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Application of Reduced Differential Transform Method for Solving Nonlinear Reaction-Diffusion-Convection Problems

A. Taghavi[1] , **A. Babaei**[2], **and A. Mohammadpour**[1,3]

[1]Department of Mathematics, University of Mazandaran, Babolsar, Iran
taghavi@umz.ac.ir
&
[2]Department of Mathematics, University of Mazandaran, Babolsar, Iran
babaei@umz.ac.ir
&
[3]Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran
a.mohammadpour@stu.umz.ac.ir

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Abstract

In this paper, Reduced differential transform method is presented for solving nonlinear reaction-diffusion-convection initial value problems. The methodology with some known techniques shows that the present approach is simple and effective.To show the efficiency of the present method, four interesting examples is given.

Keywords: Nonlinear reaction-diffusion-convection problems; reduced differential transform method

MSC 2010 No.: 34K17; 35G25

1. Introduction

Nonlinear reaction–diffusion–convection (RDC) equations of the form

$$u_t = [(\sum_{i=0}^{l} a_i u^i) u_x]_x + [\sum_{i=0}^{p} b_i u^i]_x + [\sum_{i=0}^{r} c_i u^i], \tag{1}$$

where $u = u(x,t)$ is unknown function, $l,\ p,\ r$ are nonnegative integer constants and $a_i,\ b_i, c_i$ are real constants, and the subscripts t and x denote derivatives with respect to these variables, generalizes a great number of the known nonlinear second-order equations describing various processes in biology (Murray, 1993). Actually nonlinear phenomena occurs in a wide range of apparently different contexts in nature, for instance biological, economical, chemical and physical systems (Ames, 1972; Vazquez, 2006; Murray, 1993; Murray, 1977; Witelski, 1997). There are well-known methods such as Lie and conditional symmetry methods, which successfully applied to construct exact solutions for a wide range of nonlinear equations (Bluman et al., 2010; Cherniha and Pliukhin, 2007; Cherniha, 1998). In this paper, the differential transformation method (Zhou, 1986; Arikoglu and Ozkol, 2008; Keskin and Oturanc, 2009; Jang et al., 2006; Kurnaz and Oturance, 2005) is applied to solve the nonlinear reaction–diffusion–convection equation (1) under initial condition

$$u(x,0) = f(x). \tag{2}$$

The given problem can be transformed into a recurrence relation, using differential transformation operations, which leads to a series solution.

2. Reduced differential transform method

Consider a function of two variables $u(x,t)$ which is analytic and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of one dimensional differential transform, the function $u(x,t)$ can be represented as follows:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k,$$

where $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$. The basic definitions and operations of RDTM are reviewed as follows:

Definition 1: If function $u(x,t)$ is analytic and differentiated continuosly with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{3}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{4}$$

Combining (3) and (4) gives that:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{5}$$

In fact, above definition shows that, the concept of reduced differential transform is derived from the power series expansion (Keskin and Oturanc, 2009). For example, consider $u(x,t) = e^{x+t}$. This function can be written as

$$u(x,t) = e^{x+t} = \underbrace{\left(1 + x + \frac{x^2}{2} + \cdots\right)}_{e^x} \underbrace{\left(1 + t + \frac{t^2}{2} + \cdots\right)}_{e^t} = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j$$

otherwise,

$$\begin{aligned} u(x,t) &= e^{x+t} = e^x \underbrace{\left(1 + t + \frac{t^2}{2} + \cdots\right)}_{e^t} \\ &= e^x + e^x t + \frac{e^x t^2}{2} + \cdots = \sum_{k=0}^{\infty} \underbrace{\frac{e^x}{k!}}_{U_k(x)} t^k = \sum_{k=0}^{\infty} U_k(x) t^k. \end{aligned}$$

For more clarification, suppose that we have a nonlinear partial differential equation as

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),$$

with initial condition

$$u(x,0) = f(x),$$

where $L = \frac{\partial}{\partial t}$, R is a linear operator which has partial derivatives, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term.
According to the RDTM and (T1-T10) of bellow, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x) = G_k(x) - RU_k(x) - NU_k(x), \tag{6}$$

where $U_k(x)$, $RU_k(x)$, $NU_k(x)$ and $G_k(x)$ are the transformations of the functions $Lu(x,t)$, $Ru(x,t)$, $Nu(x,t)$ and $g(x,t)$. From initial condition $u(x,0) = f(x)$, we write

$$U_0(x) = f(x). \tag{7}$$

Substituting (7) into (6) and A straightforward iterative calculation, gives the $U_k(x)$ values for $k = 1, 2, ..., n$. Then the inverse transformation of the $\{U_k(x)\}_{k=0}^{n}$ gives the approximation solutoin as:

$$\widetilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k \tag{8}$$

where n is order of approximation solution. Therefore, the exact solutoin is given by:

$$u(x,t) = \lim_{n \longrightarrow \infty} \widetilde{u}_n(x,t). \tag{9}$$

The fundamental operations of reduced differential transform are listed in below.

