Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 5, Issue 1 (June 2010) pp. 198 – 216
(Previously, Vol. 5, No. 1)

Applications and Applied Mathematics:
An International Journal
(AAM)

Convergence of the Sinc method applied to Volterra integral equations

M. Zarebnia*
Department of Mathematics
University of Mohaghegh Ardabili
56199-11367, Ardabil, Iran
zarebnia@uma.ac.ir

J. Rashidinia
School of Mathematics
Iran University of Science & Technology
Narmak, Tehran 16844, Iran
Rashidinia@iust.ac.ir

Received: July 25, 2009; Accepted: April 16, 2010

*Corresponding author

Abstract

A collocation procedure is developed for the linear and nonlinear Volterra integral equations, using the globally defined Sinc and auxiliary basis functions. We analytically show the exponential convergence of the Sinc collocation method for approximate solution of Volterra integral equations. Numerical examples are included to confirm applicability and justify rapid convergence of our method.

Keywords: Volterra, Linear, Nonlinear, Collocation, Sinc function, Exponential Convergence

MSC 2000 No.: 65R20, 45D05, 45A05, 45G10, 41A25

1. Introduction

This paper describes a collocation procedure for solving Volterra integral equations of the form

$$u(x) = f(x) + \lambda \int_a^x K(x,t) g(t,u(t)) dt, \quad x \in \Gamma = [a,b]. \tag{1}$$

The known kernel $K(x,t)$ is continuous, the functions f and g are given. In special case, if in (1) $g(t,u(t)) = u(t)$ the integral equation (1) reduce to linear Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^x K(x,t) u(t) dt, \quad x \in \Gamma = [a,b], \tag{2}$$

which has been studied in our earlier paper Rashidinia et al. (2007b). Also, if in (1), $g(t,u(t))$ be nonlinear in $u(t)$, then equation (1) is a nonlinear Volterra - Hammerstein integral equation. We assume that (1) has a unique solution u to be determined. Several numerical methods for approximating the solution of the Volterra integral equations are existed in the literature. The numerical methods for linear integral equations of the second kind studied in Delves et al. (1995). These methods transform the integral equation to a linear or nonlinear system of algebraic equations that can be solved by direct or iterative methods.

Reihani et al. (2007) applied rationalized Haar functions method for solving Fredholm and Volterra integral equations. In Blyth et al. (2002), an effective method to solve linear Volterra integral equations was introduced. Kumar et al. (1987) introduced a new collocation-type method for the solution of Fredholm- Hammerstein integral equations. Later on, Brunner (1992) applied this method to nonlinear Volterra integral and integro-differential equations and discussed its connection with the iterated collocation method. Moreover, Guoqiang (1993) studied the asymptotic error expansion of the method given in Kumar et al. (1987) for nonlinear Volterra integral equations at mesh points. The methods given in Kumar et al. (1987), transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative method. But Kumar et al. (1987), considered the solution of the definite integrals that may be evaluated analytically only in favorable cases, while in Guoqiang (1993) the solution of the integrals have to be evaluated at each time step of the iteration.

Chebyshev spectral method for the numerical solution of Equation (1) has been proposed by Elnagar et al. (1996). Sinc methods have increasingly been recognized as powerful tools for attacking problems in applied physics and engineering. The excellent overviews of methods based on Sinc functions for solving ordinary and partial differential equations are given in Lund et al. (1992), Stenger (1993). The Sinc-collocation procedures for the eigenvalue problems are presented in Eggert et al. (1987), Lund et al. (1984). The Sinc collocation method for the initial value problems using the globally defined Sinc basis functions was proposed by Carlson et al. (1997). The Sinc-Galerkin scheme has been developed to approximate solution for the

Korteweg-de Vries model equation in Al-Khaled (2001). The sinc-Galerkin method has been used to approximate solution of nonlinear fourth order boundary value problems with homogeneous and nonhomogeneous boundary conditions in El-Gamel et al. (2003). In Weber et al. (2004), an algorithm based on the Sinc function for the generation of adaptive radial grids used in density functional theory or quantum chemical calculations has been described. Their approach is general and can be applied for the integration over Slater or Gaussian type functions with only minor modifications and the relative error of the integration is fully controlled by the algorithm within a specified range of exponential parameters and for a given principal quantum number.

