



An Optimal Harvesting Strategy of a Three Species Syn-ecosystem with Commensalism and Stochasticity

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Abstract

In this paper we have studied the stability of three typical species syn-ecosystem. The system comprises of one commensal S_1 and two hosts S_2 and S_3 . Both S_2 and S_3 benefit S_1 without getting themselves affected either positively or adversely. Further S_2 is a commensal of S_3 and S_3 is a host of both S_1 and S_2 . Limited resources have been considered for all the three species in this case. The model equations of the system constitute a set of three first order non-linear ordinary differential equations. The possible equilibrium points of the model are identified. We have also studied the local and global stabilities. We have analyzed the bionomic equilibrium and optimal harvesting strategy using Pontryagin's maximum principle. We have investigated the inhabitant intensities of the fluctuations (variances) around the positive equilibrium due to noise and have investigated the stability. We have also checked the MATLAB numerical simulations for stability of the system.

Keywords: Commensal; steady states; Routh-Hurwitz criteria; Global stability; Bionomic harvesting; optimal harvesting; Pontryagin's principle; stochastic perturbation; Fourier transforms methods

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1. Introduction

Ecology is the study of the inter-relationships between creatures and their surroundings. It is usual for two or more species living in a common territories to interact in dissimilar ways. Mathematical modeling has played an important role for the last half a century in explaining several phenomena concerned with individuals and groups of populations in Nature. Lotka (1925) and Volterra (1931) established theoretical ecology in a momentous way and opened new epochs in the field of life and biological sciences. The Ecological dealings can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and Parasitism.

The general concept of modeling has been presented in the treatises of Meyer (1985), Cushing (1977) and Kapur (1985). Srinivas (1991) has studied the competitive ecosystems of two species and three species with limited and unlimited resources. Lakshmi Narayan and Pattabhi Ramacharyulu (2007) have studied prey-predator ecological model with partial cover for the prey and alternate food for the predator. Archana Reddy (2009) and Bhaskara Rama Sharma (2009) have investigated diverse problems related to two species competitive systems with time delay by employing analytical and numerical techniques. Phani Kumar (2010) studied some mathematical models of ecological commensalism, while Ravindra Reddy (2012) discussed the stability of two mutually interacting species with mortality rate for the second species. Further Srilatha et al. (2011) Shiva Reddy et al. (2011) studied stability analysis of three and four species. Hari Prasad et al. (2010, 2011) also discussed the stability of three and four species syn-ecosystems.

The present authors (2011, 2012) have investigated the stability of three species and four species with stage structure, optimal harvesting policy and stochasticity. Papa Rao et al. (2013) analyzed a three species ecological prey, predator and competitor model and discussed the stability and optimal harvesting factors. Hari Prasad et al. (2012) and Kar et al. (2006), Carletti (2006) have been the source of our inspirations to undertake the present investigation on the analytical and numerical approach of the emblematical three species (S_1, S_2, S_3) syn-ecosystem.

2. Mathematical Model

Consider a conventional syn-ecosystem which consists of three species say S_1, S_2, S_3 (Figure 2.1) where three species are living together with the following assumptions: (i) The system comprises of one commensal (S_1) and two hosts S_2 and S_3 . Both S_2 and S_3 benefiting S_1 without getting themselves affected either positively or adversely (ii) S_2 is a commensal of S_3 (iii) S_3 is a host of both S_1 and S_2 (iv) all the three species have limited resources.

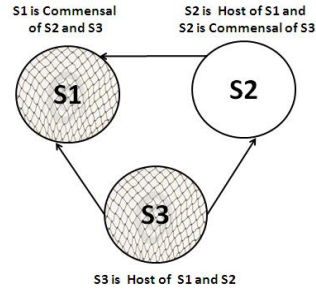


Figure 2.1. Representation of a three species commensaling system

Let $x(t)$, $y(t)$ and $z(t)$ be the population densities of species S_1 , S_2 and S_3 respectively at time instant ' t '. Let a_1 , a_2 and a_3 be the natural growth rates of species S_1 , S_2 and S_3 respectively. Keeping these in view and following Hari Prasad et al. (2012) and Tapan Kumar Kar et al. (2006), the dynamics of the system may be governed by the following first order nonlinear ordinary differential equations:

$$\frac{dx}{dt} = a_1x - a_{11}x^2 + a_{12}xy + a_{13}xz - q_1E_1x, \quad (2.1)$$

$$\frac{dy}{dt} = a_2y - a_{22}y^2 + a_{23}yz, \quad (2.2)$$

$$\frac{dz}{dt} = a_3z - a_{33}z^2 - q_2E_2z. \quad (2.3)$$

In the above model a_{ii} , $i=1,2,3$, are self-inhibition coefficients of species S_i , $i=1,2,3$, respectively, a_{12} is the interaction coefficient of S_1 due to S_2 , a_{13} is interaction coefficient of S_1 due to S_3 , a_{23} is the interaction coefficient of S_2 due to S_3 , $K_i = a_i / a_{ii}$, $i=1,2,3$, are the carrying capacities of species S_i , $i=1,2,3$, respectively, q_1 and q_2 are the catch ability coefficients of species S_1 and S_3 , respectively, E_1 and E_2 are the efforts applied to harvest the species S_1 and S_3 , respectively.

Throughout our analysis, we assume that

$$a_1 - q_1E_1 > 0, a_3 - q_2E_2 > 0. \quad (2.4)$$

3. Steady States

In this section we present the basic outcomes on the non-negative equilibriums. It can be checked that the model (2.1)-(2.3) has only three nonnegative equilibriums namely $T_0(0,0,0)$, $T_1(\bar{x}, \bar{y}, 0)$ and $T_2(x^*, y^*, z^*)$ which are attained by solving $\dot{x} = \dot{y} = \dot{z} = 0$.

Case (i): $T_0(0,0,0)$: The population is extinct but this always exists.

Case (ii): $T_1(\bar{x}, \bar{y}, 0)$: Here \bar{x} and \bar{y} are positive solutions of $\dot{x} = 0$ and $\dot{y} = 0$. We get,

$$\bar{y} = a_2 / a_{22}, \tag{3.1}$$

$$\bar{x} = (1/a_{11}) \left[(a_1 - q_1 E_1) + (a_{12} a_2) / a_{22} \right]. \tag{3.2}$$

Clearly, \bar{x} is positive due to (2.4).

Case (iii): $T_2(x^*, y^*, z^*)$ (The interior equilibrium): Here, x^*, y^* and z^* are positive solutions of the following equations:

$$a_1 - a_{11}x + a_{12}y + a_{13}z - q_1 E_1 = 0, \quad a_2 - a_{22}y + a_{23}z = 0, \quad a_3 - a_{33}z - q_2 E_2 = 0. \tag{3.3}$$

From (3.3), we get

$$z^* = (1/a_{33})(a_3 - q_2 E_2),$$

$$y^* = (1/a_{22}) \left[a_2 + (a_{23}/a_{33})(a_3 - q_2 E_2) \right],$$

$$x^* = \frac{1}{a_{11}} \left[(a_1 - q_1 E_1) + (a_{12}/a_{22}) \left\{ a_2 + (a_{23}/a_{33})(a_3 - q_2 E_2) \right\} + (a_{13}/a_{33})(a_3 - q_2 E_2) \right].$$

We clearly see that x^*, y^* and z^* are positive due to inequalities (2.4).

