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Oscillation of Neutral Partial Dynamic Equations

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Abstract

This paper is concerned with the oscillation of solutions of a certain more general neutral type dynamic equation. We establish within the necessary and sufficient conditions for the oscillation of its solutions.

Keywords: Oscillation; Partial Dynamic Equation; Neutral Type

MSC 2010: 34K40, 34N05, 35B05

1. Introduction and Preliminaries

The theory of "dynamic equations" unifies the theories of differential equations and difference equations and it also extends these classical cases to cases "in between." The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which may be an arbitrary closed subset of the reals. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. They give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or often modeled by continuous dynamic systems), die out, say in winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population Bohner and Peterson (2001). In this paper we obtain some necessary and sufficient conditions for oscillation of neutral delay partial dynamic equation of the form

$$\begin{aligned}\left[u(x,t)-\alpha u(x,\phi(t))\right]^{\Delta^2} = a(t)Lu(x,t)+\sum_{k=1}^{s} a_k(t)Lu(x,\beta_k(t)) \\ -\sum_{j=1}^{n} q_j(t)u(x,\gamma_j(t)),\end{aligned} \tag{1}$$

where $t_0\in\mathbb{T}$, $\sup\{\mathbb{T}\}=\infty$, $(x,t)\in\Omega\times[t_0,\infty)_{\mathbb{T}}\equiv G$ and L is Laplacian in $\mathbb{T}^n$. We assume throughout this paper that

(H1) α is constant with $0<\alpha<1$;

(H2) $a,\ a_k,\ q_j\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$, $k=1,2,...,s;\ j=1,2,...,n$;

(H3 $\phi,\ \beta,\ \gamma\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{T}\right)$ are unbounded increasing functions satisfying $\phi(t),\ \beta(t),\ \gamma(t)\le t$ for all sufficiently large t.

The intervals with a $\mathbb{T}$ index below are used to denote the intersection of the usual interval with $\mathbb{T}$; i.e., $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$ for convenience. Consider the following boundary condition:

$$u^{\Delta_N}(x,t)=0,\qquad (x,t)\in\partial\Omega\times[t_0,\infty)_{\mathbb{T}}, \tag{2}$$

where N is the exterior normal vektor to $\partial\Omega$.

Definition 1. The function $u\in C^2_{rd}(G)\cap C^1_{rd}(\overline{G})$ is said to be a solution of the problem (1) and (2), if it satisfies (1) in the domain G and boundary condition (2).

Definition 2. The solution $u(x,t)$ of the problem (1) and (2) is said to be oscillatory in the domain $G=\Omega\times[t_0,\infty)_{\mathbb{T}}$, if for any positive number t_μ, there exists a point $(x_0,t_0)\in\Omega\times[t_\mu,\infty)_{\mathbb{T}}$ such that the condition $u(x_0,t_0)=0$ holds.

We will give a short introduction to the time scales calculus which will be used in the next sections

1.1. Basic Definitions

We start with the definitions given by Hilger (1988) to define the exponential function on a time scale.

Definition 3. A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 4. Let $a,b\in\mathbb{T}$ *and* $f\in C_{rd}$.

i) If $\mathbb{T}=\mathbb{R}$, then $\int_a^b f(t)\Delta t=\int_a^b f(t)dt$,

where the integral on the right is the usual Riemann integral from calculus.

ii) If $[a,b]$ consist of only isolated points, then

$$\int_a^b f(t)\Delta t=\begin{cases}\sum_{t\in[a,b)}\mu(t)f(t), & a<b\\ 0, & a=b\\ -\sum_{t\in[b,a)}\mu(t)f(t), & a>b.\end{cases}$$

Some useful definitions about partial dynamic equations from Bohner, M. and Guseinov, G. Sh. (2004) are the followings.

Definition 5. Let $n\in\mathbb{N}$ be fixed. Further, for each $i\in\{1,...,n\}$ let $\mathbb{T}_i$ denote a time scale, that is, $\mathbb{T}_i$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Let us set

$$\Lambda^n=\mathbb{T}_1\, x \ldots x\, \mathbb{T}_n=\{t=(t_1,...,t_n):t_i\in\mathbb{T}_i \;\; for \;\; all \;\; i\in\{1,...,n\}\}.$$

We call Λ^n an n - dimensional time scale.

