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Fractional Integrals and Derivatives for Sumudu Transform on Distribution Spaces

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Abstract

We propose, in the present paper, the investigation of the Sumudu transformation for certain distribution spaces with regard to the fractional integral and differential operators of the transform. This paper is organized in two sections, first of which gives an abriged text on fractional operators and the Sumudu transform (which is less discussed and reserached). Basic concept in analysing the investigation is initiated by the fact that the Riemann-Liouville fractional integral can be expressed as one of the appropriate forms of the Abel integral equation, which is the second section of this paper.

Keywords: Fractional integral and derivatives, Sumudu transform, convolution, distribution spaces

MSC 2010: 26A33, 44A99, 44A35, 4605, 46F12

1. Preliminaries : Notations and Definitions

Fractional integrals are defined [cf. Samko, Kilbas and Marichev (1993, p.33)], for $\varphi(x) \in L_1(a,b)$, by

$$(I_{a+}^{\alpha}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^{\infty}(x-t)^{\alpha-1}\varphi(t)dt\ ,\ x > a \tag{1}$$

$$(I_{b-}^{\alpha}\phi)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{b}(t-x)^{\alpha-1}\phi(t)dt\ ,\ x < b, \tag{2}$$

where $\alpha > 0$ (α being the order). These integrals are also known as the Riemann-Liouville fractional integrals or the *left - sided* and *right - sided* fractional integrals, respectively. The integrals given in (1) and (2) are extensions to half and (or) whole axis finite interval $[a,b]$. These may be used on the half axis (a,∞) or $(-\infty,b)$, respectively, subject to the variable limit of integration. For the half axis, we write

$$(I_{0+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^{x}(x-t)^{\alpha-1}\varphi(t)dt \quad ,\ 0 < x < \infty \tag{3}$$

and on the whole real axis, it is given by [cf. Samko, Kilbas and Marichev (1993, pp. 93-94)]

$$(I_{+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}\varphi(t)dt \quad ,\ -\infty < x < \infty \tag{4}$$

and

$$(I_{-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}\varphi(t)dt \quad ,\ -\infty < x < \infty \tag{5}$$

Fractional derivatives of order $\alpha, 0 < \alpha < 1$, are also called Riemann-Liouville fractional derivatives or the *left - handed* and *right - handed* fractional derivatives, respectively, in the interval $[a,b]$ which are defined as [cf. Samko, Kilbas and Marichev (1993, p.35)]

$$(\mathbf{D}_{a+}^{\alpha}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^{\infty}(x-t)^{-\alpha}f(t)dt\ , \tag{6}$$

$$(\mathbf{D}_{b-}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^{b}(t-x)^{-\alpha}f(t)dt\ , \tag{7}$$

The fractional derivatives on the whole real axis is [cf. Samko, Kilbas and Marichev (1993, p.95)], $f(x) \in L_p(-\infty,\infty)$.

$$(\mathbf{D}_{+}^{\alpha}f)(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}(x-t)^{-\alpha}f(t)dt \tag{8}$$

$$(\mathbf{D}_{-}^{\alpha} f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} (t-x)^{-\alpha} f(t) dt, \tag{9}$$

Fractional integration by parts formulae [cf. Samko, Kilbas and Marichev (1993, p.96)]

$$\int_{-\infty}^{\infty} \varphi(x)(I_{+}^{\alpha}\psi)(x) dx = \int_{-\infty}^{\infty} \psi(x)(I_{-}^{\alpha}\varphi)(x) dx \tag{10}$$

$$\int_{-\infty}^{\infty} f(x)(\mathbf{D}_{+}^{\alpha} g)(x) dx = \int_{-\infty}^{\infty} g(x)(\mathbf{D}_{-}^{\alpha} f)(x) dx. \tag{11}$$

The fractional integral of Riemann-Liouville (right hand) is interpreted as one of the forms of Abel integral equation [cf. Samko, Kilbas and Marichev (1993, pp. 29-30)]

$$f(x) = (I_{0+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-t)^{\alpha-1} \varphi(t) dt \quad , \; x > 0, 0 < \alpha < 1, \tag{12}$$

solution of which is given by

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{a}^{x} (x-t)^{-\alpha} f(t) dt. \tag{13}$$

