Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 8, Issue 2 (December 2013), pp. 553 – 572

Exact Traveling Wave Solutions of Nonlinear PDEs in Mathematical Physics Using the Modified Simple Equation Method

E. M. E. Zayed
Department of Mathematics
Zagazig University
Zagazig, Egypt
e.m.e.zayed@hotmail.com

A. H. Arnous
Department of Engineering Mathematics and Physics
Higher Institute of Engineering
El Shorouk, Egypt
ahmed.h.arnous@gmail.com

Received: November 27, 2012; Accepted: August 21, 2013

Abstract

In this article, we apply the modified simple equation method to find the exact solutions with parameters of the (1+1)-dimensional nonlinear Burgers-Huxley equation, the (2+1) dimensional cubic nonlinear Klein-Gordon equation and the (2+1)-dimensional nonlinear Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation. The new exact solutions of these three equations are obtained. When these parameters are given special values, the solitary solutions are obtained.

Keywords: Nonlinear evolution equations, exact solutions, solitary wave solutions, modified simple equation method

AMS-MSC 2010 No: 35Q20, 35K99, 35P05

1. Introduction

In science, many important phenomena in various fields can be described by nonlinear partial differential equations. Searching for exact soliton solutions of these equations plays an important role in the study on the dynamics of those phenomena.

With the development of soliton theory, many powerful methods for obtaining these exact solutions are presented, such as the inverse scattering transformation method [Ablowitz and Clarkson (1991)], the Backlund transformation method [Miura (1978)], the Hirota bilinear method [Hirota (1971)], the extended tanh-function method [Fan (2000)], the sine-cosine method [Wazwaz (2004)], the exp-function method [He and Wu (2006)], the F-expansion method [Zhang and Xia (2006)], the Jacobi-elliptic function method [Lu (2005)], the (G'/G)-expansion method [Wang et al. (2008)], the modified simple equation method [Jawad et al. (2010), Zayed (2011), Zayed and Ibrahim (2012), Zayed and Arnous (2012)], and so on.

To exemplify the application of the modified simple equation method, we will consider the exact wave solutions of three nonlinear partial differential equations, namely, the (1+1) nonlinear Burgers-Huxley equation, the (2+1)-dimensional cubic nonlinear Klein-Gordon equation and the (2+1)-dimensional nonlinear KP-BBM equation.

The rest of this article is organized as follows: In Section 2, the description of the modified simple equation method is given. In Section 3, we apply this method to the three nonlinear equations indicated above. In Section 4, conclusions are given.

2. Description of the Modified Simple Equation Method

Suppose we have a nonlinear evolution equation in the form:

$$F(u, u_t, u_x, u_{xx}, ...) = 0, \tag{1}$$

where F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved.

In the following, we give the main steps of this method [Jawad et al. (2010), Zayed (2011), Zayed and Ibrahim (2012), Zayed and Arnous (2012)]:

Step 1. We use the wave transformation:

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{2}$$

where c is a nonzero constant, to reduce equation (1) to the following ODE:

$$P(u, u', u'') = 0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = d/d\xi$.

Step 2. We suppose that Equation (3) has the formal solution

$$u(\xi)=\sum_{k=0}^{N} A_k \left[\frac{\psi'(\xi)}{\psi(\xi)}\right]^k, \tag{4}$$

where A_k are constants to be determined, such that $A_N \neq 0$. The function $\psi(\xi)$ is an unknown function to be determined later, such that $\psi' \neq 0$.

Step 3. We determine the positive integer N in Equation (4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in the equation (3).

Step 4. We substitute (4) into (3), calculate all the necessary derivatives $u', u'', \ldots$, of the unknown function $u(\xi)$ and account for the function $\psi(\xi)$. As a result of this substitution, we get a polynomial of ψ^{-j}, $(j=0,1,\ldots)$. In this polynomial, we gather all the terms of the same power of ψ^{-j}, $(j=0,1,\ldots)$, and we equate with zero all the coefficients of this polynomial. This operation yields a system of equations which can be solved to find A_k and $\psi(\xi)$. Consequently, we can get the exact solutions of the equation (1).

3. Applications

In this section, we apply the modified simple equation method to find the exact wave solutions and then the solitary wave solutions of the following nonlinear partial differential equations:

Example: 1. The (1+1)-dimensional nonlinear Burgers-Huxley Equation

This equation is well known [Yefimova and Kudryashov (2004), Kheiri et al. (2011), Kudryashov and Loguinova (2008)] and has the form:

$$u_t + u_{xx} + 3uu_x + \alpha u + u^2 + u^3 = 0, \tag{5}$$

where α is a nonzero constant. The solutions of the equation (5) have been investigated by using the Cole-Hopf transformation [Yefimova and Kudryashov (2004)], the (G'/G) -expansion method [Kheiri et al. (2011)] and the extended simple equation method [Kudryashov and Loguinova (2008)]. Let us now investigate the equation (5) using the modified simple equation method. To this end, we use the transformation (2) to reduce the equation (5) to the following ODE:

$$-cu' + u'' + 3uu' + \alpha u + u^2 + u^3 = 0. \tag{6}$$

Balancing u'' with u^3 yields $N=1$. Consequently, the equation (6) has the formal solution

$$u(\xi) = A_0 + A_1\left(\frac{\psi'}{\psi}\right), \tag{7}$$

where A_0 and A_1 are constants to be determined such that $A_1 \neq 0$ and $\psi' \neq 0$. It is easy to see that

$$u' = A_1\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right), \tag{8}$$

and

$$u'' = A_1\left(\frac{\psi'''}{\psi} - \frac{3\psi'\psi''}{\psi^2} + \frac{2\psi'^3}{\psi^3}\right). \tag{9}$$