A block matrix formulation has been presented for the Sinc-Galerkin technique applied to the wind-driven current problem from oceanography in Koonprasert et al. (2004). In Rashidinia et al. (2005), we used a Sinc-collocation procedure for numerical solution of linear Fredholm integral equations of the second kind and also in Rashidinia et al. (2007a) we applied Sinc method for solving system of linear Fredholm integral equations. The paper is organized into five sections. Section 2 outlines some of the main properties of Sinc function and Sinc method that are necessary for the formulation of the discrete system. In sections 3 and 4, we illustrate how the Sinc collocation method may be used to replace (1) by an explicit system of linear or nonlinear algebraic equations. Also, in sections 3 and 4 the convergence analysis of the method has been discussed for linear and nonlinear Volterra integral equations generally. Finally we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples in section 5.

2. Sinc Basis Functions for the Collocation Method

In this section, we will review Sinc function properties. These are discussed thoroughly in Stenger (1993) and Lund et al. (1992). The Sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$Sinc(x) = \begin{cases} \dfrac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{3}$$

For any $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(j,h)(x) = Sinc\left(\frac{x - jh}{h}\right), \qquad j = 0, \pm 1, \pm 2, \cdots. \tag{4}$$

The Sinc function for the interpolating points $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{5}$$

Let

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} dt. \tag{6}$$

If u is defined on the real line, then for $h > 0$ the series

$$C(u,h)(x) = \sum_{j=-\infty}^{\infty} u(jh) Sinc(\frac{x - jh}{h}) \tag{7}$$

is called the Whittaker cardinal expansion of u, whenever this series converges. They are based in the infinite strip D_s in the complex plane

$$D_s = \{ w = u + iv : |v| < d \le \pi / 2 \}. \tag{8}$$

To construct approximation on the interval $[a,b]$, we consider the conformal map

$$\phi(z) = \ln(\frac{z-a}{b-z}). \tag{9}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z-a}{b-z} \right) \right| < d \le \frac{\pi}{2} \right\}. \tag{10}$$

The basis functions for $z \in D_E$ are derived from the composite translated Sinc functions,

$$S(j,h) \circ \phi(x) = Sinc\left(\frac{\phi(x) - jh}{h} \right). \tag{11}$$

The function

$$z = \phi^{-1}(w) = \frac{a + be^w}{1 + e^w} \tag{12}$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{ \psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty \}. \tag{13}$$

The Sinc grid points $z_k \in (a,b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \qquad k = 0, \pm 1, \pm, 2\cdots. \tag{14}$$

Definition 1:

Let $L_\alpha(D_E)$ be the set of all analytic functions, such that

$$|u(z)| \le C\frac{|\rho(z)|^\alpha}{\left[1+|\rho(z)|\right]^{2\alpha}} \quad \forall z \in D_E, \qquad 0 < \alpha \le 1, \tag{15}$$

where $\rho(z) = e^{\phi(z)}$ and C is a constant.

Theorem 1:

Let $u \in L_\alpha(D_E)$, let N be a positive integer, and let h be selected by the formula

$$h = \left(\pi d / \alpha N\right)^{\frac{1}{2}}. \tag{16}$$

Then, there exists positive constant C_1, independent of N, such that

$$\sup_{x\in\Gamma} | u\left(z\right) - \sum_{j=-N}^{N} u(z_j)S(j,h)o\phi(z) | \le C_1 e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \tag{17}$$

Theorem 2:

Let $(u/\phi') \in L_\alpha(D_E)$, $\delta_{kj}^{(-1)}$ be defined as (6), and $h = \left(\pi d / \alpha N\right)^{\frac{1}{2}}$. Then, there exists a constant, C_2, which is independent of N, such that

$$| \int_a^{z_k} u(t)dt - h\sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{u(z_j)}{\phi'(z_j)} | \le C_2 e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \tag{18}$$

3. Linear Volterra Integral Equation of the Second Kind

3.1.

If in Equation (1), $g(t,u)=u(t)$, then the problem reduce to develop the approximate solution of linear Volterra integral equation of the second kind

$$u(x)=f(x)+\lambda\int_a^x K(x,t)u(t)dt, \quad x\in\Gamma=[a,b], \tag{19}$$

where the known kernel $K(x,t)$ is continuous, the function $f(x)$ and the parameter λ are given, and $u(x)$ is the solution to be determined Delves et al. (1985). In earlier paper Rashidinia et al. (2007b), we studied this type of linear Volterra integral equation. However, here we developed our method in more general case which is necessary for convergence analysis of equation (1).