4. Local Stability

We first consider the local stability of the interior steady state. The Variational matrix of the system (2.1)-(2.3) is

$$J = \begin{pmatrix} a_1 - 2a_{11}x + a_{12}y + a_{13}z - q_1 E_1 & a_{12}x & a_{13}x \\ 0 & a_2 - 2a_{22}y + a_{23}z & a_{23}y \\ 0 & 0 & a_3 - 2a_{33}z - q_2 E_2 \end{pmatrix}. \tag{4.1}$$

At the interior equilibrium $T_2(x^*, y^*, z^*)$, the characteristic equation of (4.1) is in the form of

$$\mu^3 + A\mu^2 + B\mu + C = 0, \tag{4.2}$$

where

$$A = a_{11}x^* + a_{22}y^* + a_{33}z^*, \quad B = a_{11}a_{22}x^*y^* + a_{22}a_{33}y^*z^* + a_{11}a_{33}x^*z^*, \quad C = a_{11}a_{22}a_{33}x^*y^*z^*.$$

The system is locally asymptotically stable if all the eigenvalues of the above characteristic equation have negative real parts. By Routh- Hurwitz criteria, it follows that all eigenvalues of the above characteristic equation have negative real parts if and only if $A > 0, C > 0, AB - C > 0$. Obviously $A > 0, C > 0$ and

$$AB - C = a_{11}x^* \left(a_{11}a_{22}x^*y^* + a_{11}a_{33}x^*z^* \right) \\ + \left(a_{22}y^* + a_{33}z^* \right) \left(a_{11}a_{22}x^*y^* + a_{22}a_{33}y^*z^* + a_{11}a_{33}x^*z^* \right) > 0.$$

5. Global Stability

We now discuss the global stability of the equilibrium points $T_1(\bar{x}, \bar{y}, 0)$ and $T_2(x^*, y^*, z^*)$ of the system (2.1)-(2.3).

Theorem 5.1.

The equilibrium point $T_1(\bar{x}, \bar{y}, 0)$ is globally asymptotically stable.

Proof:

Let us consider the following Lyapunov function

$$L(x, y) = [(x - \bar{x}) - \bar{x} \ln(x / \bar{x})] + l_1 [(y - \bar{y}) - \bar{y} \ln(y / \bar{y})],$$

where l_1 is the positive constant.

$$(dL)/(dt) = [(x - \bar{x})/x](dx)/(dt) + l_1 [(y - \bar{y})/y](dy)/(dt),$$

$$(dL)/(dt) = (x - \bar{x})[-a_{11}(x - \bar{x}) + a_{12}(y - \bar{y})] + l_1 (y - \bar{y})[-a_{22}(y - \bar{y})].$$

By choosing $l_1 = 1/a_{22}$, we get

$$(dL)/(dt) = - \left[a_{11}(x - \bar{x})^2 - a_{12}(x - \bar{x})(y - \bar{y}) + (y - \bar{y})^2 \right],$$

which is in the form $-Y^T P Y$, where

$$Y^T = (x - \bar{x} \quad y - \bar{y}), \quad P = \begin{pmatrix} a_{11} & -a_{12}/2 \\ -a_{12}/2 & 1 \end{pmatrix}.$$

The equilibrium point $T_1(\bar{x}, \bar{y}, 0)$ is globally asymptotically stable if $\frac{dL}{dt} < 0$. This is possible only when the matrix P is positive definite. We observe clearly that this is the case since all the principal minors of P are positive.

Theorem 5.2.

The interior equilibrium point $T_2(x^*, y^*, z^*)$ is globally asymptotically stable if $4a_{22} > a_{23}^2$, $4a_{11} > a_{13}^2$ and $4a_{11}a_{22} > a_{12}^2$ hold.

Proof:

To find the condition for global stability at $T_2(x^*, y^*, z^*)$, we construct the Lyapunov function

$$L(x, y, z) = [(x - x^*) - x^* \ln(x/x^*)] + l_1 [(y - y^*) - y^* \ln(y/y^*)] + l_2 [(z - z^*) - z^* \ln(z/z^*)],$$

where l_1 and l_2 are positive constants.

$$(dL)/(dt) = [(x - x^*)/x](dx)/(dt) + l_1 [(y - y^*)/y](dy)/(dt) + l_2 [(z - z^*)/z](dz)/(dt),$$

$$(dL)/(dt) = (x - x^*)[-a_{11}(x - x^*) + a_{12}(y - y^*) + a_{13}(z - z^*)] + l_1 (y - y^*)[-a_{22}(y - y^*) + a_{23}(z - z^*)] + l_2 (z - z^*)[-a_{33}(z - z^*)],$$

$$(dL)/(dt) = - \left(\begin{aligned} & [a_{11}(x - x^*)^2 - a_{12}(x - x^*)(y - y^*) - a_{13}(x - x^*)(z - z^*)] \\ & + l_1 [a_{22}(y - y^*)^2 - a_{23}(y - y^*)(z - z^*)] + l_2 [a_{33}(z - z^*)^2] \end{aligned} \right).$$

By choosing $l_1 = 1, l_2 = 1/a_{33}$ we get

$$(dL)/(dt) = - \left(\begin{aligned} & [a_{11}(x - x^*)^2 - a_{12}(x - x^*)(y - y^*) - a_{13}(x - x^*)(z - z^*)] \\ & + [a_{22}(y - y^*)^2 - a_{23}(y - y^*)(z - z^*)] + (z - z^*)^2 \end{aligned} \right),$$

which is in the form of $-X^T M X$, where

$$X^T = (x - x^* \quad y - y^* \quad z - z^*)$$

and

$$M = \begin{pmatrix} a_{11} & -a_{12}/2 & -a_{13}/2 \\ -a_{12}/2 & a_{22} & -a_{23}/2 \\ -a_{13}/2 & -a_{23}/2 & 1 \end{pmatrix}.$$

The interior equilibrium point $T_2(x^*, y^*, z^*)$ is globally asymptotically stable if $\frac{dL}{dt} < 0$. This is possible only when M is positive definite. We observe clearly that M is positive definite if the hypotheses of the theorem are satisfied.

6. Bionomic Equilibrium

This is the combination of biological equilibrium and economic equilibrium. In section (3) we have discussed the biological equilibrium which is given by $\dot{x} = \dot{y} = \dot{z} = 0$. When the total revenue obtained by selling the harvested biomass equals the total cost utilized in harvesting it, we say that the bionomic equilibrium is achieved. Let c_1 be the constant fishing cost of species S_1 per unit effort and c_2 be the constant fishing cost of species S_3 per unit effort. Let p_1 be the constant price of species S_1 per unit biomass and p_2 be the constant price of species S_3 per unit biomass. Then the revenue at any time is given by

$$A(x, y, z, E_1, E_2) = [(p_1 q_1 x - c_1)E_1 + (p_2 q_2 z - c_2)E_2]. \quad (6.1)$$

If $c_1 > p_1 q_1 x$ and $c_2 > p_2 q_2 z$ then the economic rent obtained from the fishery becomes negative and the fishery will be closed. Hence for the existence of bionomic equilibrium, it is assumed that

$$c_1 < p_1 q_1 x, c_2 < p_2 q_2 z. \quad (6.2)$$

The bionomic equilibrium $((x)_\infty, (y)_\infty, (z)_\infty, (E_1)_\infty, (E_2)_\infty)$ is the positive solution of

$$\dot{x} = \dot{y} = \dot{z} = A = 0. \quad (6.3)$$

By solving (6.3) we get

$$(x)_\infty = c_1 / (p_1 q_1), (z)_\infty = c_2 / (p_2 q_2),$$

$$(y)_\infty = (1/a_{22})[a_2 + a_{23}(z)_\infty] = (1/a_{22})[a_2 + [(a_{23}c_2)/(p_2 q_2)]],$$

$$(E_1)_\infty = \frac{1}{q_1} [a_1 - (a_{11}c_1)/(p_1 q_1) + (a_{12}a_2)/a_{22} + (a_{12}a_{23}c_2)/(a_{22}p_2 q_2) + (a_{13}c_2)/p_2 q_2],$$

$$(E_2)_\infty = (1/q_2)[a_3 - (a_{33}c_2/p_2 q_2)],$$

$$(E_1)_\infty > 0 \text{ when } (x)_\infty < (a_1/a_{11}) \text{ and } (E_2)_\infty > 0 \text{ when } (z)_\infty < (a_3/a_{33}).$$