The Riemann-Liouville fractional integral (left hand) in the form of the Abel integral equation is given by

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} \phi(t) dt, \; 0 < \alpha < 1 \tag{14}$$

whose solution is obtained as

$$\phi(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{b} (t-x)^{-\alpha} f(t) dt. \tag{15}$$

The solution obtained for fractional integrals by considering the Abel integral equations are equivalent to the fractional derivatives. In addition to the brief note (that is given), one may refer to Miller and Ross (1993) and Podlubny (1999) among other monographs.

For generalized functions, fractional integrals and derivatives have two approaches, the *first* is based on Schwartz theory, by virtue of the definition of a fractional integral as the *convolution*

$$\frac{1}{\Gamma(\alpha)} x_{\pm}^{\alpha-1} * f \tag{16}$$

of the function $\frac{1}{\Gamma(\alpha)} x_{\pm}^{\alpha-1}$, with the generalized function f .

In other words, the fractional integrals of a generalized function $f \in K_1'$ is

$$(I_{0+}^{\alpha} f, \phi) = (I_{+}^{\alpha} f, \phi) = \left(\frac{1}{\Gamma(\alpha)} (x_{+}^{\alpha-1}) * f, \phi \right), \tag{17}$$

when f is supported on the half axis $x > 0$, and K_1' is the dual of the test function space $K_1 = C_0^{\infty}(R_1)$, which is also known as the space of generalized functions. For such functions, $I_{+}^{\alpha} f$ is also supported on the half axis. Equation (17) is applicable for all α with the generalized function $x_{+}^{\alpha-1}$.

The *second* approach is based on using the *adjoint operators*. By employing fractional integration by parts, we get

$$(I_{a+}^{\alpha} f, \varphi) = (f, I_{b-}^{\alpha} \varphi) \tag{18}$$

The approach via (18) will be justified if I_{b-}^{α} continuously maps the space of test functions X into itself . Sometimes a more general treatment is admitted when f and $I_{a+}^{\alpha} f$ are considered as generalized functions on different test function spaces X and Y, respectively such that $f \in X'$ (the dual of the test function space X), $I_{a+}^{\alpha}(f) \in Y'$ (the dual of test function space Y). Then I_{b-}^{α} must map continuously Y into X. Based on the same approach the fractional derivatives for generalized functions are proved. Let $(H; \langle \cdot,\cdot \rangle)$ be an inner product space over the real or complex number field K.

In the early 90's, Watugala (1993) christened the Sumudu transform. Related formulation, called the $s-$ multiplied Laplace transform, was announced as early as 1948 [Belgacem et al. (2003)]. Weerakoon (1998), using Watugala's work, introduced a complex inversion formula of the same. One may refer to Ali and Kalla (2007), Belgacem and Karaballi (2006), Eltayeb and Kilicman (2010a), Eltayeb and Kilicman (2010b), Eltayeb, Kilicman and Fisher (2010) for more text on the Sumudu transform and its applications.

Consider functions in the set A , we have M a constant and τ_1 and τ_2 need not exist simultaneously (each may be infinite),

$$A = \{ f(t) \Big| M e^{|t|/\tau_j} ; \text{if } t \in (-1)^j \times 0, \infty) \}, \tag{19}$$

which initiates the definition of the Sumudu transform, see [Belgacem, Karaballi and Kalla (2003)], in the form

$$F(u) = S[f(t)] = \begin{cases} \int_0^\infty f(ut)e^{-t}dt; & 0 \le u < \tau_2 \\ \int_0^\infty f(ut)e^{-t}dt; & \tau_1 < u \le 0. \end{cases} \tag{20}$$

Two parts arise in (20), because, in the domain of f , the variable t may not change sign. Infact, the Sumudu transform, which is itself linear, preserves linear functions and hence (in particular) does not change units.