We assume $u(x)$ be the exact solution of the integral equation (19) and let $u(x)\in L_\alpha(D_E)$. We approximate the solution of (19) by the following linear combinations of the Sinc functions and auxiliary functions:

$$u(x)=\sum_{j=-N}^{N} u(x_j)\gamma_j(x), \qquad x\in[a,b], \tag{20}$$

where

$$\gamma_i(x)=\begin{cases}\omega_a(x), & j=-N,\\ S(j,h)o\phi(x), & j=-N+1,\cdots,N-1,\\ \omega_b(x), & j=N.\end{cases} \tag{21}$$

In the above relation, auxiliary basis functions $\omega_a(x)$ and $\omega_b(x)$ are defined by

$$\omega_a(x)=\frac{1}{1+\rho(x)}, \qquad \omega_b(x)=\frac{\rho(x)}{1+\rho(x)}, \tag{22}$$

and satisfied the following conditions:

$$\lim_{x\to\Gamma_a}\omega_a(x)=1, \qquad \lim_{x\to\Gamma_b}\omega_a(x)=0, \qquad \lim_{x\to\Gamma_a}\omega_b(x)=0, \qquad \lim_{x\to\Gamma_b}\omega_b(x)=1. \tag{23}$$

Lemma 1:

$u(x) \in L_\alpha(D_E)$, let N be a positive integer and $h = \left(\pi d / \alpha N\right)^{\frac{1}{2}}$. Then

$$\sup_{x\in\Gamma} | u(x) - \sum_{j=-N}^{N} u(x_j)\gamma_j(x) | \le C_3 e^{-(\pi d \alpha N)^{\frac{1}{2}}}, \tag{24}$$

where $\gamma_j(x)$ is defined in (21) and C_3 is a positive constant, independent of N.

Proof:

By theorem 1, we have:

$$\sup_{x\in\Gamma} | u(x) - \sum_{j=-N}^{N} u(x_j)\gamma_j(x) | \le S_0 + S_1 + S_2, \tag{25}$$

where

$$S_0 = \sup_{x\in\Gamma} | u(x) - \sum_{j=-N+1}^{N-1} u(x_j) S(j,h) o\phi(x) |, \quad S_1 = \sup_{x\in\Gamma} | u(x_{-N})\omega_a(x) |, \quad S_2 = \sup_{x\in\Gamma} | u(x_N)\omega_b(x) |. \tag{26}$$

By assumption $u(x) \in L_\alpha(D_E)$ and Theorem 1, and also by using the relation (15) we obtain

$$\begin{aligned} S_0 &\le \sup_{x\in\Gamma} | u(x) - \sum_{j=-N}^{N} u(x_j) S(j,h) o\phi(x) | + \sup_{x\in\Gamma} | u(x_{-N}) S(-N,h) o\phi\left(x\right) | \\ &+ \sup_{x\in\Gamma} | u(x_N) S(N,h) o\phi\left(x\right) | \le C_4 e^{-(\pi d \alpha N)^{\frac{1}{2}}} + C_5 e^{-\alpha N h} + C_6 e^{-\alpha N h}, \end{aligned} \tag{27}$$

where C_4, C_5 and C_6 are constants. Similarly, by considering the relations (15) and (22), we obtain the following bounds for S_1 and S_2.

$$\mathrm{S}_1 \le C_7 e^{-\alpha N h}, \qquad \mathrm{S}_2 \le C_8 e^{-\alpha N h}. \tag{28}$$

By using the relations (27), (28), and taking h as in (16), we conclude that the relation (24) is hold. This completes the proof.

Lemma 2:

For $u(x)$ defined in (20), let $(K/\phi')\gamma_j(x) \in L_\alpha(D_E)$, and let h be selected from (16) then

$$\int_a^{x_k} K(x_k,t)u(t)dt = hu(t_{-N})\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j) + h\sum_{j=-N+1}^{N-1}\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}u(t_j) \\ + hu(t_N)\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j) + O(\exp(-(\pi d\alpha N)^{\frac{1}{2}})). \tag{29}$$

Proof:

Applying Theorem 2 and $(K/\phi')\gamma_j(x) \in L_\alpha(D_E)$, we have

$$\int_a^{x_k} K(x_k,t)u(t)dt = h\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K(x_k,t_l)}{\phi'(t_l)}u(t_l) + O(\exp(-(\pi d\alpha N)^{\frac{1}{2}})). \tag{30}$$