If $(E_1) > (E_1)_\infty$ and $(E_2) > (E_2)_\infty$, then the total cost utilized in harvesting the fish population would exceed the total revenues obtained from the fishery. Thus, some of the fishermen would face a loss and naturally they would withdraw their participation from the fishery. Hence, $(E_1) > (E_1)_\infty$ and $(E_2) > (E_2)_\infty$ cannot be maintained indefinitely. If $(E_1) < (E_1)_\infty$ and $(E_2) < (E_2)_\infty$, then the fishery is more profitable and, hence, in an open access fishery, it would attract more and more fishermen. This will have an increasing effect on the harvesting effort. Hence, $(E_1) < (E_1)_\infty$ and $(E_2) < (E_2)_\infty$ cannot be continued indefinitely.

7. Optimal Harvesting Strategy

Our objective now is to select a harvesting strategy that maximizes the present value

$$Q = \int_0^{\infty} e^{-\delta t} \left[(p_1 q_1 x - c_1) E_1(t) + (p_2 q_2 z - c_2) E_2(t) \right] dt, \quad (7.1)$$

of a continuous time stream of revenues. Here δ is the instantaneous annual discount rate. The problem (7.1) subject to population equations (2.1)-(2.3) and control constraints $0 \leq E_1 \leq (E_1)_{\max}$ and $0 \leq E_2 \leq (E_2)_{\max}$ can be explained by applying Pontryagin's maximum principle. The Hamiltonian is given by

$$H = e^{-\delta t} \left[(p_1 q_1 x - c_1) E_1 + (p_2 q_2 z - c_2) E_2 \right] + \lambda_1 \left[a_1 x - a_{11} x^2 + a_{12} xy + a_{13} xz - q_1 E_1 x \right] \\ + \lambda_2 \left[a_2 y - a_{22} y^2 + a_{23} yz \right] + \lambda_3 \left[a_3 z - a_{33} z^2 - q_2 E_2 z \right], \quad (7.2)$$

where λ_1, λ_2 and λ_3 are adjoint variables and the switching functions are

$$\mu_1(t) = e^{-\delta t} (p_1 q_1 x - c_1) - \lambda_1 q_1 x, \quad (7.3)$$

$$\mu_2(t) = e^{-\delta t} (p_2 q_2 z - c_2) - \lambda_3 q_2 z. \quad (7.4)$$

Since the Hamiltonian H is linear in the control variable, the optimal control will be an amalgamation of the extreme controls and the singular control. The Optimal controls $E_1(t)$ and $E_2(t)$ that maximize H must satisfy the subsequent conditions:

$$(E_1) = (E_1)_{\max}, \text{ where } \mu_1(t) > 0, \text{ i.e., } \lambda_1(t) e^{\delta t} < [p_1 - (c_1 / q_1 x)],$$

$$(E_2) = (E_2)_{\max}, \text{ where } \mu_2(t) > 0, \text{ i.e., } \lambda_3(t) e^{\delta t} < [p_2 - (c_2 / q_2 z)].$$

$\lambda_i(t) e^{\delta t}$, $i=1,3$, is the shadow price and $p_1 - (c_1 / q_1 x)$ is the net economic income on a unit harvest of species S_1 , $p_2 - (c_2 / q_2 z)$ is the net economic income on a unit harvest of species S_3 .

This shows that $(E_i) = (E_i)_{\max}$, $i = 1, 2$, or zero, according to the shadow price is not more than or superior to the net economic revenue on a unit harvest. Economically, the first condition implies that if the profit, after paying all the expenses is positive then it is beneficial to harvest up to the limit of the available effort. The second condition implies that when the shadow price exceeds the fisherman's net economic revenue on a unit harvest then the fisherman will not exert any effort. When $\mu_i(t) = 0$, $i = 1, 2$, i.e., when the shadow price equals the net economic revenue on a unit harvest then the Hamiltonian H becomes self-governing of the control variable $E_i(t)$, i.e.,

$$\frac{\partial H}{\partial E_i} = 0.$$

This is, an obligatory condition for the singular control $E_i^*(t)$ to be optimal over the control set $0 < E_i^*(t) < (E_i)_{\max}$. Thus, the optimal harvesting strategy is

$$E_i(t) = \begin{cases} (E_i)_{\max}, & \mu_i(t) > 0, \\ 0, & \mu_i(t) < 0, \\ (E_i)^*, & \mu_i(t) = 0, \end{cases} \quad (7.5)$$

when $\mu_i(t) = 0$, $i = 1, 2$, it follows that

$$\lambda_1 q_1 x = e^{-\delta t} (p_1 q_1 x - c_1) = (\partial A) / (\partial E_1) e^{-\delta t}, \quad (7.6)$$

$$\lambda_3 q_2 z = e^{-\delta t} (p_2 q_2 z - c_2) = (\partial A) / (\partial E_2) e^{-\delta t}. \quad (7.7)$$

This implies that the user's cost of harvest per unit of effort equals the concession value of the future marginal profit of the effort at the steady state level. Now the adjoint equations are

$$(d\lambda_1)/(dt) = -(\partial H)/(\partial x) = -\left[e^{-\delta t} p_1 q_1 E_1 + \lambda_1 (a_1 - 2a_{11}x + a_{12}y + a_{13}z - q_1 E_1) \right], \quad (7.8)$$

$$(d\lambda_2)/(dt) = -(\partial H)/(\partial y) = -\left[\lambda_1 (a_{12}x) + \lambda_2 (a_2 - 2a_{22}y + a_{23}z) \right], \quad (7.9)$$

$$(d\lambda_3)/(dt) = -(\partial H)/(\partial z) = -e^{-\delta t} p_2 q_2 E_2 - \lambda_1 (a_{13}x) - \lambda_2 (a_{23}y) - \lambda_3 (a_3 - 2a_{33}z - q_2 E_2). \quad (7.10)$$

We now seek to find the optimal equilibrium solution of the problem so that x , y , z , E_1 and E_2 can be treated as constants.