In other words, the Sumudu transform can also be written as [Belgacem, Karaballi and Kalla (2003), Eltayeb and Kilicman (2010a)]

$$F(u) = S[f(t);u] = \frac{1}{u}\int_0^\infty e^{-(\frac{t}{u})} f(t)dt, \quad u \in (-\tau_1, \tau_2) \tag{21}$$

inversion formula of which is given by

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(u)e^{\frac{t}{u}}du. \tag{22}$$

The Sumudu transform of the n *th order derivative* of $f(t)$ is defined by

$$S[f^{(n)}(t)] = F_n(u) = \frac{F(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}}. \tag{23}$$

Convolution of the Sumudu transform is

$$S(f * g)(t) = uS[f(t)]S[g(t)] = uF(u)G(u). \tag{24}$$

If $F(u)$ is the Sumudu transform of a function $f(t)$ in A , $f^{(n)}(t)$ is the nth derivative of $f(t)$ with respect to t, and $F^{(n)}(u)$ is *the* n *th derivative of* $F(u)$ *with respect to* u , then the Sumudu transform of the prescribed function is

$$S[t^n f^{(n)}(t)] = u^n \frac{d^n F(u)}{du^n} = u^n F^{(n)}(u). \tag{25}$$

In what follows, are mentioned properties of the Sumudu transform, see [Eltayeb and Kilicman (2010a), Eltayeb and Kilicman (2010b), Eltayeb, Kilicman and Fisher (2010), Kilicman and Eltayeb (2010)].

Theorem 1. Existence of Sumudu transform [Eltayeb and Kilicman (2010a)**]**

If f is of exponential order, then its Sumudu transform $F(u)$ exists, which is given by

$$F(u) = \frac{1}{u}\int_0^\infty f(t)e^{-\left(\frac{t}{u}\right)}dt,$$

where $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$. The defining integral for F exists at point $\frac{1}{u} = \frac{1}{\eta} + \frac{i}{\tau}$ in the right hand plane $\eta > K$ and $\zeta > L$.

In [Eltayeb, Kilicman and Fisher (2010)], the Sumudu transform is extended to the distribution spaces and some other properties have been formulated. Authors of the present paper Loonker and Banerji (2011) have proved the Parseval equation of the Sumudu transform for distribution spaces and obtained solution of the Abel integral equation related to the distribution spaces. The fractional integral for the Sumudu transform is given by [see Ali and Kalla (2007)]

$$S[{}_0D_t^{-\upsilon} f(t)] = u^{\upsilon}G(u) \ . \tag{26}$$

In terms of Equation (3), the fractional integrals for the Sumudu transform, can be defined as

$$S[I_{0+}^{\alpha}\varphi](x) = \frac{1}{\Gamma(\alpha)} S[x^{\alpha-1}].S[\varphi(x)]$$

$$= u^{\alpha}G(u) = u^{\alpha}S[\phi(x)]. \tag{27}$$

Definition 1. [Eltayeb and Kilicman (2010b), Eltayeb, Kilicman and Fisher (2010)]

The space W of test functions of exponential decay is the space of complex valued functions $\varphi(t)$ satisfying the following properties

(i) $\varphi(t)$ is infinitely differentiable; i.e., $\varphi(t) \in C^{\infty}(R^n)$

(ii) $\varphi(t)$ and its derivatives of all orders vanish at infinity faster than the reciprocal of the exponential of order $1/\omega$; that is,

$$\left|e^{t/\omega} D^k \varphi(t)\right| < M, \quad \forall 1/\omega, k \ . \tag{28}$$

Then a function $f(t)$ is said to be of exponential growth if and only if $f(t)$ together with all its derivatives grow more slowly than the exponential function of order $1/\omega$, that is, there exists a real constant $1/\omega$ and M such that $\left|D^k\varphi(t)\right| < Me^{\frac{t}{\omega}}$. A linear continuous functional over the space W of test functions is called a distribution of exponential growth and its dual space is denoted by W'.

