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The Investigation of Surplus of Energy and Signal Propagation at Time-Domain Waveguide Modes

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Abstract

Classical waveguide theory has been developed bearing on Bernoulli's product method which results in separation of space and time variables in Maxwell's equations. The time-harmonic waveguide modes have been stated mathematically for transmitting signals along the waveguides. As a starting point, present studies on transverse-electric (TE) and transverse-magnetic (TM) waveguide modes with previous results are taken and exhibited in an advanced form. They have been obtained within the framework of an evolutionary approach to solve Maxwell's equations with time derivative. As a result every modal field is obtained in the form of a product of vector functions of transverse coordinates and modal amplitudes. The modal amplitudes which are the functions of axial coordinate z and time t are the central matter of this study. The amplitudes are generated by a potential which is governing by Klein-Gordon equation (KGE). The KGE has remarkable properties of symmetry which has great importance in our analytical studies of the transient modes. Ultimately, in this study, a time-domain waveguide problem is solved analytically in accordance with the causality principle. Moreover, the graphical results are shown for the case when the *energy* and *surplus of the energy* for the time-domain waveguide modes are represented via Airy functions.

Keywords: Time Domain Waveguide Modes; Maxwell Equations; Energy; Surplus of Energy

MSC 2010 No.: 35Q60, 35Q61, 83C50

1. Introduction

Time-harmonic waveguide modes are usually interpreted for signal transmission along waveguides. However this model has two main physical drawbacks. First, the time-harmonic signals are non-casual. It means that their propagation starts at time $t = -\infty$ and continues up to time $t = \infty$. Second, these signals have their frequency bandwidth equal to zero. Any time-harmonic wave has frequency bandwidth equal to zero and therefore cannot transmit information per se.

A new alternative approach in studying the time-domain modes were developed within the framework of four-dimensional relativistic formalism in electrodynamics (Gabriel, 1980). One more alternative approach was suggested in the 80s both for the cavity and waveguide modes. As the basic data for this article, previous results obtained within the framework of Evolutionary Approach to Electromagnetics (EAE) (see, for instance, Aksoy and Tretyakov (2003); Tretyakov (1990); Tretyakov (1993); Tretyakov (1994); Tretyakov and Akgün (2010)) are used. The other set of important publications on this topic is based on the different techniques (see, for instance, Borisov (1987); Geyi (2006); Kristensson (1995); Miller (1977); Shvartsburg (1998); Shlivinski and Heyman (1999)).

The Time-domain waveguide problem may be considered as two autonomous problems. The first one is a modal basis problem. The modal basis involves two complete sets of the modes: the TE and TM ones. They are generated by the scalar potentials which are actually the same as for the time-harmonic modes. The second one is a modal amplitude problem. They are generated by a scalar potential, which is individual for the TE and TM modes and which is governed by the KGE with the time and axial coordinate as independent variables.

Even though energy and the surplus of energy were considered in Eroğlu et al. (2012), our goal is to distinguish the formulation and display the graphical properties of surplus of energy for Airy functions.

2. Formulation of the Problem

Electromagnetic wave propagation is modeled by some PDE systems for electric and magnetic fields. The mentioned system is the Maxwell system. Therefore Maxwell's equations are the starting point of this study. A standard formulation of the boundary-value problem for the system of Maxwell's equations with time derivative is given. Thus, the following vectorial Maxwell's equations have to be solved

$$\nabla \times \vec{\mathcal{E}}(\vec{\boldsymbol{R}}, t) = -\mu_0 \partial t \vec{\mathcal{H}}(\vec{\boldsymbol{R}}, t), \tag{2.1}$$

$$\nabla \times \vec{\mathcal{H}}(\vec{\boldsymbol{R}}, t) = \epsilon_0 \partial t \vec{\mathcal{E}}(\vec{\boldsymbol{R}}, t), \tag{2.2}$$

where $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are electric and magnetic field strength respectively. (2.1)-(2.2) should be solved simultaneously with the scalar Maxwell's equations