From (7.9),

$$(d\lambda_2)/(dt) + \lambda_2 (a_2 - 2a_{22}y + a_{23}z) = -e^{-\delta t} [p_1 - (c_1 / q_1 x)] a_{12} x,$$

which is in the form of

$$(d\lambda_2)/(dt) + M_1\lambda_2 = -M_2e^{-\delta t},$$

where

$$M_1 = a_2 - 2a_{22}y^* + a_{23}z^*, M_2 = [p_1 - (c_1/q_1x)]a_{12}x^*,$$

and its solution is given by

$$\lambda_2 = [M_2/(\delta - M_1)]e^{-\delta t}. \quad (7.11)$$

From (7.8),

$$(d\lambda_1)/(dt) + \lambda_1(a_1 - 2a_{11}x + a_{12}y + a_{13}z - q_1E_1) = -e^{-\delta t} p_1q_1E_1,$$

which is in the form of

$$(d\lambda_1)/(dt) + M_3\lambda_1 = -M_4e^{-\delta t},$$

where

$$M_3 = a_1 - 2a_{11}x^* + a_{12}y^* + a_{13}z^* - q_1E_1, M_4 = p_1q_1E_1,$$

and its solution is given by

$$\lambda_1 = [M_4/(\delta - M_3)]e^{-\delta t}. \quad (7.12)$$

From (7.10), (7.11) and (7.12),

$$(d\lambda_3)/(dt) + M_5\lambda_3 = -M_6e^{-\delta t},$$

where

$$M_5 = a_3 - 2a_{33}z^* - q_2E_2,$$

$$M_6 = p_2q_2E_2 + [(M_4a_{13}x^*)/(\delta - M_3)] + [(M_2a_{23}y^*)/(\delta - M_1)],$$

and its solution is given by

$$\lambda_3 = [M_6/(\delta - M_5)]e^{-\delta t}. \quad (7.13)$$

From (7.6) and (7.12) we get the singular path

$$p_1 - (c_1 / q_1 x^*) = -[M_4 / (M_3 - \delta)]. \quad (7.14)$$

From (7.7) and (7.13) we get the singular path

$$p_2 - [c_2 / (q_2 z^*)] = -[M_6 / (M_5 - \delta)]. \quad (7.15)$$

At the point $T_2(x^*, y^*, z^*)$, M_i , $i=1, 2, \dots, 6$, can be written as follows

$$\begin{aligned} M_1 &= -a_2 - (a_{23} / a_{33})(a_3 - q_2 E_2), \\ M_2 &= \frac{p_1 a_{12}}{a_{11}} \left[(a_1 - q_1 E_1) + \frac{a_{12}}{a_{22}} \left\{ a_2 + \frac{a_{23}}{a_{33}} (a_3 - q_2 E_2) \right\} + \frac{a_{13}}{a_{33}} (a_3 - q_2 E_2) \right] - \frac{c_1 a_{12}}{q_1}, \\ M_3 &= -(a_1 - q_1 E_1) - \frac{a_{12}}{a_{22}} \left[a_2 + \frac{a_{23}}{a_{33}} (a_3 - q_2 E_2) \right] - \frac{a_{13}}{a_{33}} (a_3 - q_2 E_2), \\ M_4 &= p_1 q_1 E_1, \quad M_5 = -(a_3 - q_2 E_2), \\ M_6 &= \frac{M_4 a_{13}}{(\delta - M_3) a_{11}} \left[(a_1 - q_1 E_1) + \frac{a_{12}}{a_{22}} \left\{ a_2 + \frac{a_{23}}{a_{33}} (a_3 - q_2 E_2) \right\} + \frac{a_{13}}{a_{33}} (a_3 - q_2 E_2) \right] \\ &\quad + p_2 q_2 E_2 + \frac{M_2 a_{23}}{(\delta - M_1) a_{22}} \left[a_2 + \frac{a_{23}}{a_{33}} (a_3 - q_2 E_2) \right]. \end{aligned}$$

Thus, (7.14) and (7.15) can be written as

$$F(x^*) = [p_1 - (c_1 / q_1 x^*)] + [M_4 / (M_3 - \delta)] = 0, \quad (7.16)$$

$$G(z^*) = [p_2 - (c_2 / q_2 z^*)] + [M_6 / (M_5 - \delta)] = 0. \quad (7.17)$$

There exists a unique positive root $x^* = x_\delta$ of $F(x^*) = 0$ in the interval $0 < (x)_\infty < K_1$, if the following inequalities hold: $F(0) < 0, F(K_1) > 0, F'(x^*) > 0$ for $x^* > 0$, where $K_1 = a_1 / a_{11}$.

There exists a unique positive root $z^* = z_\delta$ of $G(z^*) = 0$ in the interval $0 < (z)_\infty < K_2$, if the following inequalities hold: $G(0) < 0, G(K_2) > 0, G'(z^*) > 0$ for $z^* > 0$, where $K_2 = a_3 / a_{33}$.

For $x^* = x_\delta$, we get

$$y_\delta = (1 / a_{22}) \left[a_2 + (a_{23} / a_{33})(a_3 - q_2 E_2) \right],$$

$$z_\delta = (1/a_{33})(a_3 - q_2 E_2),$$

$$(E_1)_\delta = (1/q_1)[a_1 - a_{11}x_\delta + a_{12}y_\delta + a_{13}z_\delta],$$

$$(E_2)_\delta = (1/q_2)[a_3 - a_{33}z_\delta].$$

Here, $(E_1)_\delta > 0$ if $x_\delta < K_1$ and $(E_2)_\delta > 0$, if $z_\delta < K_2$.

From (7.11), (7.12) and (7.13), we observe that $\lambda_i(t)e^{\delta t}$, $i=1,2,3$, is independent of time and is the best possible equilibrium. Hence, they satisfy the transversality condition at ∞ , i.e., they remain bounded as $t \rightarrow \infty$.

From (7.16) and (7.17), we also have

$$p_1 q_1 x^* - c_1 = -[M_4 / (M_3 - \delta)] \rightarrow 0, \text{ as } \delta \rightarrow \infty,$$

$$p_2 q_2 z^* - c_2 = -[M_7 / (M_5 - \delta)] \rightarrow 0, \text{ as } \delta \rightarrow \infty.$$

Thus, the net economic revenue

$$A((x)_\infty, (y)_\infty, (z)_\infty, (E_1)_\infty, (E_2)_\infty) = 0.$$

This implies that an infinite concession rate pilots to the net economic revenue tending to zero and the fishery would remain closed.

8. The Stochastic Model

In this paper we assume the presence of randomly fluctuating driving forces on the deterministic growth of the species S_i , $i=1,2,3$, at time 't' so that the system (2.1)-(2.3) results in the stochastic system with 'additive noise'.

The main assumption that leads us to extend the deterministic model (2.1)-(2.3) to a stochastic counterpart is that, it is reasonable to conceive the open sea as a noisy environment. There are many ways in which environmental noise may be incorporated in system (2.1)-(2.3). Note that environmental noise should be distinguished from demographic or internal noise for which the variation over time is due. External noise may arise either from random fluctuations of one or more model parameters around some known mean values or from stochastic fluctuations of the population densities around some constant values.

In this section we compute the population intensities of fluctuations (variances) around the positive equilibrium $T_2(x^*, y^*, z^*)$ due to noise according to the method introduced by Nisbet et al. (1982). Such a method was also successfully applied by Tapaswi et al. (1999). Now we assume the presence of randomly fluctuating driving forces on the deterministic growth of the

species S_i , $i = 1, 2, 3$, at time ' t ' so that the system (2.1)-(2.3) results in the stochastic system with 'additive noise':

$$\frac{dx}{dt} = a_1x - a_{11}x^2 + a_{12}xy + a_{13}xz - q_1E_1x + \eta_1\xi_1(t), \quad (8.1)$$

$$\frac{dy}{dt} = a_2y - a_{22}y^2 + a_{23}yz + \eta_2\xi_2(t), \quad (8.2)$$

$$\frac{dz}{dt} = a_3z - a_{33}z^2 - q_2E_2z + \eta_3\xi_3(t), \quad (8.3)$$

where $x(t)$, $y(t)$ and $z(t)$ be the population densities of species S_1, S_2 and S_3 , respectively, at time instant ' t '. $\alpha_1, \alpha_2, \alpha_3$ are real constants and $\xi(t) = [\xi_1(t), \xi_2(t), \xi_3(t)]$ is a three dimensional Gaussian white noise process satisfying

$$E[\xi_i(t)] = 0, \quad i = 1, 2, 3, \quad (8.4)$$

$$E[\xi_i(t)\xi_j(t')] = \delta_{ij}\delta(t-t'), \quad i, j = 1, 2, 3, \quad (8.5)$$

where δ_{ij} is the Kronecker symbol and δ is the δ -Dirac function.

