Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 8, Issue 1 (June 2013), pp. 305- 317

Applications and Applied Mathematics:
An International Journal
(AAM)

Generalised Separable Solution of Double Phase Flow through Homogeneous Porous Medium in Vertical Downward Direction Due to Difference in Viscosity

A.K. Parikh
Department of Mathematics & Humanities
Institute of Technology, Nirma University
Ahmedabad-382481, India
amit.parikh@nirmauni.ac.in

M.N. Mehta and V.H. Pradhan
S.V. National Institute of Technology
Surat-395 007
mnm@ashd.svnit.ac.in; pvh@ashd.svnit.ac.in

Received: April 05, 2012; Accepted: December 05, 2012

Abstract

In this paper the instability (fingering) phenomenon in a double phase immiscible (oil and water) flow through the homogeneous porous medium with mean capillary pressure in the vertical downward direction is discussed. The mathematical formulation of this problem yields a non-linear partial differential equation and the generalised separable solution is given in the exponential form. The numerical solution and graphical presentation is given using MAT LAB coding.

Keywords: Instability phenomenon, Immiscible, Homogeneous porous medium, Capillary pressure

MSC 2010 No.: 76S05, 76T99

1. Introduction

When water is injected in the vertical downward direction then oil is displaced by water of lesser viscosity and instead of regular displacement of the whole front, protuberance takes place which shoot through the porous medium at a relatively very high speed. Due to the gravitational effect in the vertical downward direction the velocity of the injected water will increase and the oil from the formatted region will push towards the bottom of the cylindrical porous matrix which is connected by the pipe with the production well as shown in Figure 1.

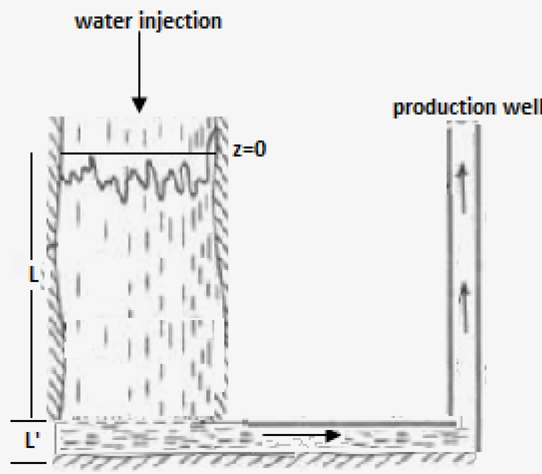

Figure- 1

Many researchers have discussed this phenomenon with various points of view. Some of these are summarised here. Scheidegger (1960) considered the average cross-sectional area occupied by the fingers while the size and shape of the individual fingers were neglected. It was shown by Scheidegger and Johnson (1961) that treatment of motion with a concept of fictitious relative permeability is formally identical to the Buckley Leverett (1942) description of two immiscible fluids flowing through the porous medium. Most of the earlier researcher such as Saffman and Taylor (1958), Scheidegger and Johnson (1961), Wodding (1962) have completely neglected the capillary pressure. Verma (1970) included capillary pressure in the analysis of fingers. Verma (1969) has discussed the statistical behaviour of the fingering phenomenon in a displacement process in heterogeneous porous medium with capillary pressure using perturbation solution. Kataria and Mehta (2001) found an analytical solution of fingering in terms of infinite series. They assumed that the saturation of injected fluid is expressible as a sum of the steady state part and transient part.

Mishra (1977), Mehta (1977) and Patel (1997) have taken the mean pressure into account in their respective researches and have used different mathematical techniques to obtain more accurate results. Swaroop and Mehta (2002) obtained a numerical solution of this phenomenon by the finite element method. Liu (1997) Putra and Schechter (1999); Tang and Firoozabadi, (2001) investigated the effects of the injection rate, the initial water saturation and gravity on water injection in slightly water-wet fractured porous media. Mehta and Kinjal (2011) discussed this phenomenon in heterogeneous porous media without inclination and with inclination; Nisha and Mehta (2011) discussed this phenomenon under a magnetic field effect and magnetic fluid effect. Recently Kinjal and Mehta (2011) have given a power series solution of this phenomenon in homogeneous porous media in the horizontal direction.