The Sumudu transform of the function f for the space W' is given by

$$f(t)=(t^{\eta})_{+}=H(t)t^{\eta}, \text{where}\, \eta\neq -1,-2,-3,\cdots, \tag{29}$$

since $(t^{\eta})_{+}\in W'$,

$$S[f(t)]=\int_0^{\infty}(t\upsilon)^{\eta}e^{-t}dt=\upsilon^{\eta}\Gamma(\eta+1). \tag{30}$$

2. Distributional Fractional Integrals and Derivatives of Sumudu Transform

In this section we will define the fractional integral and differential operators of the Sumudu transform for distribution (or generalized functions) spaces. Applications of integral transform to the fractional integrals and derivatives for generalized functions may be seen in [Erdelyi (1972), Ross (1975), Rubin (1995)].

Let space $\mathcal{S}=\mathcal{S}(R)$ be the Schwartz space of infinitely differentiable rapidly decreasing complex valued function on the real line R. Let $\mathcal{S}=\mathcal{S}(R_{+})$ be the linear topological space of the restricted Schwartz function of the half line $R_{+}=[0,\infty]$ with the topology $\mathcal{T}$ represented by the sequence of norms

$$\|f:S\|_k=\sup_{m\leq k}\sup_{x>0}(1+x)^k\left|f^{(m)}(x)\right|, k=0,1,2,\cdots. \tag{31}$$

Further, considering subspaces as linear topological spaces with the topology induced by $\mathcal{T}$, we mention

$$\mathcal{S}_{+}=\{f:f\in\mathcal{S}(R),\lim_{x\to 0}x^{-k}f^{(m)}(x)=0,\forall k,m=0,1,2,\ldots\} \tag{32}$$

and

$$\Phi=\{f:f\in S_{+},\int_0^{\infty}x^k f(x)dx=0,\forall k=0,1,2,\cdots\}. \tag{33}$$

The Sumudu transform defined for the test function space W can also be considered for the Schwartz spaces $\mathcal{S}$, possessing similar properties. Employing the Sumudu transform $S[g(z)]$ for $g\in\mathcal{S}_{+},\Phi,$ the topology in space W is defined by the norm

$$\|g:W\|_k=\sup_{|Rez|\leq k+1}(1+x)^k\left|g(z)\right|, k=0,1,2,\ldots \tag{34}$$

The subspace W_0 of W is defined by

$$W_0 = \{g : g \in W; g(z) = 0 \quad \text{for} \quad z = 1,2,\ldots\} \tag{35}$$

with the norms generated by the topology

$$\|g : W_0\| = \sup_{|Rez| \le k+1} M_k(z)|g(z)| \quad , \; k = 0,1,2,\cdots. \tag{36}$$

Theorem 2. The Sumudu transform is an isomorphism from S_+ onto W and from Φ onto W_0.

Proof:

Let $g(z) = S[f(t)], f \in \mathcal{S}_+$ and $\frac{1}{z} = \frac{1}{\xi} + \frac{i}{\eta}$. Then from Theorem 1 and using Definition 1, we have

$$\|g; W\| \le c_k \|f; \mathcal{S}\|. \tag{37}$$

Similarly, by the inverse Sumudu transform, we get

$$\|f; \mathcal{S}\| \le c_m \|S[f(t)]; W\| \tag{38}$$

with the convergence of the Sumudu transform, the isomorphism from one space to another is proved.

Corollary 1. The space S_+ (the space Φ) is an algebra (with respect to the convolution Sumudu transform) which is isomorphic to the multiplicative algebra $W(W_0)$. The space Φ is an ideal in the algebra S_+.

Theorem 3. The operator $I_{0+}^{\alpha}, Re\alpha > 0$ is an automorphism of the space Φ. There is no subspace $\text{X} \ne \{0\}$ of the space S_+ such that X is invariant under I_-^{α} for all $\alpha > 0$.