By using (20), we get the collocation result

$$\int_a^{x_k} K(x_k,t)u(t)dt = hu(t_{-N})\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K(x_k,t_l)}{\phi'(t_l)}\omega_a(t_l) \\ + h\sum_{j=-N+1}^{N-1}u(t_j)\{\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K(x_k,t_l)}{\phi'(t_l)}S(j,h)o\phi(t_l)\} \\ + hu(t_N)\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K(x_k,t_l)}{\phi'(t_l)}\omega_b(t_l) + O(\exp(-(\pi d\alpha N)^{\frac{1}{2}})). \tag{31}$$

By using Sinc function properties and setting $l = j$, we obtain the relation (29), which completes the proof.

Now, let $u(x)$ be the exact solution of (19) that is approximated by following expansion

$$u_N(x) = \sum_{j=-N}^{N} u_j\gamma_j(x), \quad x \in \Gamma = [a,b]. \tag{32}$$

Upon replacing $u(x)$ in the Volterra integral equation (19) by $u_N(x)$, applying Lemma 1 and Lemma 2, setting Sinc collocation points x_k and then considering $\delta_{kj}^{(0)} = \delta_{jk}^{(0)}$, we obtain the following system

$$u_{-N}\{\omega_a(x_k)-\lambda h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}\omega_a(t_j)\}+\sum_{j=-N+1}^{N-1}\{S(j,h)o\phi(x_k)-\lambda h\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}\}u_j$$
$$+u_N\{\omega_b(x_k)-\lambda h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}\omega_b(t_j)\}=f(x_k),\qquad k=-N,\cdots,N. \tag{33}$$

We write the above system of equations in the matrix form as:

$$AU=[B_{n\times1}\,|\,C_{n\times(n-2)}\,|\,D_{n\times1}]U=P,\qquad n=2N+1, \tag{34}$$

where

$$B=[\omega_a(x_{-N})-\lambda h\sum_{j=-N}^{N}\delta_{-Nj}^{(-1)}\frac{K(x_{-N},t_j)}{\phi'(t_j)}\omega_a(t_j),\cdots,\omega_a(x_N)-\lambda h\sum_{j=-N}^{N}\delta_{Nj}^{(-1)}\frac{K(x_N,t_j)}{\phi'(t_j)}\omega_a(t_j)]^T,$$

$$C=[\delta_{kj}^{(\circ)}-\lambda h\delta_{kj}^{(-1)}\frac{K(x_k,t_j)}{\phi'(t_j)}],\quad k=-N,\cdots,N,\quad j=-N+1,\cdots,N-1,$$

$$D=[\omega_b(x_{-N})-\lambda h\sum_{j=-N}^{N}\delta_{-Nj}^{(-1)}\frac{K(x_{-N},t_j)}{\phi'(t_j)}\omega_b(t_j),\cdots,\omega_b(x_N)-\lambda h\sum_{j=-N}^{N}\delta_{Nj}^{(-1)}\frac{K(x_N,t_j)}{\phi'(t_j)}\omega_b(t_j)]^T,$$

$$P=[f(x_{-N}),f(x_{-N+1}),\cdots,f(x_{N-1}),f(x_N)]^T,\quad U=[u_{-N},u_{-N+1},\cdots,u_{N-1},u_N]^T.$$

By solving the above system, we can the unknown vector U. Then, by using such solution we can obtain the approximate solution U_N as

$$U_N=T_u.U,\qquad T_u=\begin{pmatrix}\omega_a(x_{-N}) & 0 & \dots & 0 & \omega_b(x_{-N})\\ \omega_a(x_{-N+1}) & 1 & \dots & 0 & \omega_b(x_{-N+1})\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \omega_a(x_{N-1}) & 0 & \cdots & 1 & \omega_b(x_{N-1})\\ \omega_a(x_N) & 0 & \cdots & 0 & \omega_b(x_{N-1})\end{pmatrix}. \tag{35}$$

3.2. Convergence Analysis

We discuss the convergence of the method for the Volterra integral equation (19).