Definition 6. Let σ_i and ρ_i denote, respectively, the forward and backward jump operators in $\mathbb{T}_i$. Remember that for $u\in\mathbb{T}_i$ the forward jump operator $\sigma_i:\mathbb{T}_i\to\mathbb{T}_i$ is defined by

$$\sigma_i(u)=\inf\{v\in\mathbb{T}_i : v>u\}$$

and the backward jump operator $\rho_i:\mathbb{T}_i\to\mathbb{T}_i$ is defined by

$$\rho_i(u)=\sup\{v\in\mathbb{T}_i : v<u\}.$$

If $\mathbb{T}_i$ has a left-scattered maximum M, then we define $\mathbb{T}_i^{\kappa} = \mathbb{T}_i \setminus \{M\}$, otherwise $\mathbb{T}_i^{\kappa} = \mathbb{T}_i$. If $\mathbb{T}_i$ has a right-scattered minimum m, then we define $(\mathbb{T}_i)_{\kappa} = \mathbb{T}_i \setminus \{m\}$, otherwise $(\mathbb{T}_i)_{\kappa} = \mathbb{T}_i$.

Definition 7. Let $f : \Lambda^n \to \mathbb{R}$ be a function. The partial delta derivative of f with respect to $t_i \in \mathbb{T}_i^{\kappa}$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1,\dots,t_{i-1},\sigma_i(t_i),t_{i+1},\dots t_n) - f(t_1,\dots,t_{i-1},s_i,t_{i+1},\dots t_n)}{\sigma_i(t_i) - s_i},$$

provided that this limit exists as a finite number, and is denoted by any of the following symbols:

$$\frac{\partial f(t_1,\dots,t_n)}{\Delta_i t_i},\ \frac{\partial f(t)}{\Delta_i t_i},\ \frac{\partial f}{\Delta_i t_i}(t),\ f_{t_i}^{\Delta_i}(t).$$

2. Main results

To obtain our main results, we need the following lemmas.

Lemma 1. Assume that $y(t) > 0$, $y^{\Delta}(t) < 0$ and $q_i(s) \neq 0$ for $s \in [t, T^*]$ and $t \geq T$ where T^* satisfies that $\gamma_i(T^*) = t$. Let $b = \max\{\phi(t), T - \min\{\gamma_i(t)\}\}$, and assume that the integral inequality

$$z(t) \geq \alpha z(\phi(t)) + \int_t^{\infty} q_i(s) z(\gamma_i(s)) \Delta s,\ t \geq T \tag{3}$$

has a continuous positive solution $y : [T-b, \infty)_{\mathbb{T}} \to (0, \infty)$ with $T - b \in \mathbb{T}$. Then the corresponding integral equation

$$x(t) = \alpha x(\phi(t)) + \int_t^{\infty} q_i(s) x(\gamma_i(s)) \Delta s,\ t \geq T \tag{4}$$

has a continuous positive solution $x : [T-b, \infty)_T \to (0, \infty)$.

Proof:

Define a set of functions Λ and a mapping $\hbar$ as follows:

$$\Lambda = \left\{ w \in C_{rd}\left([T-b, \infty)_{\mathbb{T}}, \mathbb{R}^+\right) : 0 \leq w(t) \leq 1,\ t \geq T - b \right\},$$

$$(\hbar w)(t)=\begin{cases}\dfrac{1}{z(t)}\left[\alpha w(\phi(t))z(\phi(t))+\displaystyle\int_t^{\infty} q_i(s)w(\gamma_i(s))z(\gamma_i(s))\Delta s\right], & t\geq T\\ \dfrac{t-T+b}{b}(\hbar w)(T)+1-\dfrac{t-T+b}{b}, & T-b\leq t\leq T.\end{cases}$$