$$\nabla . \vec{\mathcal{E}}(\vec{\boldsymbol{R}}, t) = 0, \tag{2.3}$$

$$\nabla.\vec{\mathcal{H}}\left(\vec{\boldsymbol{R}},t\right)=0. \tag{2.4}$$

A hollow rectangular waveguide with its cross-section domain *S*, bounded by a closed singly connected smooth *L* contour is considered. Introduce a right-handed triplet of the mutually orthogonal unit vectors $(\vec{\boldsymbol{z}},\vec{\boldsymbol{l}},\vec{\boldsymbol{n}})$ where $\vec{\boldsymbol{z}}\times\vec{\boldsymbol{l}}=\vec{\boldsymbol{n}}$ and so on. The vector $\vec{\boldsymbol{z}}$ is oriented along the axis *Oz*, the vector $\vec{\boldsymbol{l}}$ is tangential to the contour *L*, and $\vec{\boldsymbol{n}}$ is the outer normal to the domain *S*.

The three-component position vector $\vec{\boldsymbol{R}}$, the Nabla operator ∇ and real-valued electromagnetic field vectors are decomposed to transverse and longitudinal parts as follows

$$\vec{\boldsymbol{R}}=\vec{\boldsymbol{r}}+\vec{\boldsymbol{z}}z,\ \nabla=\nabla_{\perp}+\vec{\boldsymbol{z}}\partial_z, \tag{2.5.a}$$

$$\vec{\mathcal{E}}\left(\vec{\boldsymbol{R}},t\right)=\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}},z,t)+\vec{\boldsymbol{z}}E_z(\vec{\boldsymbol{r}},z,t), \tag{2.5.b}$$

$$\vec{\mathcal{H}}\left(\vec{\boldsymbol{R}},t\right)=\vec{\boldsymbol{H}}(\vec{\boldsymbol{r}},z,t)+\vec{\boldsymbol{z}}H_z(\vec{\boldsymbol{r}},z,t), \tag{2.5.c}$$

where $\vec{\boldsymbol{r}}$ is a position vector within the domain.

Due to the perfect electric conductor surface of the waveguide, the field components are subjected to the following boundary conditions

$$\vec{\boldsymbol{n}}.\vec{\mathcal{H}}\Big|_L=0,\quad \vec{\boldsymbol{l}}.\vec{\mathcal{E}}\Big|_L=0,\quad \vec{\boldsymbol{z}}.\vec{\mathcal{E}}\Big|_L=0. \tag{2.6}$$

3. Modal Field Components

After decomposition of Maxwell's equations (2.1)-(2.4), two subsystems are obtained that induce two kinds of solutions as TE and TM waveguide mode.

$$\nabla_{\perp}H_z=\epsilon_0\partial_t\left[\vec{\boldsymbol{z}}\times\vec{\boldsymbol{E}}\right]+\partial_z\vec{\boldsymbol{H}}, \tag{3.1.a}$$

$$\mu_0\partial_tH_z=\nabla_{\perp}.\left[\vec{\boldsymbol{z}}\times\vec{\boldsymbol{E}}\right], \tag{3.1.b}$$

$$\partial_zH_z=-\nabla_{\perp}.\vec{\boldsymbol{H}}. \tag{3.1.c}$$

In subsystem (3.1) the *z* component of the electric field vector is zero ($E_z=0$) and generate TE modes in waveguide.

$$[\vec{\boldsymbol{z}}\times\nabla_{\perp}]E_z=\mu_0\partial_t\vec{\boldsymbol{H}}+\partial_z\left[\vec{\boldsymbol{z}}\times\vec{\boldsymbol{E}}\right], \tag{3.2.a}$$

$$\varepsilon_0\partial_tE_z=\nabla_{\perp}.\left[\vec{\boldsymbol{H}}\times\vec{\boldsymbol{z}}\right], \tag{3.2.b}$$

$$\partial_zE_z=-\nabla_{\perp}.\vec{\boldsymbol{E}}. \tag{3.2.c}$$

In subsystem (3.2) the z component of the magnetic field vector is zero ($H_z = 0$) and generate TM modes in waveguide.