Substituting (7) - (9) into (6) and equating all the coefficients of $\psi^0, \psi^{-1}, \psi^{-2}, \psi^{-3}$ to zero, we respectively obtain

$$\psi^0 : A_0(A_0^2 + A_0 + \alpha) = 0, \tag{10}$$

$$\psi^{-1} : A_1\psi''' + A_1\psi''(3A_0 - c) + A_1\psi'(3A_0^2 + 2A_0 + \alpha) = 0, \tag{11}$$

$$\psi^{-2} : A_1\psi'^2(c - 3A_0 + A_1 + 3A_0A_1) + 3\psi'\psi''A_1(A_1 - 1) = 0, \tag{12}$$

$$\psi^{-3} : A_1\psi'^3(A_1^2 - 3A_1 + 2) = 0. \tag{13}$$

Since $A_1 \neq 0$ and $\psi' \neq 0$, we deduce from (10) and (13) that

$$A_0 = 0, \quad A_0^2 + A_0 + \alpha = 0, \quad A_1 = 1, \quad A_1 = 2. \tag{14}$$

Let us now discuss the following cases:

Case 1. If $A_0 = 0, \ A_1 = 1$

In this case, equations (11) and (12) reduce to

$$\psi''' - c\psi'' + \alpha\psi' = 0, \tag{15}$$

$$(c+1)\psi'^2 = 0. \tag{16}$$

From equations (15) and (16), we get $c=-1$ and

$$\psi'''+\psi''+\alpha\psi'=0, \tag{17}$$

which has the solution

$$\psi=c_1+c_2\exp\left[(-1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]+c_3\exp\left[-(1+\sqrt{1-4\alpha})\frac{\xi}{2}\right], \tag{18}$$

where c_1 and c_2 are constants and $\alpha\le\frac{1}{4}$. Differentiating (18) with respect to ξ, we have

$$\begin{aligned}\psi'=&\frac{c_2(-1+\sqrt{1-4\alpha})}{2}\exp\left[(-1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]-\\ &\frac{c_3(1+\sqrt{1-4\alpha})}{2}\exp\left[-(1+\sqrt{1-4\alpha})\frac{\xi}{2}\right].\end{aligned} \tag{19}$$

Substituting (18) and (19) into (7), we obtain

$$\begin{aligned}u(\xi)=&\frac{c_2\left(\frac{-1+\sqrt{1-4\alpha}}{2}\right)\exp\left[(-1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]}{c_1+c_2\exp\left[(-1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]+c_3\exp\left[-(1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]}-\\ &\frac{c_3\left(\frac{1+\sqrt{1-4\alpha}}{2}\right)\exp\left[-(1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]}{c_1+c_2\exp\left[(-1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]+c_3\exp\left[-(1+\sqrt{1-4\alpha})\frac{\xi}{2}\right]}.\end{aligned} \tag{20}$$

If we set $c_1=0$ and $c_3=\pm c_2$, we have the following solitary wave solutions

$$u_1(x,t)=-\frac{1}{2}\left\{1-\sqrt{1-4\alpha}\tanh\left[\frac{\sqrt{1-4\alpha}}{2}(x+t)\right]\right\}, \tag{21}$$

and

$$u_2(x,t) = -\frac{1}{2}\left\{1-\sqrt{1-4\alpha}\coth\left[\frac{\sqrt{1-4\alpha}}{2}(x+t)\right]\right\}. \tag{22}$$

Case 2. If $A_0 = 0, \ \ A_1 = 2.$

In this case, equations (11) and (12), respectively, reduce to

$$\psi''' - c\psi'' + \alpha\psi' = 0, \tag{23}$$

and

$$\psi'((c+2)\psi' + 3\psi'') = 0. \tag{24}$$

Since $\psi' \neq 0$, we deduce from (23) and (24) that

$$\psi''' / \psi'' = \frac{c^2 + 2c + 3\alpha}{c+2}, \quad c \neq -2. \tag{25}$$

Integrating (25) yields

$$\psi'' = c_1 \exp\left[\left(\frac{c^2+2c+3\alpha}{c+2}\right)\xi\right]. \tag{26}$$

From (24) and (26) we get

$$\psi' = \frac{-3c_1}{c+2}\exp\left[\left(\frac{c^2+2c+3\alpha}{c+2}\right)\xi\right], \tag{27}$$

and consequently,

$$\psi = c_2 - \frac{3c_1}{c^2+2c+3\alpha}\exp\left[\left(\frac{c^2+2c+3\alpha}{c+2}\right)\xi\right], \tag{28}$$

where c_1 and c_2 are constants of integration. Now, the exact wave solution of the equation (5) has the form:

$$u(x,t)=\frac{-6c_1}{c+2}\left\{\frac{\exp\left[\left(\frac{c^2+2c+3\alpha}{c+2}\right)(x-ct)\right]}{c_2-\frac{3c_1}{c^2+2c+3\alpha}\exp\left[\left(\frac{c^2+2c+3\alpha}{c+2}\right)(x-ct)\right]}\right\}. \tag{29}$$