It is easy to see from (3) that $\hbar\Lambda\subset\Lambda$ and $(\hbar w)(t)>0$ on $[T-b,T)$ for any $w\in\Lambda$. Define a sequence $\{w_k(t)\}$ in Λ as follows: $w_0(t)=1$ and $w_{k+1}(t)=(\hbar w_k)(t)$, $k=0,1,2,...$, for $t\geq T-b$. From (3), by induction, we have

$$0\leq w_{k+1}(t)\leq w_k(t)\leq 1,\ k=0,1,2,...,\ t\geq T-b.$$

Then, $w(t)=\lim_{k\to\infty} w_k(t)$, for $t\geq T-b$, exists and

$$w(t)=\frac{1}{z(t)}\left[\alpha w(\phi(t))z(\phi(t))+\int_t^{\infty} q_i(s)w(\gamma_i(s))z(\gamma_i(s))\Delta s\right],\ \text{for } t\geq T,$$

$$w(t)=\frac{t-T+b}{b}(w)(T)+1-\frac{t-T+b}{b},\ \text{for } T-b\leq t\leq T.$$

Set $x(t)=w(t)z(t)$. Then $x(t)$ satisfies (4) and $x(t)>0$ for $t\in[T-b,T)$. Clearly, $x(t)$ is continuous on $[T-b,T]$. Then, in view of (4), we see that $x(t)$ is continuous on $[T-b,\infty)$. Finally, it remains to show that $x(t)>0$ for $t\geq T-b$. Assume that there exists $t^*\geq T$ such that $x(t)>0$ for $T-b\leq t\leq t^*$ and $x(t^*)=0$. Thus by (4), we have

$$0=x(t^*)=\alpha x(\phi(t^*))+\int_{t^*}^{\infty} q_i(s)x(\gamma_i(s))\Delta s,\ t\geq T$$

which implies that $\alpha=0$ and $q_i(s)x(\gamma_i(s))=0$ for all $s\geq t^*$ which contradicts $\alpha\in(0,1)$, $q_j\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $q_i(s)\neq 0$ for $s\in[t,T^*]$. Therefore, $x(t)>0$ on $[T-b,\infty)$.

Lemma 2. Every solution of the dynamic equation

$$[v(t)-\alpha v(\phi(t))]^{\Delta}+\sum_{i=1}^{n} q_i(t)v(\gamma_i(t))=0,\ \ t\geq t_0, \tag{5}$$

where $\alpha\in(0,1)$, $q_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *and* $\gamma_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ is oscillatory, if and only if the inequality

$$\left[v(t)-\alpha v(\phi(t))\right]^{\Delta}+\sum_{i=1}^{n}q_i(t)v(\gamma_i(t))\leq 0,\quad t\geq t_0 \tag{6}$$

has no eventually positive solutions.

Proof:

The sufficiency is obvious. To prove the necessity, we assume that (5) has an eventually positive solution $v(t)$. Set $y(t)=v(t)-\alpha v(\phi(t))$. Since $0<\alpha<1$ and $v(t)$ is an eventually positive solution, it is easy to see that $y(t)>0$ eventually and

$$y^{\Delta}(t)=\left[v(t)-\alpha v(\phi(t))\right]^{\Delta}=-\sum_{i=1}^{n}q_i(t)v(\gamma_i(t))\leq 0.$$

Integrating (6) from t to ∞, we have

$$y(t)\geq\int_t^{\infty}q_i(s)v(\gamma_i(s))\Delta s.$$

That is,

$$v(t)\geq\alpha v(\phi(t))+\int_t^{\infty}q_i(s)v(\gamma_i(s))\Delta s.$$

By Lemma 1, the corresponding integral equation

$$z(t)=\alpha z(\phi(t))+\int_t^{\infty}q_i(s)v(\gamma_i(s))\Delta s$$

also has a positive solution $z(t)$. Clearly, $z(t)$ is an eventually positive solution of (5), contradicting the assumption.

Theorem 1. Every solution of the problem (1) is oscillatory in G if and only if the inequality (6) has no eventually positive solutions.