Proof:

For $f \in \Phi$, we have $S[I_0^{\alpha} f](z) = u^{\alpha} S[f(z)]$, where the function $f(z)$ is an entire function of z. The Sumudu transform converges and thus, the Sumudu transform of fractional integral converges. To obtain (or justify) $S[I_0^{\alpha} f](z) \in W_0$, from Theorem 2, we have

$$|Sf(z)| \le \|Sf; W_0\| . \tag{39}$$

Hence ,

$$\left\| S[I_{0+}^{\alpha} f, W_0 \right\| \le c_k \left\| S[f]; W_0 \right\| \;, \tag{40}$$

which implies the continuity of the mapping $S[f] \to S[I_0^{\alpha} f] \in W_0$ and simultaneously justifies $S[I_{0-}^{\alpha} f] = (-1)^{\alpha-1} u^{\alpha} S[f] \in W$.

To define fractional integrals and derivatives on the distributions spaces, the space of linear continuous functionals on Φ will be denoted by Φ'. The class $C_0^{\infty} \subset \mathcal{S}$ and $\Phi \subset \mathcal{S}$ are invariant with respect to fractional integration and differentiation. The space W is considered similar to the space $\mathcal{S}$. Therefore, the fractional integration of the Sumudu transform will be in the space W.

Let us introduce another space Ψ of functions $\psi(x) \in \mathcal{S}$ with all their derivatives

$$\Psi = \{\psi : \psi \in \mathcal{S}, \psi^{(k)}(0) = 0, k = 0,1,2,\ldots\}. \tag{41}$$

Since Ψ is closed in $\mathcal{S}$, we identify Ψ' with the quotient space of Schwartzian space $\mathcal{S}'$ modulo the space Ψ_0' of functionals in $\mathcal{S}'$, having Ψ as the null space, that is, $\Psi = \mathcal{S}'/\Psi_0'$, where $\Psi_0' = \{f : f \in \mathcal{S}',\ (f,\psi) = 0, \psi \in \Psi\}$.

If W is a closed subspace in a linear topological space E, then $W' = E'/W^1$. Following by virtue of the definition of the space Ψ, that consists of functions Ψ_0' the of combinations of the delta function and its derivative. The Sumudu transform in the sense of distribution spaces are

$$\langle Sf, S\varphi \rangle = \langle f, \varphi \rangle, \tag{42}$$

$$\langle Sf, \varphi \rangle = \langle f, S\varphi \rangle \;, \tag{43}$$

where $\varphi \in \mathcal{S}$ (or Φ or W). Consequently, $\Phi' = \mathcal{S}'/\Phi_0'$ where the space $\Phi_0' = \{f : f \in \mathcal{S}', (f,\varphi) = 0, \varphi \in \Phi\}$ consists of polynomials. Therefore, Φ' may be considered as a quotient space $\mathcal{S}'/P$ modulo the subspace P of all polynomials. We may say that Φ' is obtained from $\mathcal{S}'$ by shifting out polynomials, that is, two functionals in $\mathcal{S}'$ differing by a polynomial are indistinguishable as elements of Φ' From the definition of distribution space for fractionals integrals (17) and the Sumudu transform in distribution spaces (42) and (43), we have

$$\left\langle f, I_{\mp}^{\alpha} \varphi \right\rangle = \left\langle Sf, SI_{\mp}^{\alpha} \varphi \right\rangle = \left\langle Sf, u^{\alpha} S\varphi \right\rangle. \tag{44}$$

Since $S\varphi \in \Psi$ and u^{α} is a continuous operation in $\Psi, \left\langle f, I_{\mp}^{\alpha} \varphi \right\rangle$ is a continuous functional on Φ'. In other words, we can also say that the Sumudu transform is in the space W' and the convolution of fractional integral with the Sumudu transform, that defines the Sumudu transform $\varphi(x)$ with u^{α}, will also be in the distribution space W'.