If we set $c_1=\frac{c^2+2c+3\alpha}{3}$ and $c_2=\pm 1$ in (29) we have the following solitary wave solutions, respectively, as:

$$u_1(x,t)=\left(\frac{c^2+2c+3\alpha}{c+2}\right)\left\{1+\coth\left[\frac{1}{2}\left(\frac{c^2+2c+3\alpha}{c+2}\right)(x-ct)\right]\right\}, \tag{30}$$

and

$$u_2(x,t)=\left(\frac{c^2+2c+3\alpha}{c+2}\right)\left\{1+\tanh\left[\frac{1}{2}\left(\frac{c^2+2c+3\alpha}{c+2}\right)(x-ct)\right]\right\}. \tag{31}$$

Case 3. $A_0^2+A_0+\alpha=0,\ A_1=1$. In this case, equations (11) and (12), resprctively, reduce to

$$\psi'''+(3A_0-c)\psi''+(3A_0^2+2A_0+\alpha)\psi'=0, \tag{32}$$

and

$$(c+1)\psi'^2=0. \tag{33}$$

Since $A_0^2+A_0+\alpha=0,$ we get

$$3A_0^2+2A_0+\alpha=2A_0^2+A_0. \tag{34}$$

From equations (32), (33) and (34) we have $c=-1$ and

$$\psi'''+(3A_0+1)\psi''+A_0(2A_0+1)\psi'=0, \tag{35}$$

which has the solution

$$\psi=c_1+c_2\exp[-A_0\xi]+c_3\exp[-(2A_0+1)\xi]. \tag{36}$$

Differentiating equation (36), we get

$$\psi' = -A_0 c_2 \exp[-A_0\xi] - (2A_0+1)c_3 \exp[-(2A_0+1)\xi]. \tag{37}$$

Substituting equations (36) and (37) into (7), we have

$$u(\xi) = A_0 - \frac{A_0 c_2 \exp[-A_0\xi] + (2A_0+1)c_3 \exp[-(2A_0+1)\xi]}{c_1 + c_2 \exp[-A_0\xi] + c_3 \exp[-(2A_0+1)\xi]}. \tag{38}$$

If we set $c_1 = 0$ and $c_3 = \pm c_2$ in (38), we obtain the following solitary wave solutions, respectively, as:

$$u_1(x,t) = \frac{-(A_0+1)}{2}\left\{1 - \tanh\left[\frac{1}{2}(A_0+1)(x+t)\right]\right\}, \tag{39}$$

and

$$u_2(x,t) = \frac{-(A_0+1)}{2}\left\{1 - \coth\left[\frac{1}{2}(A_0+1)(x+t)\right]\right\}. \tag{40}$$

Case 4. $A_0^2 + A_0 + \alpha = 0, \quad A_1 = 2$

In this case, equations (11) and (12), respectively, reduce to

$$\psi''' + (3A_0 - c)\psi'' + (3A_0^2 + 2A_0 + \alpha)\psi' = 0, \tag{41}$$

and

$$\psi'[(3A_0 + c + 2)\psi' + 3\psi''] = 0. \tag{42}$$

Since $\psi' \neq 0$, we deduce from (41), (42) and using (34) that

$$\psi''' / \psi'' = \frac{3\alpha + c(c+2)}{3A_0 + c + 2}. \tag{43}$$

Integrating (43) yields

$$\psi'' = c_1 \exp\left[\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)\xi\right]. \tag{44}$$

From (42) and (44) we have

$$\psi' = \frac{-3}{3A_0 + c + 2}\psi'' = \frac{-3c_1}{3A_0 + c + 2}\exp\left[\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)\xi\right], \tag{45}$$

and consequently, we get

$$\psi = c_2 - \frac{3c_1}{3\alpha + c(c+2)}\exp\left[\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)\xi\right], \tag{46}$$

where c_1 and c_3 are arbitrary constants of integration. Now, the exact wave solution of equation (5) has the form:

$$u(x,t) = A_0 - \frac{6c_1}{3A_0 + c + 2}\left\{\frac{\exp\left[\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)(x - ct)\right]}{c_2 - \left(\frac{3c_1}{3\alpha + c(c+2)}\right)\exp\left[\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)(x - ct)\right]}\right\}, \tag{47}$$

where $A_0 = \frac{1}{2}(-1 \pm \sqrt{1 - 4\alpha})$ and $\alpha \le \frac{1}{4}$. If we set $c_1 = \frac{3\alpha + c(c+2)}{3}$ and $c_2 = \pm 1$ in (47), we have respectively the following solitary wave solutions:

$$u_1(x,t) = A_0 + \left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)\left\{1 + \coth\left[\frac{1}{2}\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)(x - ct)\right]\right\}, \tag{48}$$

$$u_2(x,t) = A_0 + \left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)\left\{1 + \tanh\left[\frac{1}{2}\left(\frac{3\alpha + c(c+2)}{3A_0 + c + 2}\right)(x - ct)\right]\right\}. \tag{49}$$