Proof:

i) Sufficiency. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1) and (2) which has no zero in $\Omega\times[t_0,\infty)_{\mathbb{T}}$ for some $t_0\geq 0$. Without loss of generality, we may assume that $u(x,t)>0$, $u(x,\phi(t))>0$, $u(x,\beta_k(t))>0$ and $u(x,\gamma_j(t))>0$ in

$\Omega\times\mathbb{T}_1=[t_1,\infty)_{\mathbb{T}}$, $t_1\ge t_0$, $k=1,2,...,s; j=1,2,...,n$. Integrating (1) with respect to x over the domain Ω, we have

$$\left[\int_\Omega u(x,t)\Delta x-\alpha\int_\Omega u\big(x,\phi(t)\big)\Delta x\right]^{\Delta_2}=a(t)\int_\Omega Lu(x,t)\Delta x+\sum_{k=1}^{s}a_k(t)\int_\Omega Lu\big(x,\beta_k(t)\big)\Delta x-\sum_{j=1}^{n}q_j(t)\int_\Omega u\big(x,\gamma_j(t)\big)\Delta x, \tag{7}$$

for $t\ge t_1$. From the Green's formula on time scales, Bohner et al. (2007) and boundary condition (2), it follows that

$$\int_\Omega Lu(x,t)\Delta x=\int_{\partial\Omega}u^{\Delta_N}(x,t)\Delta S=0,\quad t\ge t_1 \tag{8}$$

and

$$\int_\Omega Lu\big(x,\beta_k(t)\big)\Delta x=\int_{\partial\Omega}u^{\Delta_N}\big(x,\beta_k(t)\big)\Delta S=0,\quad t\ge t_1, \tag{9}$$

where ΔS is the surface element on $\partial\Omega$. Let

$$v(t)=\int_\Omega u(x,t)\Delta x \ \text{ for } t\ge t_1.$$

Combining (7)-(9), we have

$$\big[v(t)-\alpha v\big(\phi(t)\big)\big]^{\Delta}+\sum_{i=1}^{n}q_i(t)v\big(\gamma_i(t)\big)=0,\ \ t\ge t_1. \tag{10}$$

This shows that $v(t)>0$ is a solution of (5). But by lemma 2 and the condition that the inequality (6) has no eventually positive solutions, we obtain that every solution of (5) is oscillatory. This is a contradiction.

ii) Necessity. Suppose that inequality (6) has an eventually positive solution. By lemma 2, we obtain that (5) has a nonoscillatory solution. Without loss of generality, we may assume that $\bar{v}(t)$ for $t\ge t_0$ is a solution of (5). From (5), we have

$$\big[\bar{v}(t)-\alpha\bar{v}\big(\phi(t)\big)\big]^{\Delta}+\sum_{i=1}^{n}q_i(t)\bar{v}\big(\gamma_i(t)\big)=0,\ \ t\ge t_0.$$

Notice that $L\bar{v}(t)=0$, $L\bar{v}\big(\beta_k(t)\big)=0$ for $t\ge t_0$, from (10) we get

$$\left[\overline{v}(t)-\alpha\overline{v}(\phi(t))\right]^{\Delta_2}=a(t)L\overline{v}(t)+\sum_{k=1}^{s}a_k(t)L\overline{v}(\beta_k(t))-\sum_{j=1}^{n}q_j(t)\overline{v}(\gamma_i(t)).$$

This shows that $\overline{u}(x,t)=\overline{v}(t)>0$ satisfies (1). It is easy to see that $\overline{u}^{\Delta_N}=0$, $(x,t)\in\Omega\times[t_0,\infty)_{\mathbb{T}}$. Hence $\overline{u}(x,t)=\overline{v}(t)>0$ is a nonoscillatory solution of the problem (1) and (2), which is a contradiction. This completes the proof.

Using lemma 2 and Theorem 1, we can obtain the following corollary.

Corollary 1. Every solution of the problem (1) and (2) is oscillatory in G if and only if every solution of the dynamic equation (5) oscillates.

By (10) and

$$w(t)=v(t)-v(\phi(t)), \tag{12}$$

we have

$$w^{\Delta}(t)\le-\sum_{j=1}^{n}q_j(t)v(\gamma_j(t))\le-\sum_{j=1}^{n}q_j(t)w(\gamma_j(t)).$$

for $t\ge t_1$. Then we obtain the inequality

$$w^{\Delta}(t)+\sum_{j=1}^{n}q_j(t)w(\gamma_j(t))\le 0,\quad t\ge t_1. \tag{13}$$

Our aim is to establish a sufficient condition for the oscillation of all solutions of the inequality (13).