As a result of decomposition, Neumann and Dirichlet boundary-value problems together with the normalization conditions as stated below, should be solved for the transverse Laplacian $\nabla_{\perp}^2$

$$(\nabla_{\perp}^2 + \upsilon_m^2)\psi_m(\vec{\boldsymbol{r}}) = 0,\ \left.\frac{\partial\psi_m}{\partial\vec{\boldsymbol{n}}}\right|_L = 0,\ \frac{\upsilon_m^2}{S}\int_s |\psi_m(\vec{\boldsymbol{r}})|^2\, ds = 1N, \tag{3.3.a}$$

$$(\nabla_{\perp}^2 + \kappa_m^2)\phi_m(\vec{\boldsymbol{r}}) = 0,\ \phi_m|_L = 0, \frac{\kappa_m^2}{S}\int_s |\phi_m(\vec{\boldsymbol{r}})|^2\, ds = 1N, \tag{3.3.b}$$

where $\partial_{\vec{\boldsymbol{n}}} = \vec{\boldsymbol{n}}.\nabla_{\perp}$ is the normal derivative on the contour L; $\upsilon_m^2 > 0$ and $\kappa_m^2 > 0$, $m = 1,2,3, \ldots$ are the eigenvalues. The corresponding eigenvectors $\psi_m(\vec{\boldsymbol{r}})$ and $\phi_m(\vec{\boldsymbol{r}})$, respectively, are the elements of the basis of solution space $L_2(S)$.

The solutions of the Neumann and Dirichlet problems (3.3.a)-(3.3.b), generate the TE and TM time-domain modal fields, respectively, with the following components

$$\vec{\boldsymbol{E}}_m^{TE}(\vec{\boldsymbol{r}}, z, t) = -\varepsilon_0^{-1/2}\partial_{ct}h_m(z,t)[\nabla_{\perp}\psi_m(\vec{\boldsymbol{r}}) \times \vec{\boldsymbol{z}}], \tag{3.4.a}$$

$$E_{zm}^{TE}(\vec{\boldsymbol{r}}, z, t) = 0, \tag{3.4.b}$$

$$\vec{\boldsymbol{H}}_m^{TE}(\vec{\boldsymbol{r}}, z, t) = \mu_0^{-1/2}\partial_z h_m(z,t)\nabla_{\perp}\psi_m(\vec{\boldsymbol{r}}), \tag{3.4.c}$$

$$H_{zm}^{TE} = \mu_0^{-1/2}\upsilon_m^2 h_m(z,t)\psi_m(\vec{\boldsymbol{r}}), \tag{3.4.d}$$

and

$$\vec{\boldsymbol{H}}_m^{TM}(\vec{\boldsymbol{r}}, z, t) = -\mu_0^{-1/2}\partial_{ct}e_m(z,t)[\vec{\boldsymbol{z}} \times \nabla_{\perp}\phi_m(\vec{\boldsymbol{r}})], \tag{3.5.a}$$

$$H_{zm}^{TM}(\vec{\boldsymbol{r}}, z, t) = 0, \tag{3.5.b}$$

$$\vec{\boldsymbol{E}}_m^{TM} = \varepsilon_0^{-1/2}\partial_z e_m(z,t)\nabla_{\perp}\phi_m(\vec{\boldsymbol{r}}), \tag{3.5.c}$$

$$E_{zm}^{TM} = \varepsilon_0^{-1/2}\kappa_m^2 e_m(z,t)\phi_m(\vec{\boldsymbol{r}}), \tag{3.5.d}$$

where $\partial_{ct} = \partial/c\partial_t$ and $c = 1/\sqrt{\varepsilon_0\mu_0}$.

4. Klein Gordon Equation

The potentials $h_m(z,t)$ and $e_m(z,t)$ in equations (3.4)-(3.5) are governed by the Klein Gordon equation

$$\left(\partial_{\upsilon_m ct}^2 - \partial_{\upsilon_m z}^2 + \upsilon_m^2\right)h_m(z,t) = 0, \tag{4.1.a}$$

$$\left(\partial_{\kappa_m ct}^2 - \partial_{\kappa_m z}^2 + \kappa_m^2\right)e_m(z,t) = 0, \tag{4.1.b}$$

which is known as the generalized wave equation (see, for instance, Aksoy and Tretyakov (2003); Eroğlu et al. (2012)).