Let us consider the technique of perturbations as

$$x(t) = u_1(t) + S^*, \quad y(t) = u_2(t) + P^*, \quad z(t) = u_3(t) + T^*, \quad (8.6)$$

$$\frac{dx}{dt} = \frac{du_1(t)}{dt}, \quad \frac{dy}{dt} = \frac{du_2(t)}{dt}, \quad \frac{dz}{dt} = \frac{du_3(t)}{dt}. \quad (8.7)$$

Using equations (8.6) and (8.7), equation (8.1) becomes

$$\begin{aligned} \frac{du_1(t)}{dt} = & a_1u_1(t) + a_1S^* - a_{11}u_1^2(t) - a_{11}(S^*)^2 - 2a_{11}u_1(t)S^* + a_{12}u_1(t)u_2(t) \\ & + a_{12}u_1(t)P^* + a_{12}u_2(t)S^* + a_{12}S^*P^* + a_{13}u_1(t)u_3(t) + a_{13}u_1(t)T^* + a_{13}u_3(t)S^* \\ & + a_{13}S^*T^* - q_1E_1u_1(t) - q_1E_1S^* + \eta_1\xi_1(t). \end{aligned} \quad (8.8)$$

The linear part of (8.8) is

$$\frac{du_1(t)}{dt} = -a_{11}u_1(t)S^* + a_{12}u_2(t)S^* + a_{13}u_3(t)S^* + \eta_1\xi_1(t). \quad (8.9)$$

Using equations (8.6) and (8.7), equation (8.2) becomes

$$\begin{aligned} \frac{du_2(t)}{dt} = & a_2u_2(t) + a_2P^* - a_{22}u_2^2(t) - a_{22}(P^*)^2 - 2a_{22}u_2(t)P^* \\ & + a_{23}u_2(t)u_3(t) + a_{23}u_2(t)T^* + a_{23}u_3(t)P^* + a_{23}P^*T^* + \eta_2\xi_2(t). \end{aligned} \tag{8.10}$$

The linear part of (8.10) is

$$\frac{du_2(t)}{dt} = -a_{22}u_2(t)P^* + a_{23}u_3(t)P^* + \eta_2\xi_2(t). \tag{8.11}$$

Using equations (8.6) and (8.7), equation (8.3) becomes

$$\begin{aligned} \frac{du_3(t)}{dt} = & a_3u_3(t) + a_3T^* - a_{33}u_3^2(t) - a_{33}(T^*)^2 - 2a_{33}u_3(t)T^* \\ & - q_2E_2u_3(t) - q_2E_2T^* + \eta_3\xi_3(t). \end{aligned} \tag{8.12}$$

The linear part of (8.12) is

$$\frac{du_3(t)}{dt} = -a_{33}u_3(t)T^* + \eta_3\xi_3(t). \tag{8.13}$$

Taking the Fourier transform on both sides of (8.9), (8.11) and (8.13), we get

$$\eta_1\tilde{\xi}_1(\omega) = (i\omega + a_{11}S^*)\tilde{u}_1(\omega) - a_{12}S^*\tilde{u}_2(\omega) - a_{13}S^*\tilde{u}_3(\omega), \tag{8.14}$$

$$\eta_2\tilde{\xi}_2(\omega) = (i\omega + a_{22}P^*)\tilde{u}_2(\omega) - a_{23}P^*\tilde{u}_3(\omega), \tag{8.15}$$

$$\eta_3\tilde{\xi}_3(\omega) = (i\omega + a_{33}T^*)\tilde{u}_3(\omega). \tag{8.16}$$

The matrix form of (8.14), (8.15) and (8.16) is

$$M(\omega)\tilde{u}(\omega) = \tilde{\xi}(\omega), \tag{8.17}$$

where

$$M(\omega) = \begin{pmatrix} A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega) \\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega) \\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega) \end{pmatrix}, \quad \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \\ \tilde{u}_3(\omega) \end{bmatrix}, \quad \tilde{\xi}(\omega) = \begin{bmatrix} \eta_1\tilde{\xi}_1(\omega) \\ \eta_2\tilde{\xi}_2(\omega) \\ \eta_3\tilde{\xi}_3(\omega) \end{bmatrix},$$

$$A_{11}(\omega) = i\omega + a_{11}S^*, \quad A_{12}(\omega) = -a_{12}S^*, \quad A_{13}(\omega) = -a_{13}S^*,$$

$$A_{21}(\omega) = 0, \quad A_{22}(\omega) = i\omega + a_{22}P^*, \quad A_{23}(\omega) = -a_{23}P^*,$$

$$A_{31}(\omega) = 0, A_{32}(\omega) = 0, A_{33}(\omega) = i\omega + a_{33}T^* . \quad (8.18)$$

Equation (8.17) can also be written as

$$\tilde{u}(\omega) = K(\omega)\tilde{\xi}(\omega), \quad (8.19)$$

where

$$K(\omega) = [M(\omega)]^{-1} = \frac{Adj M(\omega)}{|M(\omega)|} . \quad (8.20)$$

If the function $Y(t)$ has a zero mean value, then the fluctuation intensity (variance) of its components in the frequency interval $[\omega, \omega + d\omega]$ is $S_Y(\omega)d\omega$, where $S_Y(\omega)$ is the spectral density of Y and is defined as

$$S_Y(\omega) = \lim_{\tilde{T} \rightarrow \infty} \frac{|\tilde{Y}(\omega)|^2}{\tilde{T}} . \quad (8.21)$$

If Y has a zero mean value then the inverse transform of $S_Y(\omega)$ is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) e^{i\omega\tau} d\omega . \quad (8.22)$$

The corresponding variance of fluctuations in $Y(t)$ is given by

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega . \quad (8.23)$$

The auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} . \quad (8.24)$$

For a Gaussian white noise process, it is

$$\begin{aligned}
 S_{\xi_i \xi_j}(\omega) &= \lim_{\hat{T} \rightarrow +\infty} \frac{E[\tilde{\xi}_i(\omega) \tilde{\xi}_j(\omega)]}{\hat{T}} \\
 &= \lim_{\hat{T} \rightarrow +\infty} \frac{1}{\hat{T}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} \int_{-\frac{\hat{T}}{2}}^{\frac{\hat{T}}{2}} E[\tilde{\xi}_i(t) \tilde{\xi}_j(t')] e^{-i\omega(t-t')} dt dt' = \delta_{ij}.
 \end{aligned}
 \tag{8.25}$$

From (8.19) we have

$$\tilde{u}_i(\omega) = \sum_{j=1}^3 K_{ij}(\omega) \tilde{\xi}_j(\omega), \quad i = 1, 2, 3.
 \tag{8.26}$$

From (8.21) we have

$$S_{u_i}(\omega) = \sum_{j=1}^3 \eta_j |K_{ij}(\omega)|^2, \quad i = 1, 2, 3.
 \tag{8.27}$$

Hence by (8.23) and (8.27), the intensities of fluctuations in u_i , $i = 1, 2, 3$ are given by

$$\sigma_{u_i}^2 = \frac{1}{2\pi} \sum_{j=1}^3 \int_{-\infty}^{\infty} \eta_j |K_{ij}(\omega)|^2 d\omega, \quad i = 1, 2, 3.
 \tag{8.28}$$