Lemma 3:

Let $u(x)$ be the exact solution of the integral equation (19) and let $h=\left(\pi d/\alpha N\right)^{\frac{1}{2}}$, and $(K/\phi')\gamma_j(x)\in L_\alpha(D_E)$ then there exists a constant C_9 independent of N such that

$$\| A\tilde{U} - P \|_2 \le C_9 N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}, \tag{36}$$

where $\tilde{U} = (u(x_{-N}), u(x_{-N+1}), \cdots, u(x_{N-1}), u(x_N))^T$, and the components $u(x_j)$ for $j = -N(1)N$ are the values of the exact solution of the integral equation (19) at the Sinc points x_j.

Proof:

Using the Lemma 1 and 2 we have:

$$| v_K | = | (A\tilde{U} - P)_k | = | ([B_{n\times 1} \mid C_{n\times(n-2)} \mid D_{n\times 1}]\tilde{U} - P)_k | \le | u(x_k) - \sum_{j=-N}^{N} u(x_j)\gamma_j(x_k) |$$

$$+ | \lambda \int_a^{x_k} K(x_k, t)u(t)dt - (\lambda h u(x_{-N}) \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{K(x_k, t_j)}{\phi'(t_j)} w_a(t_j) + \lambda h \sum_{j=-N+1}^{N-1} \delta_{kj}^{(-1)} \frac{K(x_k, t_j)}{\phi'(t_j)} u(t_j)$$

$$+ \lambda h u(x_N) \sum_{j=-N}^{N} \frac{K(x_k, t_j)}{\phi'(t_j)} w_b(t_i)) | \le C_{10} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\} + | \lambda | C_{11} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}.$$

By setting $C_{12} = \max\{C_{10}, | \lambda | C_{11}\}$, we get

$$| v_k | \le C_{12} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}. \tag{37}$$

Finally, by using Euclidean norm, we have

$$\| A\tilde{U} - P \|_2 = (\sum_{k=-N}^{N} | v_K |^2)^{\frac{1}{2}} \le C_9 N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}.$$

Now, we show that the collocation method converges at the rate of $O(e^{-k\sqrt{N}})$, where $k > 0$.

Theorem 3:

Let us consider all assumptions of lemma 1 and let $u_N(x)$ be the approximate solution of integral equation (19) given by (33) then there exists a constant C_{13}, independent of N, such that

$$\sup_{x\in\Gamma} | u(x) - u_N(x) | \le C_{13} N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}. \tag{38}$$

Proof:

Let the analytic solution of equation (19) at the Sinc points x_j be denoted by β_N and defined by

$$\beta_N(x) = \sum_{j=-N}^{N} u(x_j)\gamma_j(x), \tag{39}$$

where $\gamma_j(x)$ is defined by (21). By triangle inequality, we get

$$|u(x) - u_N(x)| \le |u(x) - \beta_N(x)| + |\beta_N(x) - u_N(x)|. \tag{40}$$

By using Lemma 1 and assumption $u(x) \in L_\alpha(D_E)$, we obtain

$$\sup_{x\in\Gamma} |u(x) - \beta_n(x)| \le C_{14} N^{\frac{1}{2}} \exp\{-(\pi d\alpha N)^{\frac{1}{2}}\}, \tag{41}$$

where C_{14} is a constant independent of N. For the second term on the right-hand side of (40), we have

$$|\beta_N(x) - u_N(x)| = |\sum_{j=-N}^{N} \{u(x_j) - u_j\}\gamma_j(x)| \le \sum_{j=-N}^{N} |u(x_j) - u_j||\gamma_j(x)| = E.$$

We know that

$$\left(\sum_{j=-N}^{N} |\gamma_j(x)|^2\right)^{\frac{1}{2}} \le C_{15}, \tag{42}$$

where C_{15} is independent of N. By using Schwarz inequality, we obtain

$$E \le (\sum_{j=-N}^{N} |u(x_j) - u_j|^2)^{\frac{1}{2}} (\sum_{j=-N}^{N} |\gamma_j|^2)^{\frac{1}{2}} \le C_{15} \|\tilde{U} - U\|_2. \tag{43}$$

Following Lemma 3 and using (34) and (43), we have

$$\|\tilde{U} - U\|_2 = \|A^{-1}(A\tilde{U} - AU)\|_2 \le \|A^{-1}\|_2 \|A\tilde{U} - P\|_2 \le \|A^{-1}\|_2\, C_9 N^{\frac{1}{2}} \exp\{-(\pi d\alpha N)^{\frac{1}{2}}\}. \tag{44}$$

Therefore, we get

$$|\beta_N(x)-u_N(x)|\le C_9 C_{15}\,\|A^{-1}\|_2\, N^{\frac{1}{2}}\exp\{-(\pi d\alpha N)^{\frac{1}{2}}\}. \tag{45}$$

Now, from (41), (45) and assumption $C_{13}=\max\{C_{14}, C_9C_{15}\|A^{-1}\|_2\}$, we conclude that the relation (38) is hold. This will complete the proof.