Example: 2. The (2+1)-Dimensional Cubic Nonlinear Klein-Gordon Equation

This equation is well known [Wang and Zhang (2007), Zayed (2011)] and has the form:

$$u_{xx} + u_{yy} - u_{tt} + \alpha u - \beta u^3 = 0, \tag{50}$$

where α and β are nonzero constants. The solution of the equation (50) has been investigated using the multi-function expansion method [Wang and Zhang (2007)] and the (G'/G) -expansion method [Zayed (2011)]. In this section we investigate the equation (50) by the modified simple equation method. To this end, we use the wave transformation

$$u(x, y, t) = u(\xi), \quad \xi = x + y - ct, \tag{51}$$

to reduce the equation (50) to the following ODE :

$$(2 - c^2)u'' + \alpha u - \beta u^3 = 0, \tag{52}$$

where $(2 - c^2) \neq 0$. Balancing u'' with u^3 yields $N = 1$. Consequently, the equation (52) has the formal solution (7). Substituting (7) - (9) into (52) and equating all the coefficients of $\psi^0, \psi^{-1}, \psi^{-2}, \psi^{-3}$ to zero, we, respectively, obtain

$$\psi^0 : A_0(\alpha - \beta A_0^2) = 0, \tag{53}$$

$$\psi^{-1} : A_1(2 - c^2)\psi''' + A_1\psi'(\alpha - 3\beta A_0^2) = 0, \tag{54}$$

$$\psi^{-2} : 3A_1\psi'((c^2 - 2)\psi'' - \beta A_0 A_1 \psi') = 0, \tag{55}$$

and

$$\psi^{-3} : A_1\psi'^3(2(2 - c^2) - \beta A_1^2) = 0. \tag{56}$$

Since $A_1 \neq 0$ and $\psi' \neq 0$, we deduce from (53) and (56) that

$$A_0 = 0, \quad A_0 = \pm\sqrt{\frac{\alpha}{\beta}}, \quad A_1 = \pm\sqrt{\frac{2(2 - c^2)}{\beta}}, \tag{57}$$

where $\dfrac{\alpha}{\beta} > 0$ and $\dfrac{2 - c^2}{\beta} > 0$.

Let us now discuss the following cases:

Case 1. $A_0 = 0, \quad A_1 = \pm\sqrt{\dfrac{2(2 - c^2)}{\beta}}$. In this case the equations (54) and (55) yield $\psi' = 0$. This case is rejected.

Case 2. $A_0 = \pm\sqrt{\dfrac{\alpha}{\beta}}, \quad A_1 = \pm\sqrt{\dfrac{2(2 - c^2)}{\beta}}$. Since $\psi' \neq 0$, we deduce from (54) and (55) that

$$(2 - c^2)\psi''' - 2\alpha\psi' = 0, \tag{58}$$

$$(c^2-2)\psi''-\sqrt{2\alpha(2-c^2)}\psi'=0. \tag{59}$$

From (58) and (59) we deduce that

$$\psi'''/\psi''=-\sqrt{\frac{2\alpha}{2-c^2}}. \tag{60}$$

Consequently, we get

$$\psi''=c_1\exp\left[-\sqrt{\frac{2\alpha}{2-c^2}}\xi\right]. \tag{61}$$

From (59) and (61) we have

$$\psi'=-\sqrt{\frac{2-c^2}{2\alpha}}\psi''=-c_1\sqrt{\frac{2-c^2}{2\alpha}}\exp\left[-\sqrt{\frac{2\alpha}{2-c^2}}\xi\right], \tag{62}$$

and consequently, we have

$$\psi=c_2+\frac{c_1(2-c^2)}{2\alpha}\exp\left[-\sqrt{\frac{2\alpha}{2-c^2}}\xi\right], \tag{63}$$

where c_1 and c_2 are arbitrary constants of integration.

Now, the exact wave solution of the equation (50) has the form

$$u(\xi)=\pm\sqrt{\frac{\alpha}{\beta}}\mp\frac{(2-c^2)}{\sqrt{\alpha\beta}}\left\{\frac{c_1\exp\left[-\sqrt{\left(\frac{2\alpha}{2-c^2}\right)}\xi\right]}{c_2+\frac{c_1(2-c^2)}{2\alpha}\exp\left[-\sqrt{\left(\frac{2\alpha}{2-c^2}\right)}\xi\right]}\right\}. \tag{64}$$

If we set $c_1=\frac{2\alpha}{2-c^2}$ and $c_2=\pm 1$ in (64) we have, respectively, the following solitary wave solutions

$$u_1(x,y,t)=\pm\sqrt{\frac{\alpha}{\beta}}\tanh\left[\sqrt{\frac{\alpha}{2(2-c^2)}}(x+y-ct)\right], \tag{65}$$

$$u_2(x,y,t) = \pm\sqrt{\frac{\alpha}{\beta}}\coth\left[\sqrt{\frac{\alpha}{2(2-c^2)}}(x+y-ct)\right]. \tag{66}$$