From (8.20) we obtain

$$\begin{aligned}
 \sigma_{u_1}^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \eta_1 \left| \frac{Adj(1)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \eta_2 \left| \frac{Adj(2)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \eta_3 \left| \frac{Adj(3)}{|M(\omega)|} \right|^2 d\omega \right\}, \\
 \sigma_{u_2}^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \eta_1 \left| \frac{Adj(4)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \eta_2 \left| \frac{Adj(5)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \eta_3 \left| \frac{Adj(6)}{|M(\omega)|} \right|^2 d\omega \right\}, \\
 \sigma_{u_3}^2 &= \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \eta_1 \left| \frac{Adj(7)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \eta_2 \left| \frac{Adj(8)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \eta_3 \left| \frac{Adj(9)}{|M(\omega)|} \right|^2 d\omega \right\},
 \end{aligned}
 \tag{8.29}$$

where $|M(\omega)| = R(\omega) + iI(\omega)$,

$$R(\omega) = -\omega^2 (a_{11}S^* + a_{22}P^* + a_{33}T^*) + a_{11}a_{22}a_{33}S^*P^*T^*,
 \tag{8.30}$$

$$I(\omega) = -\omega^3 + \omega(a_{11}a_{22}S^*P^* + a_{22}a_{33}P^*T^* + a_{11}a_{33}S^*T^*), \tag{8.31}$$

$$|Adj(1)|^2 = X_1^2 + Y_1^2, |Adj(2)|^2 = X_2^2 + Y_2^2, |Adj(3)|^2 = X_3^2 + Y_3^2,$$

$$|Adj(4)|^2 = X_4^2 + Y_4^2, |Adj(5)|^2 = X_5^2 + Y_5^2, |Adj(6)|^2 = X_6^2 + Y_6^2,$$

$$|Adj(7)|^2 = X_7^2 + Y_7^2, |Adj(8)|^2 = X_8^2 + Y_8^2, |Adj(9)|^2 = X_9^2 + Y_9^2,$$

where

$$X_1 = -\omega^2 + a_{22}a_{23}P^*T^*, Y_1 = \omega(a_{22}P^* + a_{33}T^*), X_2 = a_{12}a_{33}S^*T^*, Y_2 = \omega a_{12}S^*,$$

$$X_3 = (a_{12}a_{23} + a_{13}a_{22})S^*P^*, Y_3 = \omega a_{13}S^*, X_4 = 0, Y_4 = 0, X_5 = -\omega^2 + a_{11}a_{33}S^*T^*,$$

$$Y_5 = \omega(a_{11}S^* + a_{33}T^*), X_6 = a_{11}a_{23}S^*P^*, Y_6 = \omega a_{23}P^*,$$

$$X_7 = 0, Y_7 = 0, X_8 = 0, Y_8 = 0, X_9 = -\omega^2 + a_{11}a_{22}S^*P^*, Y_9 = \omega(a_{11}S^* + a_{22}P^*).$$

Thus, (8.29) becomes

$$\sigma_{u_1}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\eta_1(X_1^2 + Y_1^2) + \eta_2(X_2^2 + Y_2^2) + \eta_3(X_3^2 + Y_3^2) \right] d\omega \right\},$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\eta_2(X_5^2 + Y_5^2) + \eta_3(X_6^2 + Y_6^2) \right] d\omega \right\},$$

$$\sigma_{u_3}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[\eta_3(X_9^2 + Y_9^2) \right] d\omega \right\}.$$

If we are interested in the dynamics of system (8.1)-(8.3) with either $\eta_1 = 0$ or $\eta_2 = 0$ or $\eta_3 = 0$, then the population variances are

If $\eta_1 = \eta_2 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_3}{2\pi} \int_{-\infty}^{\infty} \frac{(X_3^2 + Y_3^2)}{R^2(\omega) + I^2(\omega)} d\omega, \sigma_{u_2}^2 = \frac{\eta_3}{2\pi} \int_{-\infty}^{\infty} \frac{(X_6^2 + Y_6^2)}{R^2(\omega) + I^2(\omega)} d\omega,$$

$$\sigma_{u_3}^2 = \frac{\eta_3}{2\pi} \int_{-\infty}^{\infty} \frac{(X_9^2 + Y_9^2)}{R^2(\omega) + I^2(\omega)} d\omega.$$

If $\eta_2 = \eta_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_1}{2\pi} \int_{-\infty}^{\infty} \frac{(X_1^2 + Y_1^2)}{R^2(\omega) + I^2(\omega)} d\omega, \quad \sigma_{u_2}^2 = 0, \quad \sigma_{u_3}^2 = 0.$$

If $\eta_1 = \eta_3 = 0$, then

$$\sigma_{u_1}^2 = \frac{\eta_2}{2\pi} \int_{-\infty}^{\infty} \frac{(X_2^2 + Y_2^2)}{R^2(\omega) + I^2(\omega)} d\omega, \quad \sigma_{u_2}^2 = \frac{\eta_2}{2\pi} \int_{-\infty}^{\infty} \frac{(X_5^2 + Y_5^2)}{R^2(\omega) + I^2(\omega)} d\omega, \sigma_{u_3}^2 = 0.$$

The equations in (8.29) give three variations of the inhabitants. The integrals over the real line can be estimated which gives the variations of the inhabitants.

9. Computer Simulation

In this section we demonstrate as well as boost up, our analytical findings through numerical simulations considering the following parameters:

Example 1.

$$a_1 = 4.8, a_{11} = 0.5, a_{12} = 0.02, a_{13} = 0.15, q_1 = 0.15, E_1 = 10, a_2 = 3.5, a_{22} = 0.8, a_{23} = 0.48, a_3 = 9, a_{33} = 0.02, q_2 = 0.98, E_2 = 15.$$

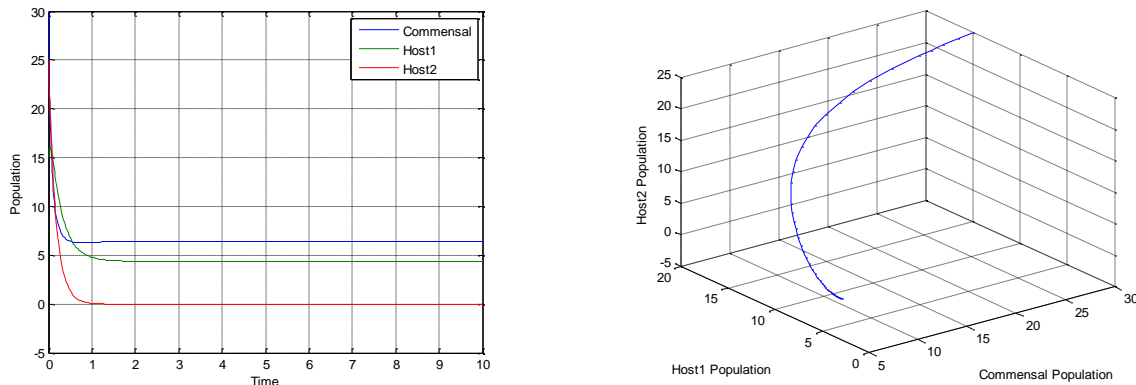


Figure 9.1. The variation of population against time, initially with $x = 30, y = 15, z = 25$, and the variation of population among commensal population, host1 population, and host2 population.