4. Nonlinear Volterra integral Equation

4.1.

In this section, we consider the nonlinear Volterra-Hammerstein integral equation of the form

$$u(x)=f(x)+(KGu(t))(x),\quad x\in\Gamma=[a,b],\qquad (KGu(t))(x)=\int_a^x K(x,t)g(t,u(t))dt, \tag{46}$$

In Equation (46), f, g and the kernel K are continuous functions, and $g(t,u)$ is nonlinear in u. Now, in this case the problem is to approximate the solution of nonlinear Volterra-Hammerstein integral equation of the second kind. We assume that $u(x)$ be the exact solution of (46) which is approximated by (20). Let $u(x)\in L_\alpha(D_E)$, and furthermore let $(K/\phi')u\in L_\alpha(D_E)$. By replacing $u_N(x)$ in the Equation (46), setting $x=x_k$, $k=-N,\dots,N$, then by applying the Theorem 2, we get the following collocation solution

$$u_{-N}\omega_a(x_k)+\sum_{j=-N+1}^{N-1}u_jS(j,h)o\phi(x_k)+u_N\omega_b(x_k)-h\sum_{j=-N}^{N}\delta_{kj}^{(-1)}\frac{(KGu_N(t_j))(x_k)}{\phi'(t_j)}=f(x_k). \tag{47}$$

We then rewrite these equations in matrix form which are the nonlinear system

$$WU-h(I^{(-1)}D(\frac{1}{\phi'}))\circ K=\Phi. \tag{48}$$

The notation "$\circ$" denotes the Hadamard matrix multiplication. $I^{(-1)}=[\delta_{kj}^{(-1)}]$, $K=[(KGu_N(t_j))(x_k)]$ and $D(1/\phi')=diag(1/\phi'(x_{-N}),\dots,1/\phi'(x_N))$,

$$W=\begin{pmatrix}\omega_a(x_{-N}) & 0 & \dots & 0 & \omega_b(x_{-N})\\ \omega_a(x_{-N+1}) & 1 & \dots & 0 & \omega_b(x_{-N+1})\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \omega_a(x_{N-1}) & 0 & \cdots & 1 & \omega_b(x_{N-1})\\ \omega_a(x_N) & 0 & \cdots & 0 & \omega_b(x_{N-1})\end{pmatrix},\quad U=[u_{-N},\cdots,u_N]^T,\quad \Phi=\left[f_{-N},\cdots,f_N\right]^T. \tag{49}$$

The above nonlinear system consists of $2N+1$ equations of the $2N+1$ unknown, namely, $\{u_j\}_{j=-N}^{N}$. Solving this nonlinear system by Newton's method, we can obtain an approximate solution of (46).

4.2. Convergence Analysis

In this section, we consider the convergence analysis of nonlinear Volterra- Hammerstein integral equation.

Lemma 4:

Let u be the exact solution of the nonlinear integral equation

$$u=f+KG(u), \tag{50}$$

where the nonlinear operator K is defined by (46). Let $u\in L_\alpha(D_E)$, $G'(u)=\partial G/\partial u$, and $G''(u)=\partial^2G/\partial u^2$ be well defined and bounded on the ball $B(u^0,r)$. Also, let $(I-KG'(u))^{-1}$ and $(I-KG'(u^0))^{-1}(KG(u^0)+f-u^0)$ be bounded on $B(u^0,r)$. Assume that

$$\left|KG(u^0)+f-u^0\right|\le H_0,\qquad \sup_{u\in B(u^0,r)}\left|(I-KG'(u))^{-1}\right|\le H_1,\qquad \sup_{u\in B(u^0,r)}\left|KG''(u)\right|\le H_2. \tag{51}$$