Most of the researchers have considered the injection of water in a horizontal direction but here we consider that water is injected in oil formation in the vertical downward direction. So the additional gravitational effect will increase the velocity of the injected water and hence more oil can be displaced during a secondary oil recovery process.

2. Statement of the Problem

In this present problem it is considered that there is a uniform water injection into an oil saturated porous medium of homogeneous physical characteristics. When water is injected in oil formatted porous media in the vertical downward direction in a cylindrical piece of porous matrix surrounded by impermeable surface and bottom is also impermeable as shown in figure (1) connected with production well, oil will displace at common interface $z = 0$ in downward direction. The protuberances may occur due to the viscosity of oil and water. The water will shoot through inter connected capillaries in downward direction to drag the oil toward the bottom.

To understand this phenomenon of fingering, we consider the cross sectional area of an actual formation of fingers in porous media as a rectangle. The shape and size of the fingers are different. Therefore for mathematical formulation, we consider the average cross sectional area of the fingers by using the different size of rectangle as shown in Figure (2). The saturation of the water in all the fingers of the schematic diagram is assumed to be $S_w(z, t)$ for $t > 0$.

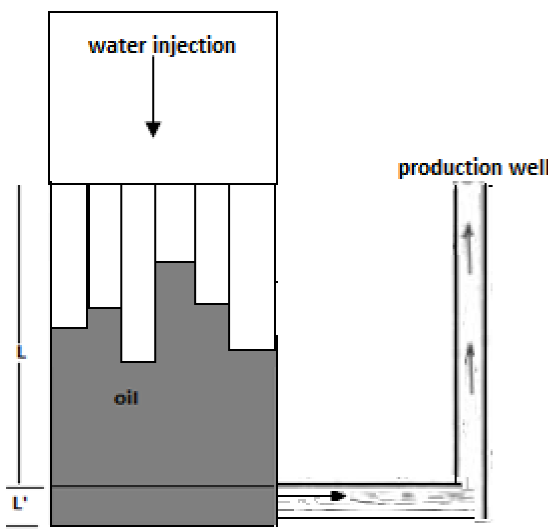

Figure - 2 Schamatic Diagram

The water and oil both are flowing in homogeneous porous media with constant porosity (P) and permeability (K). For a low Reynolds's number in the fluid flow through porous media the Darcy's law will be applicable to measure the velocity of water (V_w) and velocity of oil (V_o). When water and oil are flowing in vertical downward direction, the gravitational effect plays important role is to increase the velocity of the water and oil by the component $\rho g h$ where $h = (L + L') - z$, z is measured from z = 0 in downward direction.

Many researchers have considered injection of water in horizontal direction but here we consider that water is injected in oil formation in vertical downward direction. Hence, the additional gravitational effect will increase velocity of injected water hence more oil can be displaced during secondary oil recovery process.

In this case the saturation of the injected water (S_w) is defined as the average cross-sectional area occupied by it at level z. If the displacement processes are in a z-direction with time t then it is given as $S_w(z, t)$.

Thus the saturation of displacing fluid in a porous medium represents the average cross-sectional area occupied by fingers.

In this chapter the instability (fingering) phenomenon in double phase immiscible flow through homogeneous porous media (oil and water) with mean capillary pressure in vertical downward direction has been discussed. This phenomenon has great importance in petroleum industry. Our attempt is to stabilise the fingers with the help of mathematical solution.

The mathematical formulation of this problem leads to non-linear partial differential equation.

