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Asymptotic Properties of Solutions of Two Dimensional Neutral Difference Systems

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Abstract

In this paper we obtain sufficient conditions for the asymptotic properties of solutions of two dimensional neutral difference systems. Our result extends some existing results in the literature. An example is given to illustrate the result.

Keywords: Difference system, neutral type, Nonoscillotory, asymptotic

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1. Introduction

The problem of oscillation and nonoscillation of second order nonlinear difference equations is of particular interest because they are discrete analogues of second order differential equations with physical applications (Kocic et al. (1993), Potts (1981)). It is an interesting problem to extent an oscillation criteria for second order nonlinear difference equations to the case of nonlinear two dimensional difference systems since such systems include, in particular, second order nonlinear, half linear and quasilinear difference equations as special cases.

In the qualitative theory of difference equation, oscillatory and nonoscillatory behavior of solutions plays an important role. Further these types of solutions are associated with many physical and biological phenomena such as vibrating mechanical systems, electrical circuits and population dynamics. Thandapani et al. (1996), considered the following neutral difference equation of the form

$$\Delta(a_n\Delta(x_n + p_n x_{n-k})) + q_n(x_{n+1-l}) = 0,\ \ n \in N(n_0).$$

They obtained the oscillation criteria of the equation. Further they classified all nonoscillatory solutions of the same equation into four classes and established conditions for the existence and nonexistence of solutions in these classes. Szafranski et al. (1990) considered the two dimensional difference system of the form

$$\Delta(x_n) = b_n y_n$$

$$\Delta y_n = a_n f(x_n)\ \ n \in N(n_0).$$

They established the condition for the oscillation and asymptotic behavior of solutions of the above system. Graef et al. (1999) considered the two dimensional difference system of the form

$$\Delta(x_n) = b_n g(y_n)$$

$$\Delta y_n = -a_n f(x_n)\ \ n \in N(n_0).$$

They obtained conditions for all solutions of the system to be oscillatory. Huo and Li (2001) considered the following Emden-Fowler difference system

$$\Delta(x_n) = b_n g(y_n)$$

$$\Delta y_n = -a_n f(x_n) + r_n\ \ n \in N(n_0).$$

They established some criteria for the oscillation of the system.

Motivated by the above, in this paper we study the asymptotic behavior of two dimensional nonlinear difference system of neutral type. Consider the nonlinear two-dimensional difference systems of neutral type

$$\Delta(x_n - a_n x_{\sigma(n)}) = p_n y_n \tag{1}$$

$$\Delta y_n = -q_n f(x_{\tau(n)}),\ \ n \in N(n_0).$$

and $(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 non-negative integer.

The following conditions are assumed:

1. $\{a_n\}$ is a positive real sequence. $\{\sigma(n)\}$ and $\{\tau(n)\}$ are sequences of integers and are increasing with $\lim_{n\to\infty}\sigma(n) = \lim_{n\to\infty}\tau(n) = \infty$.

2. $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences with $q_n \not\equiv 0$ for infinitely many values of n and $\sum_{n=n_0}^{\infty} p_n = \infty$.

3. $f : \Re \to \Re$ is continuous, $uf(u) > 0$ for $u \neq 0$ and $|f(u)| \geq K|u|$ where K is a positive constant.

Let $\theta = \max\{\inf_{n>n_0} \sigma(n), \inf_{n>n_0} \tau(n)\}$. By a solution of the system (1) we mean a real sequence $X = (\{x_n\},\{y_n\})$, which is defined for all $n \geq n_0 - \theta$ and satisfies the system (1) for all $n \in N(n_0)$.

Denote by W the set of all solutions $X = (\{x_n\},\{y_n\})$ of the system (1) which exists for $n \in N(n_0)$ and satisfy

$$sup\{|x_n| + |y_n| : n \geq N\} > 0 \text{ for any integer } N \geq N_0.$$

A real sequence defined on $N(N_0)$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

A solution $X \in W$ is said to be oscillatory if both components are oscillatory and it will be called nonoscillatory otherwise.