The set of TE and TM modes are complete due to Sturm-Liouville and Weyl theorem in functional analysis about the orthogonal detachments of Hilbert space $L_2(S)$ (see, for instance, Aksoy and Tretyakov (2003); Tretyakov and Akgün (2010)). This Hilbert space can be specified by an inner product as

$$\left(\vec{X}_1, \vec{X}_2\right) = \frac{1}{S}\int_S \left(\varepsilon_0 \vec{\boldsymbol{E}}_1 \vec{\boldsymbol{E}}_2 + \mu_0 \vec{\boldsymbol{H}}_1 \vec{\boldsymbol{H}}_2\right) ds < \infty, \tag{4.2}$$

where $\vec{X}_i = col\left(\vec{\boldsymbol{E}}_i, \vec{\boldsymbol{H}}_i\right), i = 1,2, ...$ One can verify that $\left(\vec{X}_m^{TE}, \vec{X}_n^{TM}\right) = 0$ for any combinations of m and n with the values $0,1,2,...$ independently. Therefore, any pair of the TE and TM time-domain modes is orthogonal in the sense of inner product in equation (4.2).

The KGE in (4.1.a)-(4.1.b) can be rewritten in the general form

$$\left(\partial_\tau^2 - \partial_\xi^2 + 1\right) f(\xi, \tau) = 0, \tag{4.3}$$

where $f(\xi, \tau)$ is either $h_m(z, t)$ provided that $\xi = \upsilon_m z$ and $\tau = \upsilon_m ct$ for TE-modes or $e_m(z, t)$ provided that $\xi = \kappa_m z$ and $\tau = \kappa_m ct$ for TM-modes (τ is the scaled time and ξ is the scaled coordinate).

The KGE maintains its form under an action of a *Poincare group* within the framework of the *group theory*. In this aspect, Miller (1977) established eleven so called orbits of symmetry in terms of the *group theory*. His results are crucial for development of the electromagnetic field theory in the time-domain. According to Tretyakov and Akgün (2010), with inversion of case 5, u and v can be written in terms of $\xi$and τ as $u + v = (\xi + \tau)/2$ and $u - v = \mp\sqrt{\tau + \xi}$.

After some manipulations, in line with the causality principle, $f(\xi, \tau)$ satisfies

$$f(\xi, \tau) = \begin{cases} 0 & , \quad \tau < 0, \\ [c_1 Ai(u) + c_2 Bi(u)][c_3 Ai(v) + c_4 Bi(v)] & , \; 0 \le \xi \le \tau, \\ 0 & , \quad \xi > \tau, \end{cases} \tag{4.4}$$

where $c_{1,2,3,4}$ are arbitrary constants, $Ai(\neq)$ and $Bi(\neq)$ are Airy functions.

Their arguments are the functions of time, τ and axial coordinate, ξ. All possible combinations of the Airy functions are

$$f_1(\xi, \tau) = Ai(u)Ai(v), \; f_2(\xi, \tau) = Ai(u)Bi(v),$$

$$f_3(\xi, \tau) = Bi(u)Ai(v), \; f_4(\xi, \tau) = Bi(u)Bi(v) \, .$$

5. Graphical Results

Ultimately, the *energy* density which is stored in the transverse and longitudinal part of the field is altering. *Surplus of the energy* defined in Eroğlu et al. (2012), Tretyakov and Kaya (2012; 2013), is the difference between stored *energies*. These energy quantities are presented via modal amplitudes. The *energy* and *surplus of the energy* is considered respectively,

$$W(\xi,\tau) = [A^2(\xi,\tau) + B^2(\xi,\tau) + f^2(\xi,\tau)]/2, \quad (5.1)$$

$$sW(\xi,\tau) = [A^2(\xi,\tau) - B^2(\xi,\tau)]/2, \quad (5.2)$$

where $A(\xi,\tau) = -\frac{\partial}{\partial\tau} f(\xi,\tau)$, $B(\xi,\tau) = \frac{\partial}{\partial\xi} f(\xi,\tau)$ and $f(\xi,\tau)$ is the Airy function.

In this work, the *energetic* fields are specially treated for the Airy functions as a case example. In Figure 5.1.a, Figure 5.1.b, Figure 5.2.a and Figure 5.2.b dependence on time, τ of the *energy* density, $W_{1,2}(\xi,\tau)$ and *surplus of the energy*, $sW_{1,2}(\xi,\tau)$ are exhibited for on fixed position, $\xi = \tau - 0,05$ of the cross-section.