Example 2:

$$a_1 = 4.4, a_{11} = 0.06, a_{12} = 0.02, a_{13} = 0.01, q_1 = 0.05, E_1 = 20, a_2 = 1.5, a_{22} = 0.05, a_{23} = 0.04, a_3 = 2.7, a_{33} = 0.02, q_2 = 0.1, E_2 = 15.$$

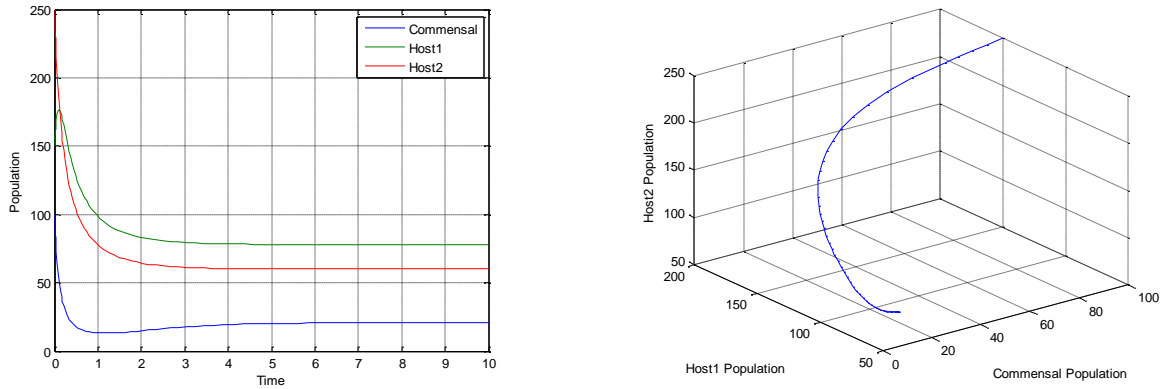


Figure 9.2. The variation of population against time, initially with $x = 100, y = 150, z = 250$ and the variation of population among commensal population, host1 population, and host2 population

Example 3.

$$a_1 = 2.4, a_{11} = 0.06, a_{12} = 0.02, a_{13} = 0.01, q_1 = 0.05, \omega = 10, E_1 = 20, a_2 = 4.5, a_{22} = 0.05, a_{23} = 0.04, \gamma = 5, a_3 = 2.7, a_{33} = 0.02, q_2 = 0.01, E_2 = 15.$$

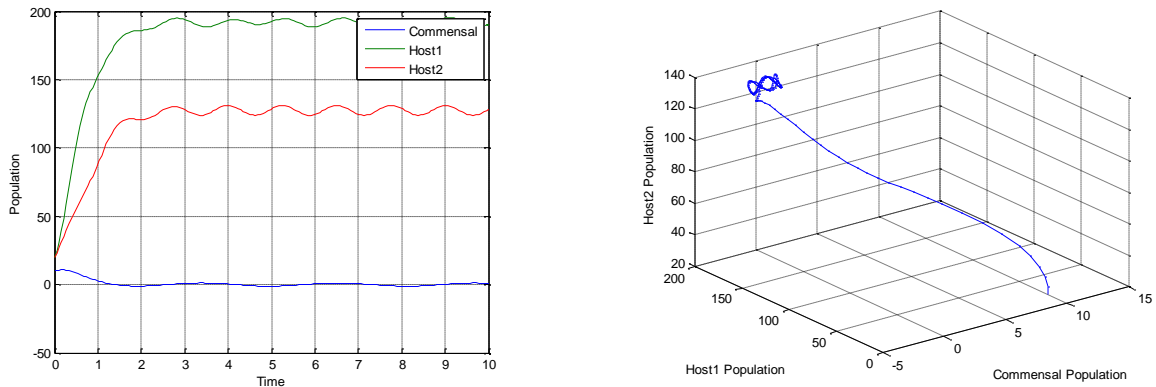


Figure 9.3. The variation of population against time, initially with $x = 10, y = 20, z = 20$ and the variation of population among commensal population, host1 population, and host2 population

Example 4.

$$a_1 = 5.4, a_{11} = 0.06, a_{12} = 0.2, a_{13} = 0.1, q_1 = 0.05, \omega = 10, E_1 = 20, a_2 = 4.5, a_{22} = 0.05, a_{23} = 0.04, \gamma = 5, a_3 = 2.7, a_{33} = 0.02, q_2 = 0.01, E_2 = 15.$$

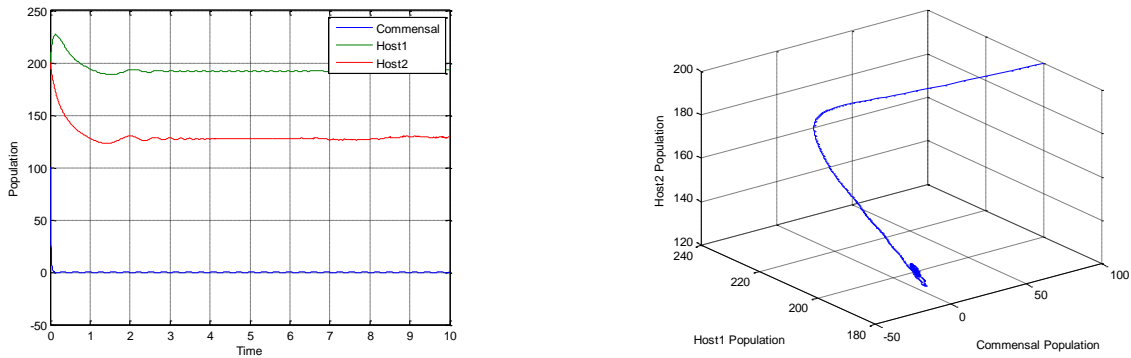


Figure 9.4. The variation of population against time, initially with $x = 100, y = 200, z = 200$ and the variation of population among commensal population, host1 population, and host2 population

Example 5.

$$a_1 = 5.4, a_{11} = 0.06, a_{12} = 0.02, a_{13} = 0.01, q_1 = 0.01, \omega = 10, E_1 = 10, a_2 = 1.5, a_{22} = 0.05, a_{23} = 0.04, \gamma = 5, a_3 = 2.7, a_{33} = 0.02, q_2 = 0.01, E_2 = 15.$$

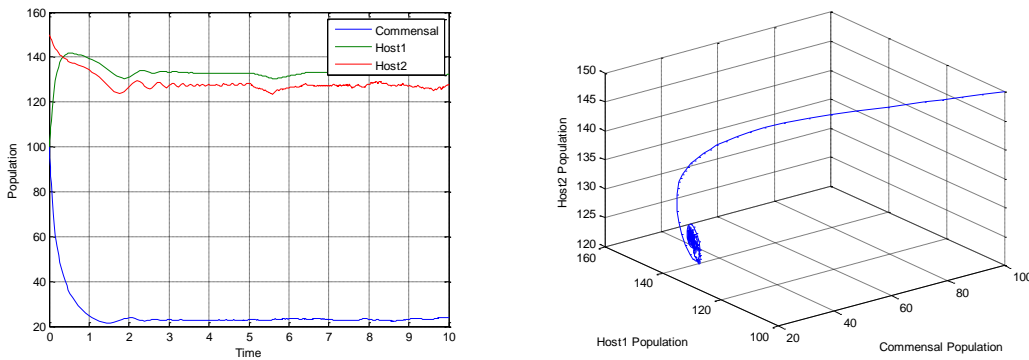


Figure 9.5. The variation of population against time, initially with $x = 100, y = 100, z = 150$ and the variation of population among commensal population, host1 population, and host2 population

Example 6.

$$a_1 = 2.4, a_{11} = 0.06, a_{12} = 0.02, a_{13} = 0.01, q_1 = 0.01, \omega = 10, E_1 = 10, a_2 = 1.5, a_{22} = 0.05, a_{23} = 0.04, \gamma = 5, a_3 = 0.7, a_{33} = 0.02, q_2 = 0.01, E_2 = 10.$$