Function Form	Transformed Form
$u(x,t)$	$U_k(x)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$T1:\ u(x,t)=c\ \ (c\ is\ a\ constant)$	$U_k(x)=\delta(k)\ =\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$
$T2:\ u(x,t)=v(x,t)+w(x,t)$	$U_k(x)=V_k(x)+W_k(x)$
$T3:\ u(x,t)=cv(x,t)$	$U_k(x)=cV_k(x)\ \ (c\ is\ a\ constant)$
$T4:\ u(x,t)=v(x,t)w(x,t)$	$U_k(x)=\sum_{k_1=0}^{k}V_{k_1}(x)W_{k-k_1}(x)$
$T5:\ u(x,t)=x^m t^n$	$U_k(x)=x^m\delta(k-n)\ =\begin{cases}x^m & k=n\\ 0 & k\neq n\end{cases}$
$T6:\ u(x,t)=x^m t^n v(x,t)$	$U_k(x)=x^m V_{k-n}(x)$
$T7:\ u(x,t)=\frac{\partial}{\partial t}v(x,t)$	$U_k(x)=(k+1)V_{k+1}(x)$
$T8:\ u(x,t)=\frac{\partial^m}{\partial x^m}v(x,t)$	$U_k(x)=\frac{\partial^m}{\partial x^m}V_k(x).$

Also

T9: if $u(x,t)=v_1(x,t)v_2(x,t)...v_{m-1}(x,t)v_m(x,t)$, then

$$U_k(x)=\sum_{k_{m-1}=0}^{k}\sum_{k_{m-2}=0}^{k_{m-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}(V_1)_{k_1}(V_2)_{k_2-k_1}\cdots(V_{m-1})_{k_{m-1}-k_{m-2}}(V_m)_{k-k_{m-1}}.$$

And finally T10: if $u(x,t)=v^m(x,t)\frac{\partial}{\partial x}w(x,t)$, then

$$U_k(x)=\sum_{k_m=0}^{k}\sum_{k_{m-1}=0}^{k_m}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\partial}{\partial x}W_{k_1}V_{k_2-k_1}V_{k_3-k_2}\cdots V_{k_{m-1}-k_{m-2}}V_{k-k_{m-1}}.$$

3. Application of reduced differential transform method

According to the RDTM and (T1-T10), we can construct the following iteration for the Eq.(1)

as:

$$
\begin{aligned}
(k+1)U_{k+1}(x) \quad = \quad & \frac{\partial}{\partial x}[a_0\frac{\partial}{\partial x}U_k(x) + a_1\sum_{k_1=0}^{k}\frac{\partial}{\partial x}U_{k_1}(x)U_{k-k_1}(x) \\
& +a_2\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2}\frac{\partial}{\partial x}U_{k_1}(x)U_{k_2-k_1}(x)U_{k-k_2}(x)+\cdots \\
& +a_l\sum_{k_l=0}^{k}\sum_{k_{l-1}=0}^{k_l}\sum_{k_{l-2}=0}^{k_{l-1}}\ldots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\partial}{\partial x}U_{k_1}(x)U_{k_2-k_1}(x)U_{k_3-k_2}(x)\cdots \\
& \qquad\qquad\qquad\qquad U_{k_l-k_{l-1}}(x)U_{k-k_l}(x)] \\
& +\frac{\partial}{\partial x}[b_0+b_1U_k(x)+b_2\sum_{k_1=0}^{k}U_{k_1}(x)U_{k-k_1}(x)+\cdots \\
& +b_p\sum_{k_{p-1}=0}^{k}\sum_{k_{p-2}=0}^{k_{p-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_{k_1}(x)U_{k_2-k_1}(x)U_{k_3-k_2}(x)\cdots \\
& \qquad\qquad\qquad\qquad U_{k_{p-1}-k_{p-2}}(x)U_{k-k_{p-1}}(x)] \\
& +[c_0+c_1U_k(x)+c_2\sum_{k_1=0}^{k}U_{k_1}(x)U_{k-k_1}(x)+\cdots \\
& +c_r\sum_{k_{r-1}=0}^{k}\sum_{k_{r-2}=0}^{k_{r-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}U_{k_1}(x)U_{k_2-k_1}(x)U_{k_3-k_2}(x)\cdots \\
& \qquad\qquad\qquad\qquad U_{k_{r-1}-k_{r-2}}(x)U_{k-k_{r-1}}(x)].
\end{aligned}
$$