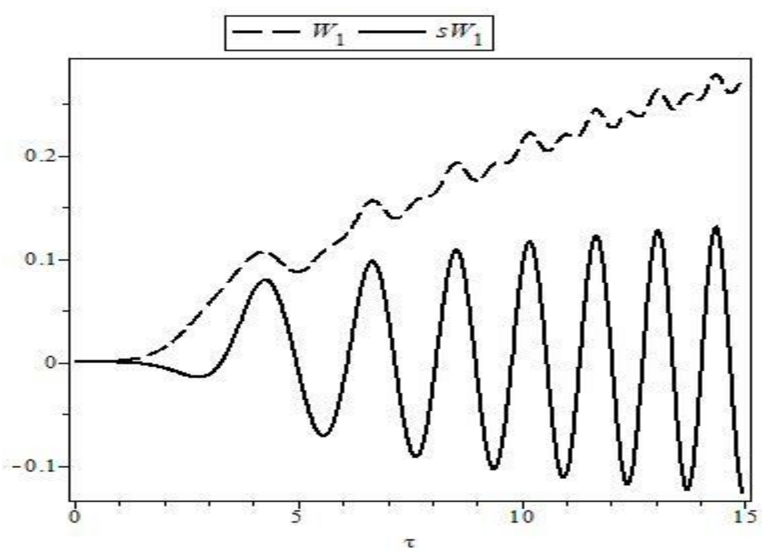

Figure5.1.a. Exchange of W_1 (_ _) and sW_1 (___) along $\tau = 15$, ξ is fixed

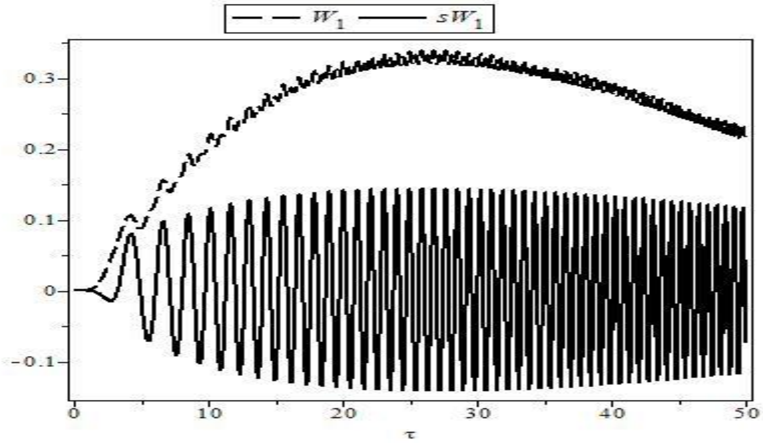

Figure5.1.b. Exchange of W_1 (_ _) and sW_1 (___) along $\tau = 50$, ξ is fixed

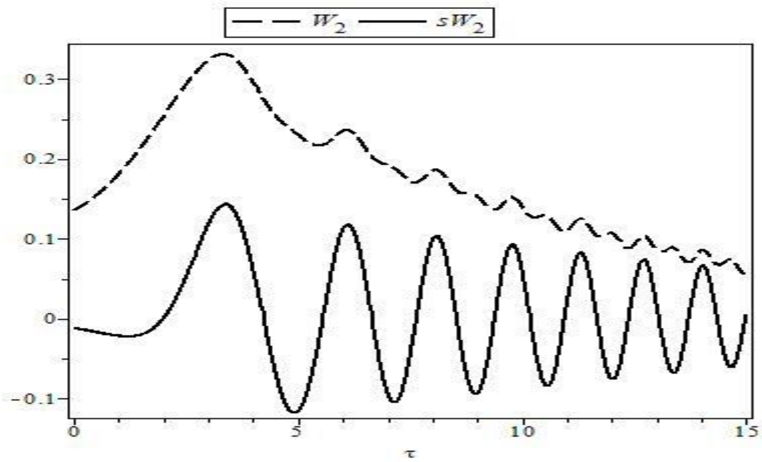

Figure5.2.a. Exchange of W_2 (_ _) and sW_2 (___) along $\tau = 15$, ξ is fixed.