Consider the linear delay dynamic equation of the form

$$w^{\Delta}(t)+\sum_{j=1}^{n}q_j(t)w(\gamma_j(t))=0. \tag{14}$$

where for $j=1,2,...,n$,

$$q_j\in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right),\text{ and }\lim_{t\to\infty}q_j(t)=q_j. \tag{15}$$

Therefore, we have the limiting equation

$$\overline{w}^{\Delta}(t)+\sum_{j=1}^{n}q_j\overline{w}(\gamma_j(t))=0. \tag{16}$$

Zhang et al. (2002) studied the oscillation of the following delay differential equations on time scales

$$x^{\Delta}(t)+p(t)x(\tau(t))=0\,, \qquad t\geq t_0\,,$$

where $p\in C_{rd}(\mathbb{T},\mathbb{R}_+)$, $\tau\in C_{rd}(\mathbb{T},\mathbb{T})$ and $\tau(t)<t$ for $t\in\mathbb{T}$ and $\sup\{\mathbb{T}\}=\infty$. They have proved the following result.

Proposition 1. Assume that $1-p(t)\mu(t)>0,\ t\in\mathbb{T}$. Define a set

$$E=\{\lambda \mid \lambda>0,\ 1-\lambda p(t)\mu(t)>0\}.$$

If

$$\lim_{t_0\to\infty}\sup_{t>t_0}\sup_{\lambda\in E}\left\{\lambda\exp\left\{\int_{\tau(t)}^{t}\xi_{\mu(s)}(-\lambda p(s))\Delta s\right\}\right\}<1,$$

then all solutions of equation are oscillatory.

By this proposition we have the following theorem.

Theorem 2. Assume that $1-\sum_{j=1}^{n}q_j(t)\mu(t)>0,\ t\in\mathbb{T}$. Define a set

$$E=\left\{\lambda \mid \lambda>0,\ 1-\lambda\sum\nolimits_{j=1}^{n}q_j(t)\mu(t)>0\right\}.$$

If

$$\lim_{t_0\to\infty}\sup_{t>t_0}\sup_{\lambda\in E}\left\{\lambda\exp\left\{\int_{\gamma_j(t)}^{t}\xi_{\mu(s)}\left(-\lambda\sum\nolimits_{j=1}^{n}q_j(s)\right)\Delta s\right\}\right\}<1,$$

then all solutions of equation (1) are oscillatory.

Corollary 2. Assume that (15) holds and that every solution of the limiting equation (16) oscillates. Then every solution of (14) also oscillates.

Example 1. Let $\mathbb{T}=\mathbb{R}$ and $t\in[0,\infty)_{\mathbb{R}}$. Consider the following neutral delay parabolic differential equation

$$\frac{\partial}{\partial t}\left[u(x,t)-\frac{1}{2}u(x,t-\pi)\right]=Lu(x,t)+e^{t}Lu\left(x,t-\frac{\pi}{2}\right)$$
$$+e^{t}Lu\left(x,t-\frac{3\pi}{2}\right)-2u\left(x,t-\frac{\pi}{2}\right), \tag{17}$$

$(x,t)\in(0,\pi)\times[0,\infty)_{\mathbb{R}}$ with boundary condition

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial u(\pi,t)}{\partial x}=0 .$$

It is easy to see that $\alpha=\frac{1}{2}$, $\phi(t)=t-\pi$, $\beta_1(t)=t-\frac{\pi}{2}$, $\beta_2(t)=t-\frac{3\pi}{2}$, $\gamma(t)=t-\frac{\pi}{2}$, $a(t)=1$, $a_1(t)=a_2(t)=e^{t}$, $\phi_1(t)=\frac{\pi}{2}$ and $q=2$. Then we see that all the assumptions of the theorems are satisfied. Thus we obtain that every solution of the problem (17) oscillates in $(0,\pi)\times[0,\infty)_{\mathbb{R}}$. In fact $u(x,t)=\cos x.\sin t$ is such a solution of the problem (17).

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