If

$$\tilde{h}=H_1^2H_2H_0<2 \tag{52}$$

and

$$r>H_0H_1\sum_{k=0}^{\infty}(\tilde{h}/2)^{2^k-1}, \tag{53}$$

then the sequence

$$u^{n+1} = u^n + (I - KG'(u^n))^{-1}(KG(u^n) + f - u^n) \tag{54}$$

is well defined. Furthermore $u^{n+1} \in L_\alpha(D_E)$ for every positive integer n and the sequence u^n converges to $u*$. Furthermore,

$$\left|u^n - u*\right|_\infty \le H_1 H_0 \frac{\left(\tilde{h}/2\right)^{2^n-1}}{1-\left(\tilde{h}/2\right)^{2^n}} \quad . \tag{55}$$

Proof:

By applying Kantorovich's theorem in Kantorovich et al. (1964), we can conclude the existence of the sequence $\{u^n\}_{n\ge 0}$ and the relation (55) is held.

From Lemma 4, we can show that the Sinc-collocation method converges at rate of $O(e^{-k\sqrt{N}})$, $k > 0$.

Theorem 5:

Let us consider all assumptions in Lemma 4 and let, the discrete equivalent of $KG'(u)$ and $KG''(u)$ be well defined and bounded on the ball $\overline{B}(u^0, r)$, and also the discrete equivalent of $(I - KG'(u))^{-1}$ be bounded on $\overline{B}(u^0, r)$. Let the sequence v_N^n be the discrete equivalent of (54). Then,

(a) $\{v_N^n\}_{n\ge 0}$ converges to v_N^* and the $v_N^n - v_N^*$ has a bound as define in (55).

(b) There exists a constant C_{16} independent of N such that

$$\sup_{x\in\Gamma} | v_N^* - u^* | \le C_{16} N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}. \tag{56}$$

Proof:

(a): Let $\{v_N^n\}_{n\ge 0}$ be the discrete sequence by the Sinc-collocation method that defined by the discrete equivalent of (54). Similarly, by using Lemma 4, the sequence $\{v_N^n\}_{n\ge 0}$ exists and converges to v_N^* and moreover we have

$$| v_N^n - v_N^* |_\infty \leq H_1 H_0 \frac{\left(\tilde{h}/2\right)^{2^n-1}}{1-\left(\tilde{h}/2\right)^{2^n}}, \tag{57}$$

where H_0, H_1 and $\tilde{h}$ are defined in (51) and (52).

(b): Let the sequence u^n defined by (54). By using Lemma 4, we know that the sequence u^n exists and converges to u^* and also

$$| u^n - u^* |_\infty \leq H_1 H_0 \frac{\left(\tilde{h}/2\right)^{2^n-1}}{1-\left(\tilde{h}/2\right)^{2^n}}. \tag{58}$$

By considering bounded inverse of $(I - KG'(u))^{-1}$ on the ball $\overline{B}(u^0, r)$ and Theorem 3, we have

$$\sup_{x\in\Gamma} | v_N^n - u^n | \leq C_{17} N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}. \tag{59}$$

Now, we consider the following inequality

$$| v_N^* - u^* | \leq | v_N^* - v_N^n | + | v_N^n - u^n | + | u^n - u^* |. \tag{60}$$

By considering assumptions in Lemma 4, for n large enough, the given bounds (57) and (58) can be made as small as you possible,

$$H_1 H_0 \frac{\left(\tilde{h}/2\right)^{2^n-1}}{1-\left(\tilde{h}/2\right)^{2^n}} \leq C_{18} N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}. \tag{61}$$

Having applied the relations (57)-(59) and (61), and also by having considered the maximum norm bounds for $v_N^* - v_N^n$, $v_N^n - u^n$ and $u^n - u^*$, we obtain

$$\sup_{x\in\Gamma} | v_N^* - u^* | \leq C_{16} N^{\frac{1}{2}} \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}. \tag{62}$$

5. Numerical Examples

In order to illustrate the performance of the Sinc method in solving linear and nonlinear Volterra integral equations and justify the accuracy and efficiency of the presented method, we consider the following examples. The examples have been solved by presented method with different values of N and α, $0 \le \alpha \le 1$. The errors are reported on the set of Sinc grid points

$$S = \{x_{-N}, ..., x_0, ..., x_{-N}\}, \quad x_k = \frac{a + be^{kh}}{1 + e^{kh}}, \; k = -N, ..., N. \tag{63}$$

The maximum error on the Sinc grid points is

$$\| E_S(h) \|_\infty = \max_{-N \le j \le N} | u(x_j) - u_N(x_j) |. \tag{64}$$

We stopped the numbers of iteration in the Newton method when we achieved the accuracy $\varepsilon = 10^{-4}$. Examples were given for different values of N, $\alpha = 1$, $d = \pi / 2$ and $h = \pi(1/2N)^{1/2}$.