3. Mathematical Formulation

Since water and oil flow through a porous media, for small Reynolds number Darcy's law is valid. According to Bear and Cheng (2010) , Li and Home (2001) and using Darcy's law, when water is injected in downward direction then the velocity of injected water (V_w) and velocity of oil (V_o) under gravitational effect will be

$$V_w = -\left(\frac{K_w}{\delta_w}\right) K \left(\frac{\partial P_w}{\partial z} + \rho g h\right) \quad \text{where } h = (L + L') - z, \tag{1}$$

$$V_o = -\left(\frac{K_o}{\delta_o}\right) K \left(\frac{\partial P_o}{\partial z} + \rho g h\right) \quad \text{where } h = (L + L') - z, \tag{2}$$

where K is the permeability of the homogeneous medium, K_w and K_o are the relative permeability's of water and oil which are functions of the saturation S_w and S_o, P_w and P_o are pressures of water and oil, δ_w and δ_o are constant kinematic viscosity of water and oil respectively.

The equation of continuity for native fluid and injected fluid when phase densities are considered as constants are given as

$$P\left(\frac{\partial S_w}{\partial t}\right) + \left(\frac{\partial V_w}{\partial z}\right) = 0, \tag{3}$$

$$P\left(\frac{\partial S_o}{\partial t}\right)+\left(\frac{\partial V_o}{\partial z}\right)=0, \tag{4}$$

where P is the constant porosity of the homogeneous porous medium and it is considered constant. It is well known fact and by definition of phase saturation, Scheidegger (1960), that

$$S_w + S_o = 1. \tag{5}$$

When water is injected at a common interface in the downward direction, water will flow through the interconnected capillaries under gravitational effect due to the difference in viscosity; the capillary pressure (P_c) (defined as the pressure difference of the flowing phase (water and oil) across their common interface) is a function of water saturation.

It may be written as,

$$P_c = P_o - P_w \qquad \text{Scheidegger (1960)}, \tag{6}$$

and

$$P_c = \beta S_w \qquad \text{where } \beta \text{ is constant, Verma (1969).} \tag{7}$$

For definiteness in the mathematical analysis, we use the standard relation between saturation of water and oil and the relative permeability of water and oil, given by Scheidegger and Johnson (1961) are

$$K_w = S_w, \tag{8}$$

and

$$K_o = S_0 = 1-\alpha S_w \quad \text{where} \quad \alpha = 1.11. \tag{9}$$

If we choose $\alpha \approx 1$ for definiteness, then

$$K_o \approx S_0 = 1 - S_w \quad \left(S_0 + S_w = 1, \text{Scheidegger (1960)}\right).$$

The equation of motion for saturation of water and oil are obtained by substituting the values of V_w and V_o from equations (1) and (2) in the equations (3) and (4), respectively,

$$P\left(\frac{\partial S_w}{\partial t}\right)=\frac{\partial}{\partial z}\left[\frac{K_w}{\delta_w}K\frac{\partial P_w}{\partial z}\right]-\rho g, \tag{10}$$

and

$$P\left(\frac{\partial S_o}{\partial t}\right)=\frac{\partial}{\partial z}\left[\frac{K_o}{\delta_o}K\frac{\partial P_o}{\partial z}\right]-\rho g. \tag{11}$$

On substituting the value of P_w from (6) to (10), we get

$$P\left(\frac{\partial S_w}{\partial t}\right) = \frac{\partial}{\partial z}\left[\frac{K_w}{\delta_w} K\left(\frac{\partial P_o}{\partial z} - \frac{\partial P_c}{\partial z}\right)\right] - \rho g. \tag{12}$$

From equation (5), we have

$$\left(\frac{\partial S_o}{\partial t}\right) + \left(\frac{\partial S_w}{\partial t}\right) = 0.$$

Combining this with (11) and (12) gives,

$$\frac{\partial}{\partial z}\left[K\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right)\frac{\partial P_o}{\partial z} - K\left(\frac{K_w}{\delta_w}\right)\frac{\partial P_c}{\partial z}\right] = 2\rho g. \tag{13}$$

