluid when $S_t = 1.2$ and $\beta = 0.0$

	Newtonian fluid film	MHD Newtonian fluid film
	$M = 0.0$	$M = 0.4$
x	$w(x)$	$w(x)$
0.0	0.00000	0.00000
0.2	0.21599	0.20425
0.4	0.38400	0.36178
0.6	0.50400	0.47358
0.8	0.57610	0.54038
1.0	0.60000	0.56260

Table 2. Velocity distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$w(x)$	$w(x)$
0.0	0.00000	0.00000
0.2	0.17845	0.18509
0.4	0.31382	0.33074
0.6	0.40689	0.43644
0.8	0.45900	0.50129
1.0	0.47394	0.52364

Table 3. Velocity distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0.2$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$w(x)$	$w(x)$
0.0	0.00000	0.00000
0.2	0.17635	0.18357
0.4	0.30990	0.32775
0.6	0.40160	0.43220
0.8	0.45290	0.49619
1.0	0.46761	0.51821

Table 4. Velocity distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0.4$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$w(x)$	$w(x)$
0.0	0.00000	0.00000
0.2	0.17067	0.17961
0.4	0.29928	0.31994
0.6	0.38729	0.42108
0.8	0.43643	0.48274
1.0	0.45056	0.50387

Table 5. Shear stress distribution for the thin film flow of MHD Sisko fluid when $\beta = 0$ and $S_t = 1.2$

	Newtonian fluid film	MHD Newtonian fluid film
	$M = 0.0$	$M = 0.4$
x	$S_{XZ}(x)$	$S_{XZ}(x)$
0.0	1.20120	1.14015
0.2	0.95732	0.90348
0.4	0.72231	0.67262
0.6	0.47999	0.44605
0.8	0.23994	0.22232
1.0	0.00000	0.00000

Table 6. Shear stress distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$S_{xz}(x)$	$S_{xz}(x)$
0.0	1.20101	1.23058
0.2	0.96113	0.97764
0.4	0.72133	0.72867
0.6	0.48168	0.48318
0.8	0.24252	0.24057
1.0	0.00000	0.00000

Table 7. Shear stress distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0.2$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$S_{xz}(x)$	$S_{xz}(x)$
0.0	1.18914	1.22083
0.2	0.95001	0.96796
0.4	0.71213	0.72011
0.6	0.47523	0.47671
0.8	0.23932	0.23701
1.0	0.00000	0.00000

Table 8. Shear stress distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0.4$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$S_{xz}(x)$	$S_{xz}(x)$
0.0	1.15692	1.19563
0.2	0.91989	0.94269
0.4	0.68731	0.69755
0.6	0.45794	0.45947
0.8	0.23087	0.22741
1.0	0.00000	0.00000

Table 9. Vorticity vector distribution for the thin film flow of MHD Sisko fluid when $\beta = 0$ and $S_t = 1.2$

	Newtonian fluid film	MHD Newtonian fluid film
	$M = 0.0$	$M = 0.4$
x	$\overline{\Omega}(x)$	$\overline{\Omega}(x)$
0.0	-1.99914	-1.13958
0.2	-0.95932	-0.90330
0.4	-0.72125	-0.67257
0.6	-0.47999	-0.44604
0.8	-0.23991	-0.22231
1.0	0.00000	0.00000

Table 10. Vorticity vector distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$\overline{\Omega}(x)$	$\overline{\Omega}(x)$
0.0	-1.00091	-1.02349
0.2	-0.78404	-0.82718
0.4	-0.57029	-0.62892
0.6	-0.36144	-0.42731
0.8	-0.16202	-0.21994
1.0	0.00000	0.00000

Table 11. Vorticity vector distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0.2$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$\overline{\Omega}(x)$	$\overline{\Omega}(x)$
0.0	-0.99010	-1.01598
0.2	-0.77405	-0.81956
0.4	-0.56217	-0.62200
0.6	-0.35591	-0.42190
0.8	-0.15946	-0.21682
1.0	0.00000	0.00000

Table 12. Vorticity vector distribution for the thin film flow of MHD Sisko fluid when $\beta = 0.2$, $S_t = 1.2$ and $M = 0.4$

	Shear thinning fluid film	Shear thickening fluid film
	$n = 0.5$	$n = 1.5$
x	$\overline{\Omega}(x)$	$\overline{\Omega}(x)$
0.0	-0.96036	-0.98389
0.2	-0.74686	-0.78721
0.4	-0.54026	-0.59237
0.6	-0.34113	-0.39838
0.8	-0.15271	-0.20300
1.0	0.00000	0.00000