Example 1:

We consider the integral equation

$$u(x) = 2x + 1 - e^{-x^2} - \int_{\circ}^{x} e^{t^2 - x^2} u(t)dt, \qquad 0 \le x \le 1,$$

with exact solution $u(x) = 2x$. The maximum of absolute error on the Sinc grid S is tabulated in Table 1. The graph of the exact and approximate solutions is shown in Figure 1.

Table 1. Results for Example 1

N	h	$\|E_S(h)\|_\infty$
5	0.993459	6.40071×10^{-4}
10	0.702481	5.69465×10^{-5}
20	0.496729	2.63171×10^{-6}
30	0.405578	2.45218×10^{-7}
40	0.351241	3.34306×10^{-8}
50	0.314159	5.82622×10^{-9}

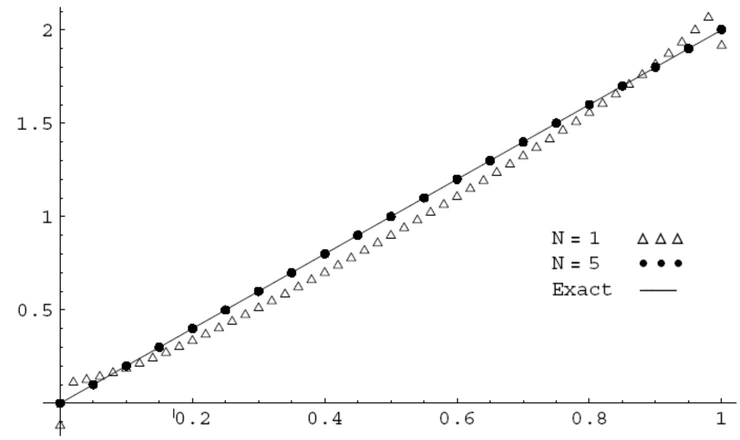

Figure 1. Exact and approximate solutions for Example 1, ($N = 1, 5$)

Example 2:

Consider the following nonlinear Volterra-Hammerstein integral equation with exact solution $u(x) = x^2 - x$.

$$u(x) = -\frac{15}{56}x^8 + \frac{13}{14}x^7 - \frac{11}{10}x^6 + \frac{9}{20}x^5 + x^2 - x + \int_0^x (x+t)[u(t)]^3\,dt, \qquad 0 \le x \le 1.$$

The maximum absolute error on the Sinc grid S is tabulated in Table 2. This Table indicates that as N increases the error is decreased more rapidly. In Figure 2 for large values of N the approximate solutions are indistinguishable from the exact solutions for the given scale.

Table 2. Results for Example 2

N	h	$\| E_S(h) \|_\infty$
5	0.993459	4.87734×10^{-5}
10	0.702481	4.01787×10^{-6}
20	0.496729	7.58342×10^{-8}
30	0.405578	2.88422×10^{-9}
40	0.351241	1.61805×10^{-10}
50	0.314159	3.86900×10^{-11}

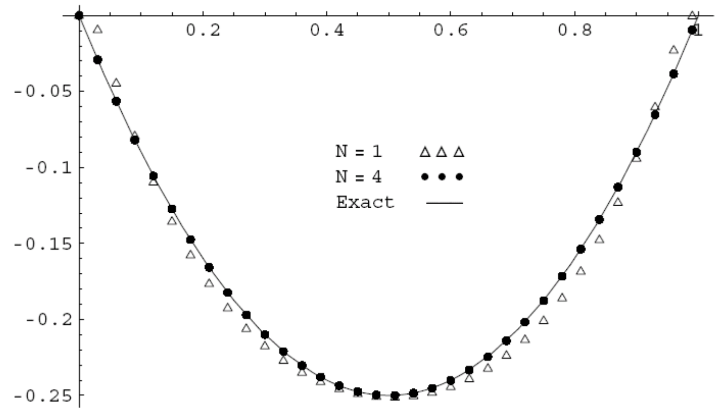

Figure 2. Exact and approximate solutions for Example 2, ($N = 1, 4$)

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