From the initial condition (2), we can get the $U_0(x)$ and afterwards the $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{n}$ gives approximation solution as:

$$\widetilde{u}_n(x,t)=\sum_{k=0}^{n}U_k(x)t^k$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t)=\lim_{n\longrightarrow\infty}\widetilde{u}_n(x,t).$$

We apply this method for solving the examples which have solved by the other methods.

Example 1. In this example, we will consider the following initial value nonlinear problem (Shidfar et al., 2011; Cherniha and Pliukhin, 2007; Duangpithak and Torvattanabun, 2012)

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)+3u\frac{\partial u}{\partial x}+2(u-u^2), \\ \\ u(x,0)=2\sqrt{e^x-e^{-4x}}, \; -\infty<x<+\infty, \end{cases} \tag{10}$$

with the exact solution $u(x,t)=2e^{2t}\sqrt{e^x-e^{-4x}}$.

By using the basic properties of the reduced differential transform and formulas (T1-T10), we can find transformed form of Equation (10) as:

$$(k+1)U_{k+1}(x) = \frac{\partial}{\partial x}\sum_{k_1=0}^{k}\frac{\partial}{\partial x}U_{k_1}(x)U_{k-k_1}(x)+3\sum_{k_1=0}^{k}\frac{\partial}{\partial x}U_{k_1}(x)U_{k-k_1}(x) +2U_k(x)-2\sum_{k_1=0}^{k}U_{k_1}(x)U_{k-k_1}(x). \tag{11}$$

Initial condition $u(x,0)=2\sqrt{e^x-e^{-4x}}$ gives:

$$U_0(x)=2\sqrt{e^x-e^{-4x}} \tag{12}$$

After substituting (12) into (11), we obtain the next terms of $U_k(x)$ as:

$$\begin{aligned} U_1(x) &= 4\sqrt{-e^{-4x}+e^x},\ U_2(x)=4\sqrt{-e^{-4x}+e^x}, \\ U_3(x) &= \frac{8}{3}\sqrt{-e^{-4x}+e^x},\cdots,\ U_k(x)=\frac{(2)^{k+1}}{k!}\sqrt{-e^{-4x}+e^x} \end{aligned}$$

and the approximation solutoin as:

$$\widetilde{u}_n(x,t)=\sum_{k=0}^{n}\frac{(2)^{k+1}}{k!}\sqrt{-e^{-4x}+e^x}\ t^k=2\sqrt{-e^{-4x}+e^x}\sum_{k=0}^{n}\frac{(2t)^k}{k!}. \tag{13}$$

At last, we get the following exact solution :

$$u(x,t)=\lim_{n\longrightarrow\infty}\widetilde{u}_n(x,t)=2\ e^{2t}\ \sqrt{-e^{-4x}+e^x}$$

Example 2. Let us consider the following reaction-diffusion equation with exponential nonlinearities (Shidfar et al., 2011; **?**)

$$\begin{cases} \frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\exp(u)\frac{\partial u}{\partial x}\right)+\exp(-u)+1-\exp(u), \\ u(x,0)=\ln\left(\frac{1}{2}+2\cosh(x)\right),\ -\infty<x<+\infty, \end{cases} \tag{14}$$

with the exact solution $u(x,t) = \ln\left((\frac{1}{2}+\frac{\sqrt{5}}{2}\tanh(\frac{\sqrt{5}t}{2})+(\frac{1}{\cosh(\frac{\sqrt{5}t}{2})})(e^{(\frac{t}{2}-x)}+e^{(\frac{t}{2}+x)})\right)$. By using the transformation $v=exp(u)$, the problem (14) becomes

$$\begin{cases} \frac{\partial v}{\partial t}=v\frac{\partial^2 v}{\partial x^2}+1+v-v^2, \\ v(x,0)=\frac{1}{2}+2\cosh(x),\ -\infty<x<+\infty. \end{cases} \tag{15}$$