Example: 3. The (2+1)-Dimensional Nonlinear KP-BBM Equation

This equation is well known [Zayed and Al-Joudi (2010), Wazwaz (2008)] and has the form

$$(u_t + u_x - \alpha(u)_x^2 - \beta u_{xxt})_x + \gamma u_{yy} = 0, \tag{67}$$

where α, β, γ are nonzero constants. The solution of the equation (67) has been investigated using the auxiliary equation method [Zayed and Al-Joudi (2010)] and the extended tanh-function method [Wazwaz (2008)]. In this section we will solve the equation (67) by using the modified simple equation method. To this end, we use the wave transformation (51) to reduce the equation (67) to the following ODE:

$$(-cu' + u' - \alpha(u^2)' + c\beta u''')' + \gamma u'' = 0. \tag{68}$$

By integrating the equation (68) twice with zero constants of integration, we get

$$(\gamma - c + 1)u - \alpha u^2 + c\beta u'' = 0. \tag{69}$$

Balancing u'' with u^2 yields $N = 2$. Consequently, the equation (69) has the formal solution:

$$u(\xi) = A_0 + A_1\left(\frac{\psi'}{\psi}\right) + A_2\left(\frac{\psi'}{\psi}\right)^2, \tag{70}$$

where A_0, A_1 and A_2 are constants to be determined such that $A_2 \neq 0$ and $\psi' \neq 0$. It is easy to see that

$$u'(\xi) = A_1\left(\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right) + 2A_2\left(\frac{\psi'\psi''}{\psi^2} - \frac{\psi'^3}{\psi^3}\right), \tag{71}$$

and

$$u''(\xi) = A_1\left(\frac{\psi'''}{\psi} - \frac{3\psi'\psi''}{\psi^2} + \frac{2\psi'^3}{\psi^3}\right) + 2A_2\left(\frac{\psi'\psi'''}{\psi^2} + \frac{\psi''^2}{\psi^2} - \frac{5\psi'^2\psi''}{\psi^3} + \frac{3\psi'^4}{\psi^4}\right). \tag{72}$$

Substituting (70) and (72) into (69) and equating all the coefficients of $\psi^{0},\psi^{-1},\psi^{-2},\psi^{-3},\psi^{-4}$ to zero, we deduce, respectively, that

$$\psi^{0}:(\gamma+1-c)A_0-\alpha A_0^2=0, \tag{73}$$

$$\psi^{-1}:A_1[(\gamma+1-c)\psi'-2\alpha A_0\psi'+c\beta\psi''']=0, \tag{74}$$

$$\psi^{-2}:-\alpha A_1^2\psi'^2+(\gamma+1-c)A_2\psi'^2-2\alpha A_0A_2\psi'^2-3c\beta A_1\psi'\psi''+2c\beta A_2[\psi''^2+\psi'\psi''']=0, \tag{75}$$

$$\psi^{-3}:2c\beta A_1\psi'^3-2\alpha A_1A_2\psi'^3-10c\beta A_2\psi'^2\psi''=0, \tag{76}$$

and

$$\psi^{-4}:6c\beta A_2\psi'^4-\alpha A_2^2\psi'^4=0. \tag{77}$$

From equations (73) and (77), we have the following results

$$A_0=0,\quad A_0=\frac{\gamma-c+1}{\alpha},\quad A_2=\frac{6c\beta}{\alpha}, \tag{78}$$

where $\gamma-c+1\neq 0$.

Let us now discuss the following cases:

Case 1. $A_0=0$ and $A_1=0$, then $\psi'=0$. This case is rejected.

Case 2. $A_0=0$ and $A_1\neq 0$, then we deduce from equations (74) - (76) that

$$(\gamma-c+1)\psi'+c\beta\psi'''=0, \tag{79}$$

$$-\alpha A_1^2\psi'^2+(\gamma-c+1)A_2\psi'^2-3c\beta A_1\psi'\psi''+2c\beta A_2(\psi''^2+\psi'\psi''')=0, \tag{80}$$

$$2c\beta A_1\psi'-2\alpha A_1A_2\psi'-10c\beta A_2\psi''=0. \tag{81}$$

From (79) and (81), we have

$$\psi'=\frac{-c\beta\psi'''}{\gamma-c+1}=\frac{-6c\beta\psi''}{\alpha A_1}, \tag{82}$$

and consequently, we get

$$\psi''' / \psi'' = \frac{6(\gamma - c + 1)}{\alpha A_1}. \tag{83}$$

Integrating (83), we get

$$\psi'' = c_1 \exp\left[\frac{6(\gamma - c + 1)}{\alpha A_1}\xi\right], \tag{84}$$

and substituting from (84) into (82), we have

$$\psi' = \frac{-6c\beta c_1}{\alpha A_1}\exp\left[\frac{6(\gamma - c + 1)}{\alpha A_1}\xi\right]. \tag{85}$$