Some oscillation results for difference system (1) when $a_n = 0$ for $n \in N(N_0)$ and $\tau(n) = n$ have been presented in Graef et al. (1999), Huo and Li (2001). In particular when $p_n > 0$ for all, $n \in N(n_0)$ the difference system reduces to the second order nonlinear difference equations

$$\Delta\left(\frac{1}{p_n}\Delta(x_n - a_n x_{\sigma(n)})\right) = -q_n f(x_{\tau(n)}). \tag{2}$$

Also if $p_n = 1$ for $n \in N(n_0)$ and if $f(u) = |u|^{\lambda}\, sgnu$, the above equation becomes

$$\Delta\left(\frac{1}{p_n}\Delta(x_n - a_n x_{\sigma(n)})\right) = |x_{\tau(n)}|\, sgnx_{\tau(n)}. \tag{3}$$

The oscillatory and asymptotic behavior of the equations of type (2) and (3) are studied by Hoker and Patula (1983), Stemal et al. (1998), Thandapani (1992), Thandapani et al. (1995), Zang (1993).

2. Some Useful Lemmas

Denote $A(n,s)=\sum_{t=s}^{n-1} p_t$, $n>s\geq n_0$. For any x_n we define z_n by

$$z_n = x_n - a_n x_{\sigma(n)}. \tag{4}$$

We begin with the following lemma

Lemma 1.

Let (C1) - (C3) hold and let $X=(\{x_n\},\{y_n\})\in W$ be a solution of the system (1) with $\{x_n\}$ either eventually positive or eventually negative for all $n\geq N_1\in N(n_0)$. Then $(\{x_n\},\{y_n\})$ is nonoscillatory and $\{z_n\}$, $\{y_n\}$ are monotone for $n\in N(N_1)$.

Proof:

Let $X=(\{x_n\},\{y_n\})\in W$ and let $\{x_n\}$ be eventually positive. Then from the second equation of the system (1) we have $\Delta y_n\leq 0$ for all $n\geq N_1\in N(n_0)$ and Δy_n and y_n are not identically zero for infinitely large values of n. Hence, $\{y_n\}$ is either eventually positive or eventually negative $n\geq N_2$. Then, $(\{x_n\},\{y_n\})$ is nonoscillatory. Further from the first equation of the system (1) we have $\Delta z_n>0$ or $\Delta z_n<0$ eventually. Hence, $\{z_n\}$ is monotone for all $n\geq N\geq N_2$. The proof is similar when $\{x_n\}$ is eventually negative.

Lemma 2.

In addition to conditions (C1) - (C3) assume that $1\leq a_n$ for all $n\in N(n_0)$. Let $\{x_n\}$ be a nonoscillatory solution of the inequality

$$x_n(x_n - a_n x_{\sigma(n)})>0 \tag{5}$$

for n sufficiently large.

1. If $\sigma(n)=n+k$ for $n\in N(n_0)$ where k is a positive integer then $\{x_n\}$ is bounded. Moreover if $1<\lambda\leq a_n$, $n\in N(n_0)$ for some constant λ then $\lim_{n\to\infty} x_n=0$.

2. If $\sigma(n)=n-k$ for, then there exists a positive constant C such that $|x_n|\geq C$ for all large n.

Proof:

Let $\{x_n\}$ be a nonoscillatory solution of the system (1). Without loss of generality we may assume that $\{x_n\}$ be an eventually positive solution of the inequality (5), the proof for the case $\{x_n\}$ eventually negative is similar.

Assume that $1 \le a_n$.