We consider the fractional integral in the form of Abel integral equation, that is given in (12), the convolution form of which is given by

$$f(x) = \frac{1}{\Gamma(\alpha)}[x^{\alpha-1}] \cdot \varphi(x)]. \tag{45}$$

Applying Sumudu transform convolution in both the sides, we have

$$S[f(x)] = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha)u \cdot u^{\alpha-1}\, S[\phi(x)]. \tag{46}$$

Setting $f(t) = \frac{t^{n-1}}{n-1!}$, the Sumudu transform is $F(u) = u^{n-1}$, which reduces to the following

$$F(u) = u^{\alpha} S[\phi(x)].$$

$$S[\phi(x)] = u^{\alpha} F(u) = \frac{1}{u}\frac{1}{\Gamma(1-\alpha)} S[x^{-\alpha} * f(t)].$$

$$S[\varphi(x)] = \frac{1}{u\Gamma(1-\alpha)} S\left[\int_0^x (x-t)^{-\alpha} f(t)dt\right],$$

i.e.,

$$S[\varphi(x)] = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{u} S[H(x)] \;, \tag{47}$$

where $H(x) = \int_0^x (x-t)^{-\alpha} f(t)dt, H(0) = 0$. $S[H'(x)] = \frac{S[H(x)] - H(0)}{u} = \frac{S[H(x)]}{u}$. Then

$$S[\phi(x)] = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{u}{u} S[H'(x)],$$

i.e.,

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[\int_0^x (x-t)^{-\alpha} f(t)dt\right]. \tag{48}$$

The solution obtained in the form of $\varphi(x)$ is equivalent to $\mathbf{D}_{+}^{\alpha} f(t)$. Considering another fractional integral form as the Abel integral equation (14) and once again using the Sumudu transform, the solution obtained will be equivalent to $\mathbf{D}_{-}^{\alpha} f(t)$.

As the Abel integral equation, $I_{+}^{\alpha}\varphi$ with $f \in \mathbf{X}'$, X= $C_{b}^{\infty}[a,b]$, has the unique solution $\varphi = \mathbf{D}_{+}^{\alpha} f$ in the space $\mathbf{X}'$ of generalized functions. This solution can be understood in the sense that

$$\left\langle \mathbf{D}_{+}^{\alpha} f, \varphi \right\rangle = \left\langle f, \mathbf{D}_{b-}^{\alpha} \varphi \right\rangle, \varphi \in \mathrm{X}. \tag{49}$$

Moreover, if the fractional integrals of the Sumudu transform are considered in the distribution spaces, then the solution thus obtained will also be interpreted in the sense of distribution spaces (or generalized functions).

Remark 1. Replacing α by $(1-\alpha)$ in (12) we notice it to reduce to another form of Abel integral equation. In other words, the fractional integrals and derivatives both can be employed to different types of distribution spaces as discussed in this paper and as cited in [Loonker and Banerji (2011)].

Applications: An Equation of the form $y^{(\alpha)} + y = f(x)$, $y(0) = 0$, $0 < \alpha < 1$ is solved by invoking the Sumudu transform and using $y(0) = 0$, which yields the solution by applying complex inversion formula of fractional Sumudu transform, see [Kilicman and Eltayeb (2010)].

A linear fractional partial differential equation, see Jumarie (2007)

$$\partial_{t}^{\alpha} z(x,t) = c\partial_{x}^{\beta} z(x,t), \quad x,t \in R^{+},$$

with boundary conditions $z(0,t) = f(t), z(x,0) = g(x)$, is solved by taking fractional double Sumudu transform, see Gupta, Sharma and Kilicman (2010).

We also can refer to [Eltayeb and Kilicman (2010b)] to study a differential equation involving the Heaviside function and the Dirac delta function, whose solution is found through Sumudu transform. Applications of Sumudu transform are also studied for population growth and in finance, see [Eltayeb and Kilicman (2010a), Kataetbeh and Belgacem (2011)].

Moreover, fractional differential equations those are solved either by using the Sumudu transform or by invoking any type of fractional derivative operators, can be extended to the distribution spaces. And (on the other hand) the distributional Sumudu transform can be used to investigate and study of several types fractional derivatives and to obtain solutions of fractional differential equations, see, for instance [Kataetbeh and Belgacem (2011)].

3. Conclusion

Since the Riemann-Liouville fractional integral is expressible as one of the forms of the Abel integral equation and solution obtained is one of the fractional derivatives, the Sumudu transform defined for a certain test function space can as well be considered for the Schwartz space (those possessing similar properties). Two results proved in Section 2, explain the investigation.

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