Integrating (85), we have

$$\psi = c_2 - \frac{c\beta}{\gamma - c + 1}c_1 \exp\left[\frac{6(\gamma - c + 1)}{\alpha A_1}\xi\right], \tag{86}$$

where c_1 and c_2 are arbitrary constants of integration. Substituting equation (82) into (80), we get

$$A_1 = \pm\frac{6}{\alpha}\sqrt{-c\beta(\gamma - c + 1)}, \tag{87}$$

where $c\beta(\gamma - c + 1) < 0$. Now, the exact wave solution of the equation (67) in this case has the form

$$\begin{aligned} u(\xi) = {} & \frac{-6c\beta c_1}{\alpha}\left\{\frac{\exp\left[\pm\sqrt{\frac{-(\gamma - c + 1)}{c\beta}}\xi\right]}{c_2 - \frac{c\beta c_1}{\gamma - c + 1}\exp\left[\pm\sqrt{\frac{-(\gamma - c + 1)}{c\beta}}\xi\right]}\right\} \\ & - \frac{6c^2\beta^2 c_1^2}{\alpha(\gamma - c + 1)}\left\{\frac{\exp\left[\pm\sqrt{\frac{-(\gamma - c + 1)}{c\beta}}\xi\right]}{c_2 - \frac{c\beta c_1}{\gamma - c + 1}\exp\left[\pm\sqrt{\frac{-(\gamma - c + 1)}{c\beta}}\xi\right]}\right\}^2. \end{aligned} \tag{88}$$

If we set $c_1 = \frac{\gamma - c + 1}{c\beta}$ and $c_2 = \pm 1$ in (88), we have the following solitary wave solutions:

$$u_1(x,y,t)=\frac{-3(\gamma-c+1)}{2\alpha}\operatorname{csc}h^2\left[\sqrt{\frac{-(\gamma-c+1)}{4c\beta}}(x+y-ct)\right], \tag{89}$$

and

$$u_2(x,y,t)=\frac{3(\gamma-c+1)}{2\alpha}\operatorname{sec}h^2\left[\sqrt{\frac{-(\gamma-c+1)}{4c\beta}}(x+y-ct)\right]. \tag{90}$$

Case 3. $A_0=\frac{\gamma-c+1}{\alpha}$ and $A_1=0$, then $\psi'=0$. This case is rejected.

Case 4. $A_0=\frac{\gamma-c+1}{\alpha}$ and $A_1\neq 0$, then we deduce from equations (74) - (76) that

$$-(\gamma-c+1)\psi'+c\beta\psi'''=0, \tag{91}$$

$$-\alpha A_1^2\psi'^2-\frac{6c\beta(\gamma-c+1)}{\alpha}\psi'^2-3c\beta A_1\psi'\psi''+\frac{12c^2\beta^2}{\alpha}[\psi''^2+\psi'\psi''']=0, \tag{92}$$

$$\psi'^2[A_1\psi'+\frac{6c\beta}{\alpha}\psi'']=0. \tag{93}$$

From equations (91) and (93), we have

$$\psi'=\frac{c\beta\psi'''}{\gamma-c+1}=\frac{-6c\beta\psi''}{\alpha A_1}, \tag{94}$$

and consequently, we get

$$\psi'''/\psi''=\frac{-6(\gamma-c+1)}{\alpha A_1}. \tag{95}$$

Integrating the equation (95), we obtain

$$\psi''=c_1\exp\left[\frac{-6(\gamma-c+1)}{\alpha A_1}\xi\right], \tag{96}$$

and substituting the equation (96) into (94), we have

$$\psi'=\frac{-6c\beta}{\alpha A_1}c_1\exp\left[\frac{-6(\gamma-c+1)}{\alpha A_1}\xi\right]. \tag{97}$$

Integrating (97), we have

$$\psi = c_2 + \frac{c\beta}{\gamma - c + 1} c_1 \exp\left[\frac{-6(\gamma - c + 1)}{\alpha A_1}\xi\right], \tag{98}$$

where c_1 and c_2 are arbitrary constants of integration. Substituting the equation (94) into (92), we get

$$A_1 = \pm\frac{6}{\alpha}\sqrt{c\beta(\gamma - c + 1)}, \tag{99}$$

where $c\beta(\gamma - c + 1) > 0$. Now, the exact wave solution of the equation (67) in this case has the form

$$u(\xi) = \frac{\gamma - c + 1}{\alpha} - \frac{6c\beta c_1}{\alpha}\left\{\frac{\exp\left[\mp\sqrt{\frac{\gamma - c + 1}{c\beta}}\xi\right]}{c_2 + \frac{c\beta c_1}{\gamma - c + 1}\exp\left[\mp\sqrt{\frac{\gamma - c + 1}{c\beta}}\xi\right]}\right\} + \frac{6c^2\beta^2 c_1^2}{\alpha(\gamma - c + 1)}\left\{\frac{\exp\left[\mp\sqrt{\frac{\gamma - c + 1}{c\beta}}\xi\right]}{c_2 + \frac{c\beta c_1}{\gamma - c + 1}\exp\left[\mp\sqrt{\frac{\gamma - c + 1}{c\beta}}\xi\right]}\right\}^2. \tag{100}$$

If we set $c_1 = \frac{\gamma - c + 1}{c\beta}$ and $c_2 = \pm 1$ in the equation (100), we have respectively the following solitary wave solutions:

$$u_1(x, y, t) = \frac{\gamma - c + 1}{\alpha}\left\{1 - \frac{3}{2}\operatorname{sec}h^2\left[\sqrt{\frac{\gamma - c + 1}{4c\beta}}(x + y - ct)\right]\right\}, \tag{101}$$

and

$$u_2(x, y, t) = \frac{\gamma - c + 1}{\alpha}\left\{1 + \frac{3}{2}\operatorname{csc}h^2\left[\sqrt{\frac{\gamma - c + 1}{4c\beta}}(x + y - ct)\right]\right\}. \tag{102}$$

4. Conclusions

In this article, we have applied the modified simple equation method to find the exact solutions of the (1+1)-dimensional nonlinear Burgers-Huxley equation, the (2+1)-dimensional cubic nonlinear Klein - Gordon equation and the (2+1)-dimensional nonlinear KP-BBM equation which play an important role in the mathematical physics.