On integrating, we get

$$K\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right)\frac{\partial P_o}{\partial z} - K\left(\frac{K_w}{\delta_w}\right)\frac{\partial P_c}{\partial z} = 2\,\rho g\, C, \tag{14}$$

where C is the constant of integration which can be evaluated later. Simplifying the above equation, we get

$$\frac{\partial P_o}{\partial z} = \frac{2\,\rho g\, C}{K\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right)} + \frac{\frac{\partial P_c}{\partial z}}{1 + \left(\frac{K_o}{K_w}\right)\left(\frac{\delta_w}{\delta_o}\right)} \tag{15}$$

From equations (12) and (15),

$$P\left(\frac{\partial S_w}{\partial t}\right) + \frac{\partial}{\partial z}\left[\frac{\left(\frac{K_o}{\delta_o}\right)\left(\frac{\partial P_c}{\partial z}\right)}{\left(\frac{K_w}{\delta_w} + \frac{K_o}{\delta_o}\right)} - \frac{2\,\rho g\, C}{1 + \left(\frac{K_o}{K_w}\right)\left(\frac{\delta_w}{\delta_o}\right)}\right] + \rho g = 0. \tag{16}$$

The pressure of oil (P_o) can be expressed as

$$P_o = \frac{P_o + P_w}{2} + \frac{P_o - P_w}{2} = \bar{P} + \frac{1}{2}P_c, \tag{17}$$

where $\bar{P}$ is the constant mean pressure.

Differentiating the above equation with respect to z, we get the following equation.

$$\frac{\partial P_o}{\partial z} = \frac{1}{2}\frac{\partial P_c}{\partial z}\,. \tag{18}$$

The concept of mean pressure is justified in the statistical treatment of fingering Verma(1969) so that on substituting the value of $\frac{\partial P_o}{\partial z}$ from (18) to (15), we get

$$2\,\rho g\,C = -\left[\frac{K}{2}\left(\frac{K_w}{\delta_w} - \frac{K_o}{\delta_o}\right)\frac{\partial P_c}{\partial z}\right]. \tag{19}$$

Also substituting value of C in equation (16), we get

$$P\left(\frac{\partial S_w}{\partial t}\right) - \frac{1}{2}\frac{\partial}{\partial z}\left[K\left(\frac{K_w}{\delta_w}\frac{dP_c}{dS_w}\frac{\partial S_w}{\partial z}\right)\right] + \rho g \;= 0, \tag{20}$$

and the value of P_c and K_w from equations (7) and (8) into the above equation, we get

$$P\left(\frac{\partial S_w}{\partial t}\right) - \frac{\beta}{2}\frac{K}{\delta_w}\frac{\partial}{\partial z}\left(S_w\frac{\partial S_w}{\partial z}\right) + \rho g = 0. \tag{21}$$

Since the gravitational effect g increases the velocity of the injected water in the downward direction it will also increase the saturation of the injected water. For particular interest we assume that

$$g \propto S_w{}^2 \Rightarrow g = \omega S_w{}^2\,, \omega \text{ is constant of proportion.}$$

Then, equation (21) can be rewritten as

$$P\left(\frac{\partial S_w}{\partial t}\right) - \frac{\beta}{2}\frac{K}{\delta_w}\frac{\partial}{\partial z}\left(S_w\frac{\partial S_w}{\partial z}\right) + \rho\,\omega S_w{}^2 = 0. \tag{22}$$

A set of boundary conditions at the common interface and at the bottom of the cylindrical porous matrix at $z = L$ is imposed:

$$S_w(0,t) = S_{w0}(t)\;, \qquad t > 0, \tag{23}$$

$$S_w(L,t) = S_{w1}(t)\;, \qquad t > 0, \tag{24}$$

where L being the length of the cylindrical porous matrix.

We choose the dimensionless variables;

$$T = \frac{K\beta}{2L^2 P\delta_w}t\,, \qquad Z = \frac{z}{L}. \tag{25}$$

Equation (22) and the boundary conditions (23) & (24) are now converted into (26), (27) and (28) as follows:

$$\frac{\partial S_w}{\partial T} = S_w \frac{\partial^2 S_w}{\partial Z^2} + \left(\frac{\partial S_w}{\partial Z}\right)^2 - A{S_w}^2 , \qquad A = \frac{2L^2 P\delta_w \rho\, \omega}{K\beta}, \tag{26}$$

$$S_w(0,T) = S_{w0}(T) , \qquad T > 0, \tag{27}$$

$$S_w(1,T) = S_{w1}(T) , \qquad T > 0. \tag{28}$$

Equation (26) is a nonlinear partial differential equation of motion of the injected water which governs phenomenon of instability.