Figures 2-6 are showing the effects of fluid behaviour index n, Sisko fluid parameter β, Stokes number S_t and Hartmann number M on the velocity profile of the MHD Sisko fluid film. It is observed in Figure 2 that the velocity increases with the increase in n and shear thickening fluid film drains down faster than the shear thinning fluid film. From Figure 3 (drawn to observe the effect of β on velocity profile), we depicted that the velocity decreases with the increase in β. The comparison of Newtonian fluid film ($\beta = 0$) and Sisko fluid film ($\beta > 0$) shows notable increase in magnitude of the velocity from Sisko fluid film to Newtonian fluid film. Figure 4 shows the effect of S_t on velocity profile, increase in velocity with the increase in S_t can be seen in it. In order to observe the effect of M on velocity, Figures 5 and 6 are plotted. It is evident that the velocity decreases with the increase in M, i.e., fluid drains down slowly in the presence of magnetic field.

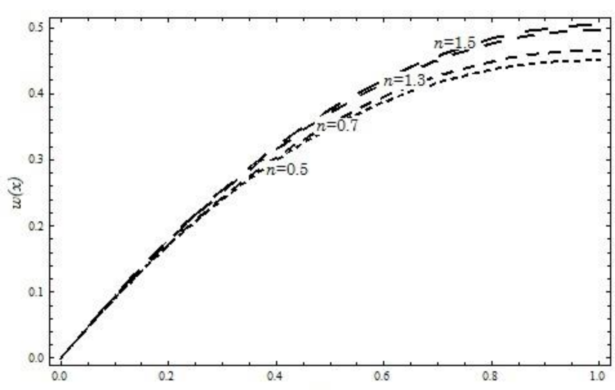

Figure 2. The effect of variation in fluid index n on velocity profile, for $S_t = 1.2$, $\beta = 0.2$ and $M = 0.4$

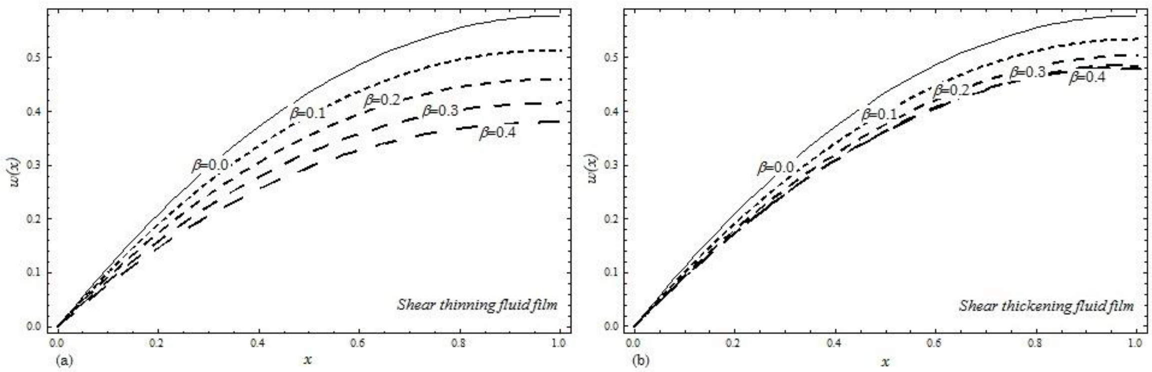

Figure 3. The effect of variation in Sisko fluid parameter β on velocity profile, for $S_t = 1.2$, $M = 0.4$, (a) $n = 0.5$ and (b) $n = 1.5$

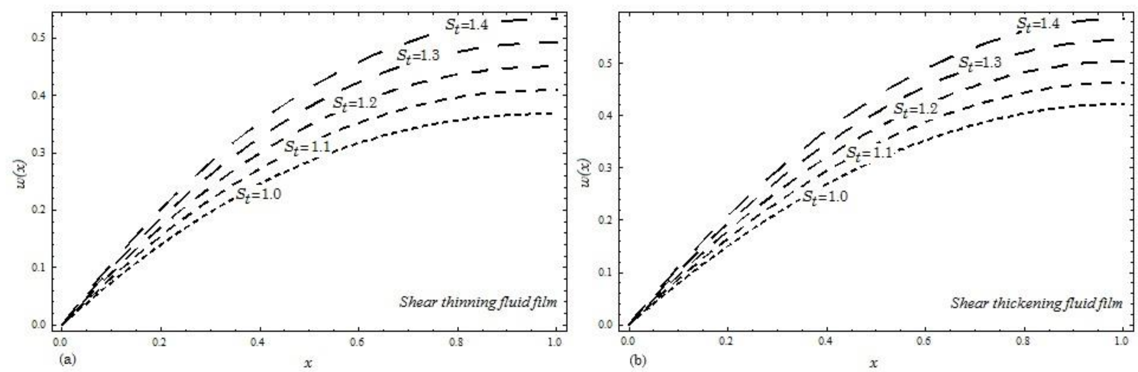

Figure 4. The effect of variation in Stokes number S_t on velocity profile for $\beta = 0.2$, $M = 0.4$, (a) $n = 0.5$ and (b) $n = 1.5$