On comparing our results of these equations using the modified simple equation method with the well- known results from other methods, we conclude that our results are different, new and not published elsewhere. Furthermore, the proposed method in this article is effective and can be applied to many other nonlinear partial differential equations.

Finally, the physical meaning of our new results in this article can be summarized as follows: The solutions (21), (31), (39), (49) and (65) represent the kink shaped solitary wave solutions, while the solutions (90) and (101) represent the bell shaped solitary wave solutions, (see also Figures 1-4).

Acknowledgment

The authors wish to thank the referees for their comments.

REFERENCES

Ablowitz, M. J. and Clarkson, P. A. (1991). *Solitons, nonlinear evolution equation and inverse scattering*, Cambridge University Press, New York.

Fan, E. G. (2000). Extended tanh-function method and its applications to nonlinear equations, *Phys. Lett.A* **277**, PP. 212 - 218.

He, J. H. and Wu, X. H. (2006). Exp-function method for nonlinear wave equations, *Chaos, solitons and Fractals* , **30**, PP. 700 - 708.

Hirota, R. (1971). Exact solutions of KdV equation for multiple collisions of solitons, *Phys.Rev.Lett.*, **27** , PP.1192 - 1194.

Jawad, A. J. M., Petkovic, M. D. and Biswas, A. (2010). Modified simple equation method for nonlinear evolution equations, *Appl. Math. Comput.*, **217**, PP. 869 - 877.

Kheiri, H. Moghaddam, M. R. and Vafael, V. (2011). Applications of the (G'/G)-expansion method for the Burgers, Burgers-Huxley and modified Burgers-KdV equations, *Pramana J.Phys.*, **76**, PP.831-842.

Kudryashov, N. A. and Loguinova, N. B. (2008). Extended simplest equation method for nonlinear differential equations, *Appl. Math. Comput.*, **205**, PP. 396 - 402.

Lu, D. (2005). Jacobi elliptic function solutions for two variant Boussinesq equations, Chaos, *Solitons and Fractals*, **24** , PP. 1373 - 1385.

Miura, M. (1978). *Backlund transformation*, Springer, Berlin.

Wang, M. Li, X. and Zhang, J. (2008). The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Phys. Lett. A.*, **372**,PP. 417 -423.

Wang, Z. and Zhang, H. Q. (2007). Many new kinds exact solutions to (2+1)-dimensional Burgers equation and Klein-Gordon equation used a new method with symbolic computation, *Appl. Math. Comput.*, **186**, PP. 693 - 704.

Wazwaz, A. M. (2008). The extended tanh -method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations, *Chaos, Solitons and Fractals*, **38**, PP. 1505 - 1516.

Wazwaz, A. M. (2004). A sine-cosine method for handling nonlinear wave equations, *Math. Comput. Model.* **40**, PP. 499 - 508.

Yefimova, O. Y. and Kudryashov, N. A. (2004). Exact solutions of the Burgers-Huxley equation, *J. Appl. Mech.*, **68**, PP. 413 - 420.

Zayed, E. M. E. (2011). A note on the modified simple equation method applied to Sharma-Tasso- Olver equation, *Appl. Math. Comput.*, **218**, PP. 3962 - 3964.

Zayed, E. M. E. and Al-Joudi, S. (2010). The auxiliary equationmethod and its applications for solving nonlinear partial differential equations, *Commu. Appl. Nonlinear Anal.*, **17**, PP.83 - 101.

Zayed, E. M. E. and Arnous, A. H. (2012). Exact solutions of the nonlinear ZK-MEW and the potential YTSF equations using the modified simple equation method, *AIP Conf. Proc.*, **1479**, PP. 2044-2048.

Zayed, E. M. E. and Ibrahim, S. A. Hoda. (2012). Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method, *Chin. Phys. Lett.*, **29**, PP. 060201-060204.

Zayed, E. M. E. (2011). The (G'/G)-expansion method combined with the Riccati equation for finding exact solutions of nonlinear PDEs, *J.Appl. Math. Informatics*, **29**, PP. 351 - 367.

Zhang, S. and Xia, T. C. (2006). A generalized F-expansion method and new exact solutions of Konopelchenko-Dukrovsky equation, *Appl. Math. Comput.*, **183**, PP. 1190 - 1200.