3. Separable Solution Involving Exponential of Z

As discussed by Galaktionav and Posashkov (1989), the exact solution of equation (26) is

$$S_w(Z,T) = \emptyset(T) + \Psi(T)\exp(\pm\lambda Z) \quad \text{where } \lambda = \left(\frac{A}{2}\right)^{1/2} , \tag{29}$$

$$S_w(Z,T) = \emptyset(T) + \Psi(T)e^{\pm\sqrt{\frac{A}{2}}Z}. \tag{30}$$

The saturation of the injected water will increase as the depth increases for any time T > 0. Therefore, the positive exponential term is considered for physical consistency.

Where $\emptyset(T)$ and $\Psi(T)$ can be determined from a system of first order ODEs as follows:

$$\emptyset'(T) = A\emptyset^2 \implies \emptyset(T) = -\frac{1}{AT + C_1} , \tag{31}$$

$$\Psi'(T) = (\lambda^2\emptyset + 2A\emptyset)\Psi = \frac{5A}{2}\,\emptyset\Psi \implies \Psi(T) = \frac{C_2}{(AT + C_1)^{5/2}}. \tag{32}$$

From (30),

$$S_w(Z,T) = -\frac{1}{AT + C_1} + \frac{C_2}{(AT + C_1)^{5/2}} e^{\sqrt{\frac{A}{2}}Z} , \tag{33}$$

where C_1 *and* C_2 are the constants of integration.

To determine constants C_1 *and* C_2 , we use conditions (27) and (28), to get

$$C_2 = \left(S_{w0}(T) + \frac{1}{AT + C_1}\right)(AT + C_1)^{5/2} . \tag{34}$$

Substituting the value of C_2 from (34) in (33), we get

$$S_w(Z,T) = -\frac{1}{AT+C_1} + \left(S_{w0}(T) + \frac{1}{AT+C_1}\right) e^{\sqrt{\frac{A}{2}}Z}. \tag{35}$$

Also using the condition, $S_w(1,T) = S_{w1}(T)$ we get

$$S_{w1}(T) = -\frac{1}{AT+C_1} + \left(S_{w0}(T) + \frac{1}{AT+C_1}\right) e^{\sqrt{\frac{A}{2}}} . \tag{36}$$

This gives

$$\frac{1}{AT+C_1} = \frac{S_{w1}(T) - S_{w0}(T)}{e^{\sqrt{\frac{A}{2}}} - 1} - S_{w0}(T) . \tag{37}$$

Substituting into (35), yields

$$S_w(Z,T) = S_{w0}(T) + \left(S_{w1}(T) - S_{w0}(T)\right)\left[\frac{e^{\sqrt{\frac{A}{2}}Z}-1}{e^{\sqrt{\frac{A}{2}}}-1}\right], \tag{38}$$

where $A = \frac{2L^2 P\delta_w \rho\,\omega}{K\beta}$.

This represents the saturation of the injected water in the downward direction for any distance Z and at any time T > 0.

4. Stability Analysis of The Solution

Here the PME is considered; it is posed in a spatial bounded domain Ω. The problem has been shown to be uniquely solvable in a class of weak solutions. It was also shown that these weak solutions are not always classical solutions (Juan Luis Vazquez (2007)). The central issue is to construct an existence theory as wide as possible and complement it with uniqueness and stability. Now, it is not automatic that the most natural class of data for existence purposes coincides with the class where uniqueness and stability can be proved. The existence of the separable variable solution in a bounded domain was rigorously proved by Aronson and Peletier in (1981).