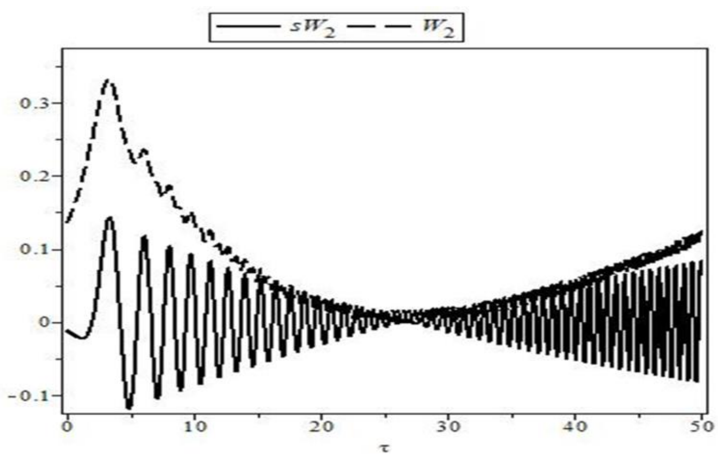

Figure5.2.b. Exchange of W_2 (_ _) and sW_2 (___) along $\tau = 50$, ξ is fixed.

6. Conclusion

In this study, the time-domain waveguide modes are expressed analytically by a method of Evolutionary Approach to Electromagnetics (EAE). The solution is obtained in Hilbert space, $L_2(S)$. A time-dependent source function is considered in a hollow rectangular waveguide with perfect electric conductor surface.

As seen from the results of the transverse-longitudinal decomposition, the signal transmission problem in waveguide is composed of two factors modal basis problem and a modal amplitude problem. In other words components of the modal field vectors are stated as the product of $\psi_m(\vec{\boldsymbol{r}})$ or $\phi_m(\vec{\boldsymbol{r}})$ and $h_m(z,t)$ or $e_m(z,t)$ according to the time domain mode in waveguide. Consequently two kinds of solution come up: the transverse electric (TE) time domain mode and the transverse magnetic (TM) time domain mode.

The basis elements are obtained via the solution of the Neumann and Dirichlet boundary value problems for transverse Laplacian together with the proper normalization conditions. The complete set of solutions $\{\psi_m(\vec{\boldsymbol{r}})\}_{m=0}^{\infty}$ and $\{\phi_m(\vec{\boldsymbol{r}})\}_{m=0}^{\infty}$ specify the field pattern in waveguide.

The other factor is a modal amplitude which is the function of axial waveguide coordinate z and time t. An evolutionary formulation for the amplitudes is obtained via projecting Maxwell's equations onto the basis elements. This evolutionary equation is the Klein Gordon equation also known as the generalized wave equation. The modal amplitudes of transverse and longitudinal coordinates $A(\xi,\tau)$, $B(\xi,\tau)$ and $f(\xi,\tau)$, respectively, are calculated for the components of TE and TM modes.

The KGE is solved analytically in compliance with the causality principle. Due to this new technique which is based on the properties of the symmetry of the KGE, it is disposed of by calculating the convolution integral as a standard solution to KGE. It writes the solution to the the KGE as $f(\xi,\tau) = U\big(u(\xi,\tau)\big)V\big(v(\xi,\tau)\big)$. The separation of variables method can only be applied if and only if the functions $u(\xi,\tau)$ and $v(\xi,\tau)$ are chosen properly. The list of inverse functions that enable factorization for solution to KGE is given in Tretyakov and Akgün (2010). In this study the fifth function pair is considered and the modal amplitudes are obtained via Airy functions (see, for instance, Tretyakov and Akgün (2010)).

Besides the time domain modal fields, energy quantities which propagate together with the field waves are studied and the results are displayed graphically. A new energy is quantity defined in Eroğlu et al. (2012), Tretyakov and Kaya (2012; 2013) and *surplus of energy* is obtained for Airy functions. Specially, *energy* and *surplus of the energy* are investigated in details via Airy functions. Graphical results of *energy* and *surplus of the energ*y are exhibited along particular time interval.

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