FIGURES

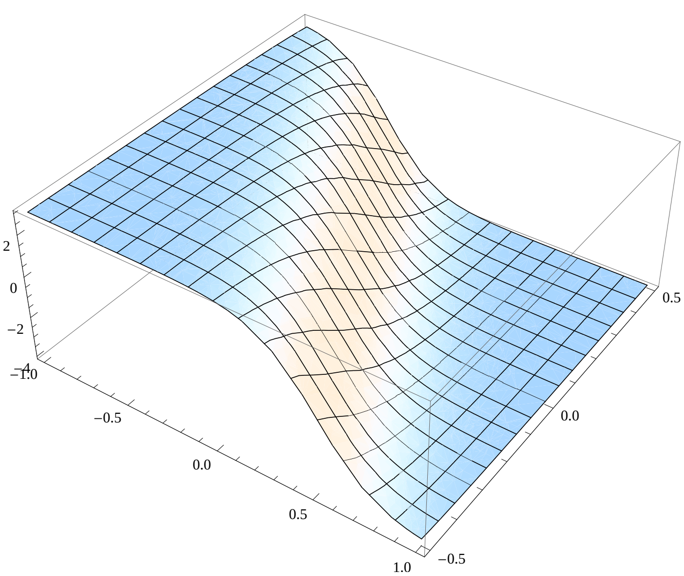

Figure 1. The plot of the solution (21), when $\alpha = -12$

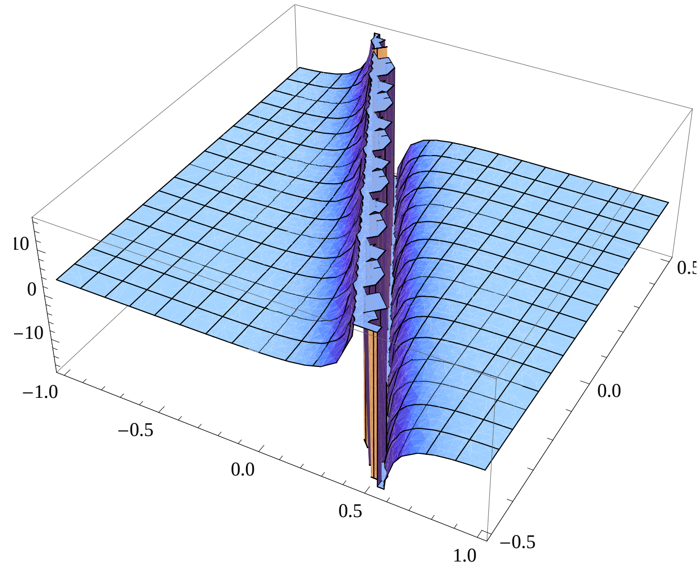

Figure 2. The plot of the solution (22), when $\alpha = -12$

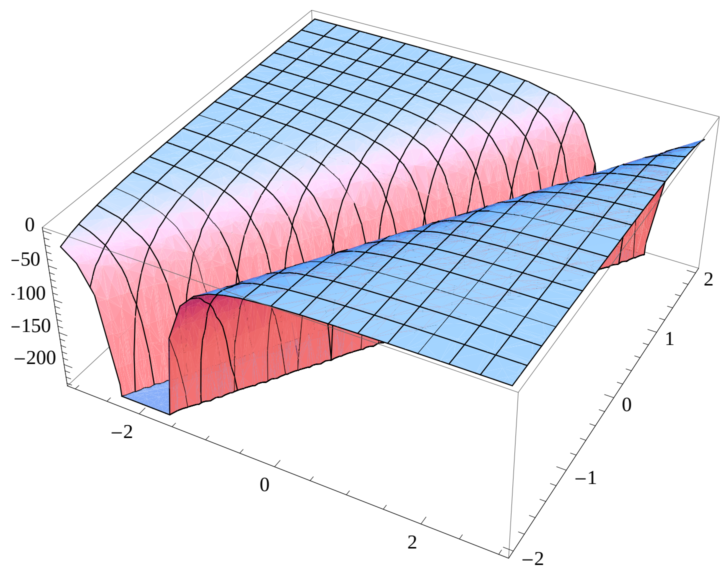

Figure 3. The plot of the solution (89), when $\alpha = 2, \beta = -9, \gamma = 4, c = 1, y = 0$

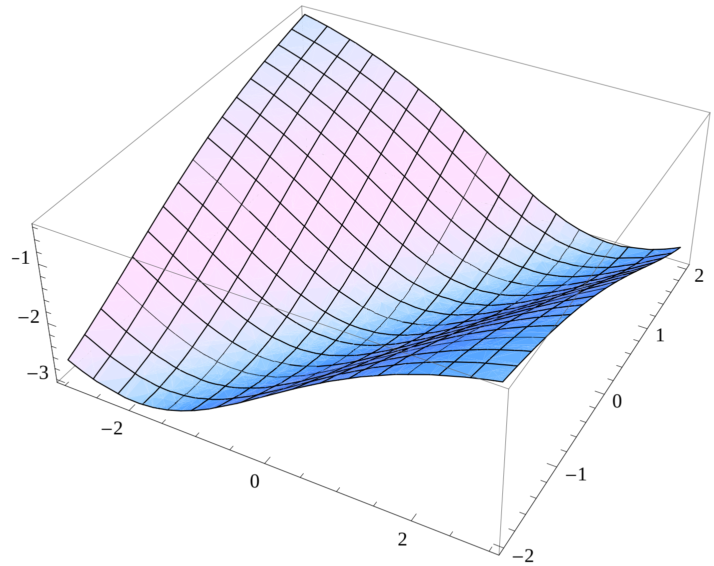

Figure 4. The plot of the solution (90), when $\alpha = 2, \beta = -9, \gamma = 4, c = 1, y = 0$