1. Let $x_n > 0$ for all $n \in N(n_0)$. In view of $1 \le a_n$ and $\sigma(n) = n + k$ we have $x_{n+k} < \frac{1}{a_n} x_n \le x_n$ for all large n, which implies $\{x_n\}$ is bounded. If $1 < \lambda \le a_n$, $n \in N(n_0)$ holds for some positive constant λ, then we have $x_{n+k} < \frac{1}{a_n} x_n < \frac{1}{\lambda} x_n$. Then, $x_{n+jk} < \lim_{j\to\infty} \left(\frac{1}{\lambda}\right)^j x_n \to 0$ as $j \to \infty$ which implies that $\lim_{n\to\infty} x_n = 0$.

2. Let $\{x_n\}$ be a nonoscillatory solution of the system (1). Let $x_n > 0$ for all $n \ge n_1 \in N(n_0)$. In view of $1 \le a_n$ and $\sigma(n) = n - k$ we have $x_n - a_n x_{n-k} > 0$. This implies $x_{n+k} > x_n$, which implies that there exists a constant $C > 0$ such that $|x_n| \ge C$ for all large n.

3. Asymptotic Behavior

In this section we present a sufficient condition for the asymptotic behavior of solutions of the system (1).

Theorem 3.

Assume that

$$1 < \lambda \le a_n, \ (\lambda \text{isaconstant}) \tag{6}$$

$$\sigma(n) = n + k \text{ and } \tau(n) = n + l \text{ with } k < l, \tag{7}$$

$$\limsup_{n\to\infty} \left(K \sum_{s=n+k-l}^{n-1} A_s(n, s+1) q_s / a_{s-k+l} \right) > 1, \tag{8}$$

and

$$\sum_{n=n_0}^{\infty} p_n \sum_{s=n}^{\infty} \frac{q_s}{a_{s-k+l}} = \infty. \tag{9}$$

Then, for every nonoscillatory solution $X = (\{x_n\}, \{y_n\}) \in W$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ holds.

Proof:

Let $(\{x_n\},\{y_n\}) \in W$ be a non oscillatory solution of the system (1). Without loss of generality we may suppose that $x_n > 0$ for all $n \geq n_1 \in N(n_0)$. By the second equation of the system (1) and the hypotheses, we have $\Delta y_n \leq 0$ for all $n \geq n_2 \in N(n_1)$.

In view of Lemma 1, we have two cases for sufficiently large $n_3 \in N(n_2)$;
(I) $y_n > 0$, for $n \geq n_3$;
(II) $y_n < 0$, for $n \geq n_3$.

Case (I) We consider two possibilities.

(A): Let $z_n < 0$ for $n \geq n_4$ where $n_4 \in N(n_3)$ is sufficiently large.

We prove $\lim_{n\to\infty} z_n = 0$. Since $\{z_n\}$ is non-decreasing

$$\lim_{n\to\infty} z_n = -L, \ L > 0 \text{ is a constant} \tag{10}$$

and

$$z_n \leq -L \text{ for } n \geq n_4.$$

Since $x_n > 0$, by (4), we have

$$-x_{n+l} < \frac{z_{n-k+l}}{a_{n-k+l}}, \ n \geq n_4,$$

and

$$-L \geq z_{n-k+l} \geq -a_{n-k+l} x_{n+l}, \ n \geq n_4. \tag{11}$$

By the hypotheses and the second equation of (1)

$$\frac{-KLq_n}{a_{n-k+l}} \geq -Kq_n x_{n+l} \geq -q_n f(x_{n+l}) = \Delta y_n \ \text{ for} n \geq n_4. \tag{12}$$

Summing (12) from n to n^* and then taking $n^* \to \infty$ we obtain

$$KL\sum_{s=n}^{\infty} \frac{q_s}{a_{s-k+l}} \leq y_n, \ n \geq n_4.$$

Multiplying the last inequality by p_n and using the first equation of the system (1), we have

$$KLp_n \sum_{s=n}^{\infty} \frac{q_s}{a_{s-k+l}} \le \Delta z_n, \; n \ge n_4.$$

Summing the last inequality from N to $n-1$ and letting $n \to \infty$, we obtain

$$KL \sum_{n=N}^{\infty} p_n \sum_{s=n}^{\infty} \frac{q_s}{a_{s-k+l}} \le -L - z_N < -z_N.$$

This contradicts (9) consequently $\lim_{n\to\infty} z_n = 0$.