For stability of the solution, we use the stability test. From equation (38),

$$|S_w(Z,T) - S_{w0}(T)| \le \left|\left(S_{w1}(T) - S_{w0}(T)\right)\left[\frac{e^{\sqrt{\frac{A}{2}}Z} - 1}{e^{\sqrt{\frac{A}{2}}} - 1}\right]\right| \le \varepsilon.$$

As $z \to \infty$ for any $T > 0$. (Yortsos and Huang (1986)). Hence, the solution of saturation $S_w(Z,T)$ is stable; that is the weak approximate solution of instability (fingering) phenomenon.

The uniqueness and existence of the Porous Medium equation is discussed by Juan Luis Vazquez (2007). We have not discussed at this stage as this is not our particular interest.

Solution (38) is in the exponential form which is convergent as per Ratio test. It satisfies both conditions (27) and (28). This shows that the fingers are stabilized for any given Z < 1; it can also be stabilized by heterogeneity of porous media, Verma (1970). Saturation of the injected water (S_w) is increasing with respect to time as well as with respect to depth which is consistent with the physical phenomenon.

5. Numerical and Graphical Solution

To use numerical values we have considered that the boundary conditions are linear functions of time.

$$S_w(0,T) = S_{w0}(T) = aT, \qquad T > 0,$$
$$S_w(1,T) = S_{w1}(T) = bT, \qquad T > 0,$$

where a and b are constants.

Here, for numerical calculation we consider the following values:

$$a = 0.1, b = 1, \varepsilon = 0.001, \omega = 9.8, P = 0.5, \delta_w = 0.05, \beta = 0.5, k = 0.1, \rho = 0.1$$
$$\Rightarrow A \approx 1$$

Numerical and graphical presentations of equation (38) have been obtained by using MATLAB coding. Figures 3.3 and 3.4 shows the graphs of S_w vs. Z for time T = 0.5, 0.6, 0.7, 0.8 and Table 1 shows the numerical values.

Table 1. Saturation of injected water (S_w) or different Z for fixed T

Time (T) →	0.5	0.6	0.7	0.8
Depth (Z) ↓	**Saturation of injected water (S_w)**			
0	0.0500	0.0600	0.0700	0.0800
0.1	0.0821	0.0985	0.1149	0.1313
0.2	0.1165	0.1398	0.1631	0.1864
0.3	0.1534	0.1841	0.2148	0.2455
0.4	0.1931	0.2317	0.2703	0.3089
0.5	0.2356	0.2828	0.3299	0.3770
0.6	0.2813	0.3376	0.3938	0.4501
0.7	0.3303	0.3964	0.4625	0.5285
0.8	0.3829	0.4595	0.5361	0.6127
0.9	0.4394	0.5273	0.6152	0.7030
1	0.5000	0.6000	0.7000	0.8000

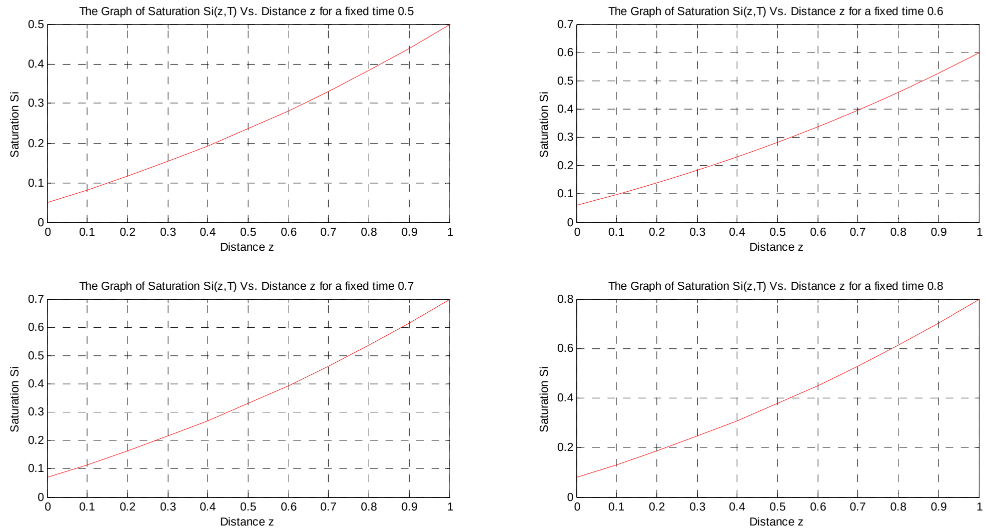

Figure 3. Graph of saturation of injected water (S_w) vs. depth (Z) for different time T > 0