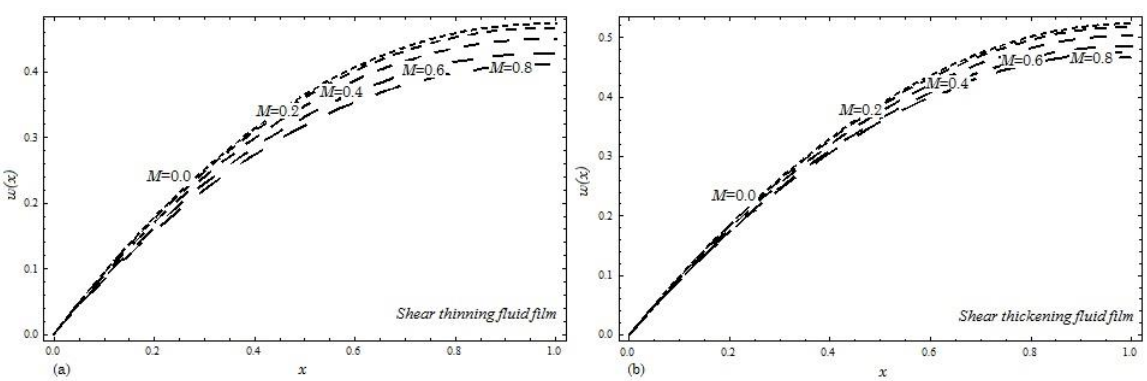

Figure 5. The effect of variation in Hartmann number M on velocity profile for $S_t = 1.2$ and $\beta = 0.2$, (a) $n = 0.5$ and (b) $n = 1.5$

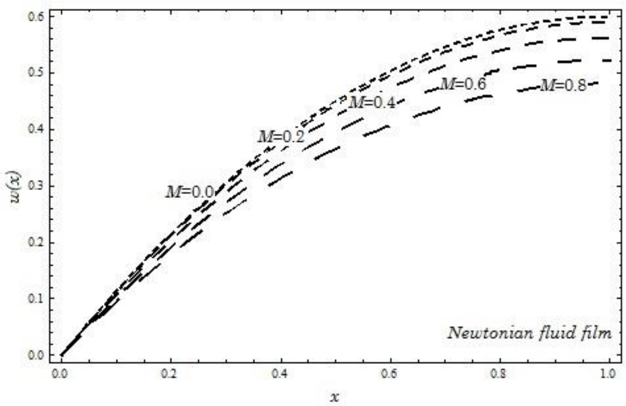

Figure 6. The effect of variation in Hartmann number M on velocity profile for $S_t = 1.2$ and $\beta = 0$

6. Concluding Remarks

In the present theoretical work, we analytically investigated the flow behaviour during the steady drainage of magnetohydrodynamic (MHD) Sisko fluid film down a vertical belt. Modeling of the problem yielded non-linear ordinary differential equation which has been analyzed using ADM. We found ADM much easier to proceed and concluded that it can provide any desired higher order solution recursively with efficiency and high accuracy. ADM does not require linearization, perturbation or any other similar restrictive assumptions which is the main advantage of this method for obtaining the approximate solutions as is shown in our present work. We observed the effect of Hartman number M on velocity profile, shear stress and vorticity vector via tables and the effects of fluid behaviour index n, Sisko fluid parameter β, Stokes number S_t and Hartman number M on velocity profile via graphs. The results obtained indicate the following findings:

- The velocity decreases monotonically with the increase in Hartman number M. The decrease in velocity with increasing M discloses the fact that transverse magnetic field damps the drainage of thin film flow of Sisko fluid down the belt.
- The velocity increases with the increase in n and S_t and decreases with increasing β.
- Shear stress exerted by the belt on Sisko fluid film decreases with increase in the Hartman number M.
- With the increase in the Hartman number M, the vorticity effect decreases. Sisko fluid film has clockwise rotational effects and these effects are maximum near the belt and minimum at the free surface.

These findings show that presence of a uniform transverse magnetic field stressed the system.

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