Since $\{z_n\}$ is bounded, there is a constant $B > 0$ such that $z_n \ge -B$ for $n \ge n_4$ and by (4), one has

$$x_n = a_n x_{n+k} + z_n \ge a_n x_{n+k} - B \ge \lambda x_{n+k} - B \;\text{ for } n \ge n_4. \tag{13}$$

We claim $\{x_n\}$ is bounded. Let $\{x_n\}$ be unbounded then $\{x_{n+k}\}$ is unbounded and there is a sequence $\{n_j\}$ such that $n_j \to \infty$ as $j \to \infty$ and $\lim_{j\to\infty} x_{n_j+k} = \infty$ and $x_{n_j+k} = \max_{n_4 \le s \le n_j+k} x_s$.

By (13),

$$\lambda x_{n_j+k} \le x_{n_j+k} + B,$$

$$x_{n_j+k} \le \frac{B}{\lambda - 1}, \quad j = 1, 2, \ldots.$$

This is a contradiction to $\lim_{j\to\infty} x_{n_j+k} = \infty$, and, hence, $\{x_n\}$ is bounded.

Next we claim that $\lim_{n\to\infty} x_n = 0$.

Let $\lim_{n\to\infty} x_{n+k} = c > 0$. Then, $\lim_{n\to\infty} \sup x_n = c$. Let $\{n_j\}$, $j = 1, 2, \ldots$ be a subsequence such that $\lim_{j\to\infty} n_j = \infty$ and $\lim_{j\to\infty} \sup x_{n_j+k} = c$. Then, $\lim_{j\to\infty} x_{n_j} \le c$. By (4), we have

$$z_{n_j} \le x_{n_j} - \lambda x_{n_j+k}, \;\; j\, 1, 2, \ldots \;\text{ and }\; x_{n_j+k} \le \frac{x_{n_j} - z_{n_j}}{\lambda}, \;\; j = 1, 2, \ldots.$$

By the last inequality, we have

$$c = \limsup_{j\to\infty} x_{n_j+k} \le \frac{\limsup_{j\to\infty} x_{n_j}}{\lambda} \le \frac{c}{\lambda},$$

which holds when $\lambda \le 1$, a contradiction to (6). This means that $\lim_{n\to\infty} x_{n+k} = 0$ and also $\limsup_{n\to\infty} x_n = 0$. Further, $x_n > 0$ holds for $n \in N(n_0)$, so $\lim_{n\to\infty}\inf x_n = 0$ and this leads to $\lim_{n\to\infty} x_n = 0$.

Next we prove $\lim_{n\to\infty} y_n = 0$. Let $\lim_{n\to\infty} y_n = D > 0$. Then,

$$y_n \ge D, \text{ for} n \ge n_3. \tag{14}$$

Summing the first equation of system (1) from n_3 to $n-1$ and using (4) and (14), we have

$$z_n - z_{n_3} \ge D \sum_{s=n_3}^{n-1} p_s. \tag{15}$$

By (15) and the hypothesis, $\lim_{n\to\infty} z_n = \infty$, which contradicts the fact $z_n < 0$ for $n \ge n_4$. Hence, $\lim_{n\to\infty} y_n = 0$.

(B): Let $z_n > 0$ for $n \ge n_4$ where $n_4 \in N(n_3)$ is sufficiently large. By Lemma 2, $\lim_{n\to\infty} x_n = 0$. holds. We prove similarly as in the above proof that $\lim_{n\to\infty} y_n = 0$.

The relation (15) implies $\lim_{n\to\infty} z_n = \infty$. Therefore, by (4) we have $z_n < x_n$ for $n \ge n_3$ and that contradicts $\lim_{n\to\infty} x_n = 0$. Hence, $\lim_{n\to\infty} y_n = 0$.