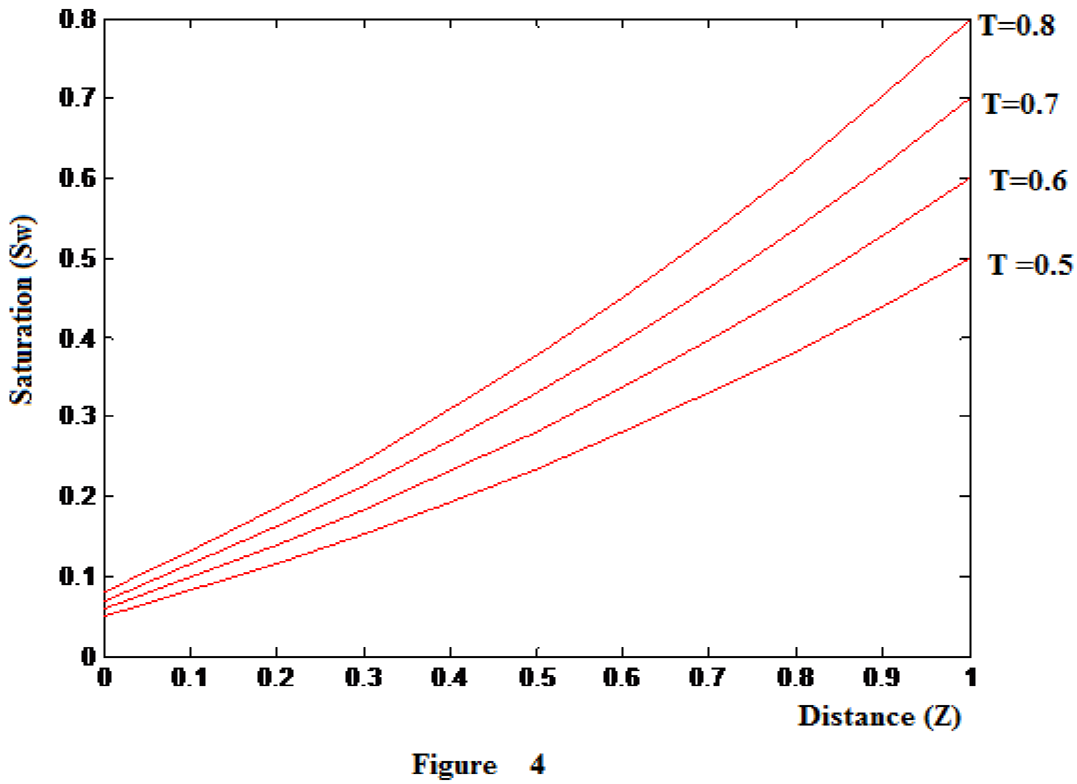

Figure 4. Graph of saturation of injected water (S_w) Vs. depth (Z) for fixed time T>0

6. Conclusion

The solution (38) represents the saturation of the injected water in the downward direction for any depth Z and for any time $T > 0$ under gravitational effect. The boundary conditions are assumed to be linear function of time. The solution is in the form of exponential function which satisfies both the boundary conditions (27) and (28) at $Z = 0$ and at $Z=L$. The saturation of water is increasing for any depth Z at any time $T > 0$ and it is consistent with physical phenomenon. As the saturation of water increases in the downward direction with respect to depth and time, it will push oil from the oil formatted region downward. Since the bottom is impermeable, the maximum amount of oil residing at the bottom is transferred towards the oil production well through the interconnected pipe of diameter L'.