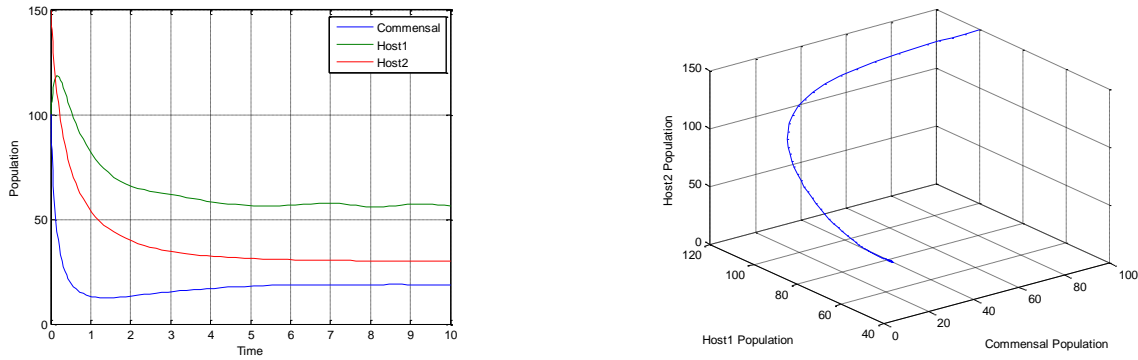


Figure 9.6. The variation of population against time, initially with $x = 100, y = 100, z = 150$ and the variation of population among commensal population, host1 population, and host2 population

Example 7.

$$a_1 = 2, a_{11} = 0.01, a_{12} = 0.45, a_{13} = 0.08, q_1 = 0.2, \omega = 10, E_1 = 10, a_2 = 1, a_{22} = 0.5, a_{23} = 0.32, \gamma = 30, a_3 = 3, a_{33} = 0.3, q_2 = 0.01, E_2 = 10.$$

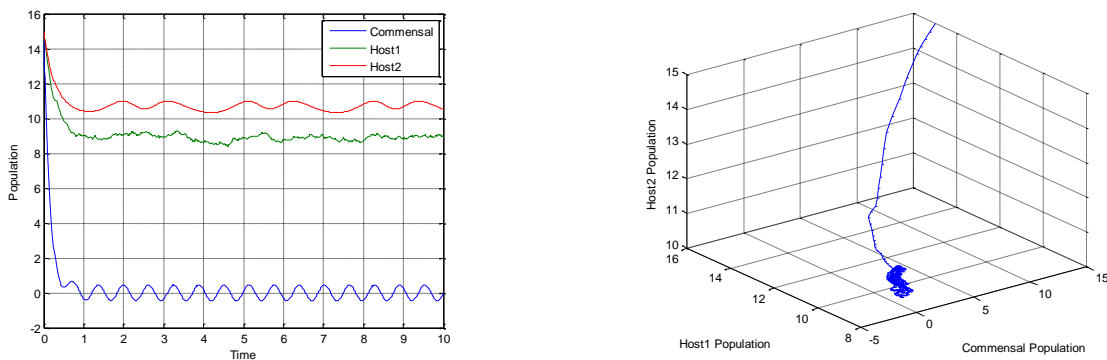


Figure 9.7. The variation of population against time, initially with $x = 15, y = 15, z = 15$ and the variation of population among commensal population, host1 population, and host 2 population

Example 8.

$$a_1 = 3, a_{11} = 0.01, a_{12} = 0.45, a_{13} = 0.08, q_1 = 0.2, \omega = 5, E_1 = 10, a_2 = 2, a_{22} = 0.5, a_{23} = 0.32, \gamma = 30, a_3 = 1, a_{33} = 0.2, q_2 = 0.01, E_2 = 10.$$

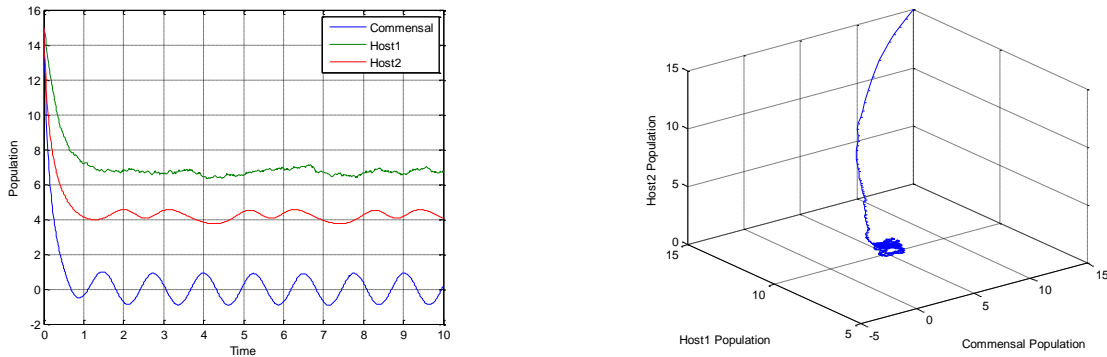


Figure 9.8. The variation of population against time, initially with $x = 15, y = 15, z = 15$ and the variation of population among commensal population, host1 population, and host 2 population

Example 9.

$$a_1 = 3, a_{11} = 0.01, a_{12} = 0.45, a_{13} = 0.08, q_1 = 0.2, \omega = 5, E_1 = 10, a_2 = 2, a_{22} = 0.5, a_{23} = 0.32, \\ \gamma = 30, a_3 = 1.2, a_{33} = 0.2, q_2 = 0.1, E_2 = 10$$

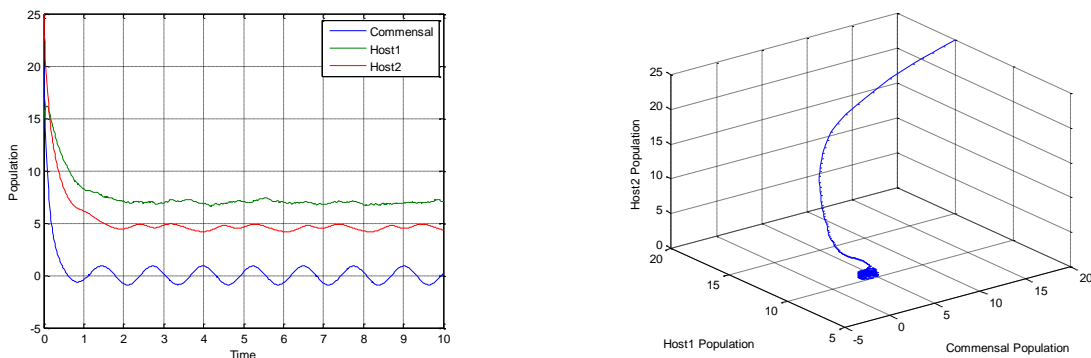


Figure 9.9. The variation of population against time, initially with $x = 20, y = 15, z = 25$ and the variation of population among commensal population, host1 population, and host 2 population

10. Conclusions

In this paper a model of a distinctive three species syn-ecosystem with a stochastic term has been invented. At first we discussed the model without the stochastic term and examined the survival of the equilibrium points as well as local and global stabilities by using Routh-Hurwitz criteria and Lyapunov function respectively. We also analyzed the idea of a bionomic equilibrium and computed optimal harvesting policy through Pontryagin’s maximum principle. Later we computed the population intensities of fluctuations (variances) around the positive equilibrium (due to noise). The given MATLAB simulations exhibit the theoretical analysis. Figures 9.1-9.4 represent the stability of the deterministic model and figures 9.5-9.9 show the fluctuations in the

population densities around the mean state. The conclusion is that the noise on the system results in immense variances of oscillations around the equilibrium point causing our system to be chaotic. So in our model, standard deviations act as an unpredictable dynamic force that influences bulky rise and fall for intensities in the region of the equilibrium point.

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