Case (II)

In this case ,

$$y_n \le -L \text{ for } n \ge n_3, L \text{ is a positive constant.} \tag{16}$$

Summing the first equation of the system (1) from n_3 to $n-1$, and using the inequality (16), we have

$$z_n - z_{n_3} \le -L \sum_{s=n_3}^{n-1} p_s \quad n \ge n_3. \tag{17}$$

By (17) and the hypothesis, it follows that $\lim_{n\to\infty} z_n = -\infty$ and $z_n < 0$ for $n \ge n_4$, where $n_4 \in N(n_3)$ is sufficiently large. By (4) we have $z_n > -a_n x_{n+k}$, $n \ge n_4$. Then,

$$x_{n+k} > \frac{z_{n-k+l}}{a_{n-k+l}}, \ n \geq n_4. \tag{18}$$

Since $|f(u)| \geq k|u|$, for $u = x_{n+k}$, we have

$$x_{n+l} \leq \frac{f(x_{n+l})}{K}, \ n \geq n_4. \tag{19}$$

Multiply (19) by Kq_n and using (18) we have

$$-Kq_n \frac{z_{n-k+l}}{a_{n-k+l}} \leq Kq_n x_{n+l} \leq q_n f(x_{n+l}), \ n \geq n_4. \tag{20}$$

Using the second equation of the system (1) and (20), we have

$$\sum_{t=s}^{n-1} A(n,t+1)\Delta y_t = -\sum_{t=s}^{n-1} A(n,t+1) q_t f(x_{t+l}) \leq K \sum_{t=s}^{n-1} A(n;t+1) q_t \frac{z_{t-k+l}}{a_{t-k+l}} \text{ , for } n > s \geq n_4. \tag{21}$$

Using summation by parts formula we have

$$\sum_{t=s}^{n-1} A(n,t+1)\Delta y_t = -z_n + z_s - A(n,s). \tag{22}$$

Combining (21) and (22), we have

$$\sum_{t=s}^{n-1} A(n,t+1)\Delta y_t = z_s - z_n - A(n,s) y_s \leq K \sum_{t=s}^{n-1} A(n,t+1) q_t \frac{z_{t-k+l}}{a_{t-k+l}} \ \ n > s \geq n_4. \tag{23}$$

Since $z_n < 0$, $y_n < 0$, and $A(n,t) \geq 0$, $n > s \geq n_4$, we have

$$z_s \leq K \sum_{t=s}^{n-1} A(n,t+1) q_t \frac{z_{t-k+l}}{a_{t-k+l}}, \ n > s \geq n_4. \tag{24}$$

Let $s = n + k - l$ and using the fact that $z_{n-k+l} < 0$ and non-increasing, by (24), we obtain

$$1 \geq K \sum_{t=n+k-l}^{n-1} A(n,t+1) q_t \frac{q_t}{a_{t-k+l}}, \ n \geq n_5.$$

This contradicts (8). The second part of the proof for the case $x_n < 0$ eventually is similar to the previous one and hence the details are omitted. Hence, the proof is complete.

Example 4.

Consider the difference system

$$\Delta(x_n - 3x_{n+1}) = 2(n+1)y_n$$

$$\Delta y_n = -\frac{3}{n}x_{n+3} \quad n \geq 1. \tag{25}$$

Here $a_n = 3$, $\sigma(n) = n+1$, $p_n = 2(n+1)$, $q_n = \frac{3}{n}$, $\tau(n) = n+3$, $f(u) = u$, $K = 1$, $A(n,s) = n(n+1) - s(s+1)$. Then all the conditions of Theorem 3 are satisfied. Hence, every nonoscillatory solution $(\{x_n\},\{y_n\})$ of (25) satisfies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

4. Conclusion

The sufficient condition for the asymptotic behavior of the two dimensional neutral difference system have been discussed. The example considered in this work supports the results of the theorem.

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