The graphical representation and numerical values are given using MATLAB coding. The graph of S_w vs. Z for any time T is steadily increasing and after some depth Z for any time T, it is likely to be constant which concludes that the fingers maybe stabilized after some depth Z in downward direction.

REFERENCES

Aronson and Peletier (1981). Large time behaviour of solutions of the porous medium equation in bounded domains. *J. Differ. Equ.*, **39**(3),378–412.

Bear and Cheng (2010). Modelling Ground water flow and contaminant Transport theory and Applications of transport in Porous Media, Springer

Bear (1972). Dynamics of fluid flow in porous media, American Elsevier

Buckley and Leverett (1942). Mechanism of fluid displacement in sands, Trans. AIME 146, p. 107.

Galaktionov and Posashkov (1989). Handbook of Nonlinear PDE, Chapmann & Hall/CRC.

Galaktionov, Posashkov and Svirshchevskii (1995). Generalized separation of variables for differential equations with polynomial right-hand sides, *Dif. Uravneniya* [in Russian],Vol. 31, No. 2, pp. 253.261.

Kataria and Mehta (2001). On an average cross-sectional area occupied by the fingers under mean pressure, *Journal of Pure and Applied Sciences*, 11, 55-62.

Lange, Schroter, Schere, Engel and Rehberg (1998). Fingering instability in water-sand mixture. Eur. Phys. J. B.Vol.4, pp. 475-484

Li, Kevin and Home, Ronald N. (2001). Characterization of spontaneous water imbibition into gas saturated rocks, SPEJ, Vol. 6(4), pp. 375-385.

Liu and Wen (1997). Nonlinear diffusive surface waves in porous media, in: J. Fluid Mech. 347 pp.119-139.

Mishra (1977). A mathematical study of multiphase fluid flow problems in porous media, Ph.D. Thesis, South Gujarat University, Surat.

Patel and Mehta (2011). The Power Series Solution of Fingering Phenomenon arising in Fluid Flow through Homogeneous Porous Media, AAM, Vol-6(11)

Scheidegger (1960). The physics flow through porous media, University of Toronto press, pp.229-231,216.

Schiedegger and Jhonson (1961). The statistical behaviour of instabilities in displacement process in porous media. Canedian J. Physics, 39,326.

Tang and Firoozabadi (2001). Effect of pressure gradient and initial water injection in water-wet and mixed-wet fractured porous media. SPE Reserv, Evalu. Eng. 516–524 (Dec.).

Vazquez, Juan Luis (2007). The Porous Medium Equation, Clarendon Press, Oxford University Press.

Verma (1970). Stability of fingers in slightly dipping heterogeneous porous medium with capillary pressure, *Revue Romanie Des sciences Techniques series Macanique Appliquee.*, 15,1197-1206.

Verma (1980). Instabilities in two phase flow through porous media with magnetic fluid, In multiphase Transport, Fundamentals, Reactor Safety, Applications (Ed. T.N.Veziroglu). Hemisphere publ. Co., Washington, 5, 1323-1335.

Verma (1969). Statistical behaviour of fingering in a displacement process in heterogeneous porous medium with capillary pressure, Canadian J. Phys., pp. 3, 47, 319.

Vyas, Patel and Mehta (2011). Power series solution of double phase instability phenomena under magnetic field effect, Int. J. of Appl. Math and Mech., 7(16), pp.1-10.

Wang and Feyen (1998). Prediction of fingering in porous media. Water resource, Vol. 9, pp. 2183-2190.

Yortsos and Huang (1986). Linear stability analysis of immiscible displacement: part I-simple basic profiles; SPE Reservoir Engineering, pp. 378-390.

Zaitsev and Polyanin (1996). Handbook of Partial Differential Equations: Exact Solutions [in Russian], Mezhdunarodnaya Programma Obrazovaniya, Moscow.

Zhan and Yortsos (2002). The shape of a gravity finger in a rectangular channel in homogeneous porous media. Trans. Porous Media, Vol.49, pp. 77-97.