Taking reduced differential transform of Problem (15), the following are obtained:

$$(k+1)V_{k+1}(x) = \sum_{k_1=0}^{k}\frac{\partial^2}{\partial x^2}V_{k_1}(x)V_{k-k_1}(x) +\delta(k)+V_k(x)-\sum_{k_1=0}^{k}V_{k_1}(x)V_{k-k_1}(x), \tag{16}$$

and

$$V_0(x) = \frac{1}{2} + 2\cosh(x). \tag{17}$$

By using the recurrence relation (16) and the transformed initial condition (17), we obtain the following terms of $V_k(x)$:

$$\begin{aligned}
V_1(x) &= -\frac{5}{4} - \cosh(x), V_2(x) = \frac{1}{2}\cosh(x)(3 + 4\cosh(x)), \\
V_3(x) &= \frac{1}{4}(-81 - 116\cosh(x) - 56\cosh(2x) - 16\cosh(3x)) \\
V_4(x) &= \frac{1}{48}(85 + 165\cosh(x) + 89\cosh(2x) + 32\cosh(3x) + 4\cosh(4x)) \\
V_5(x) &= \frac{1}{480}(-1331 - 2234\cosh(x) - 1366\cosh(2x) \\
&\qquad - 626\cosh(3x) - 160\cosh(4x) - 8\cosh(5x)) + \cdots .
\end{aligned}$$

For instance the inverse transformation of the set of values $\{V_k(x)\}_{k=0}^{5}$, gives approximation solution as:

$$\begin{aligned}
\widetilde{v}_5(x,t) &= \sum_{k=0}^{5} V_k(x)t^k = \frac{1}{480}(-1331 - 2234\cosh(x) \\
&\quad -1366\cosh(2x) - 626\cosh(3x) - 160\cosh(4x) - 8\cosh(5x))t^5 \\
&\quad +\frac{1}{48}(85 + 165\cosh(x) + 89\cosh(2x) + 32\cosh(3x) + 4\cosh(4x))t^4 \\
&\quad +\frac{1}{4}(-81 - 116\cosh(x) - 56\cosh(2x) - 16\cosh(3x))t^3 \\
&\quad +\frac{1}{2}\cosh(x)(3 + 4\cosh(x))t^2 - (\frac{5}{4} + \cosh(x))t + \frac{1}{2} + 2\cosh(x).
\end{aligned} \tag{18}$$

We indicated the approximation solution $u_{50}(x,t)$ and the exact solution of the problem in the Figure 1.

The cpu time (on a pc with core 2 duo processor) for obtaining the approximate solution $u_{50}(x,t)$ was 0.78 s. A comparison between the relative error of this approximate solution and the relative error of approximate solution which was obtained by using homotopy analysis method (HAM) (Shidfar et al., 2011) has shown in figure 2. The cpu time (on the similar system) for the HAM method was 26.84 s. Thus the time needed using the RDTM is very very shorter than using the HAM.

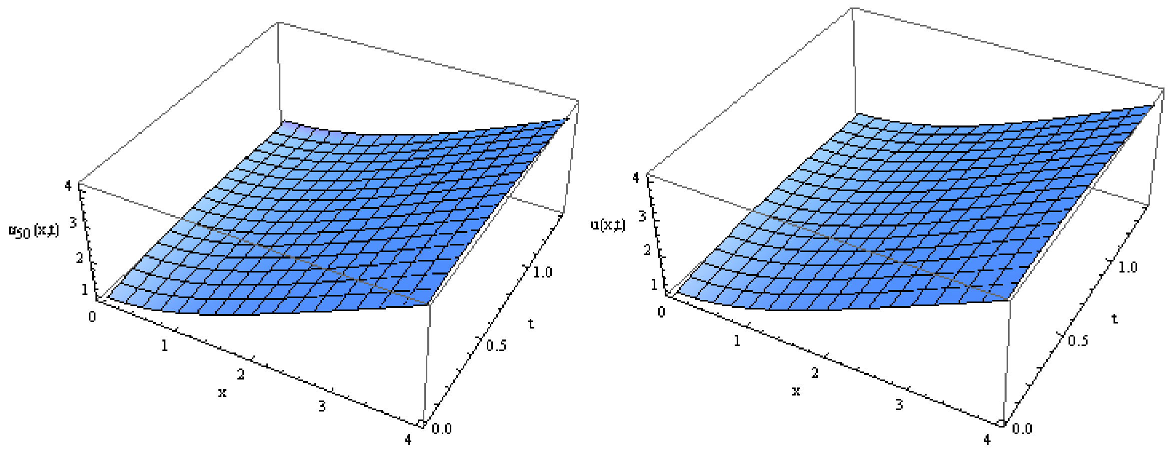

Fig. 1: The approximation solution $u_{50}(x,t)$ and the exact solution of the example 2.

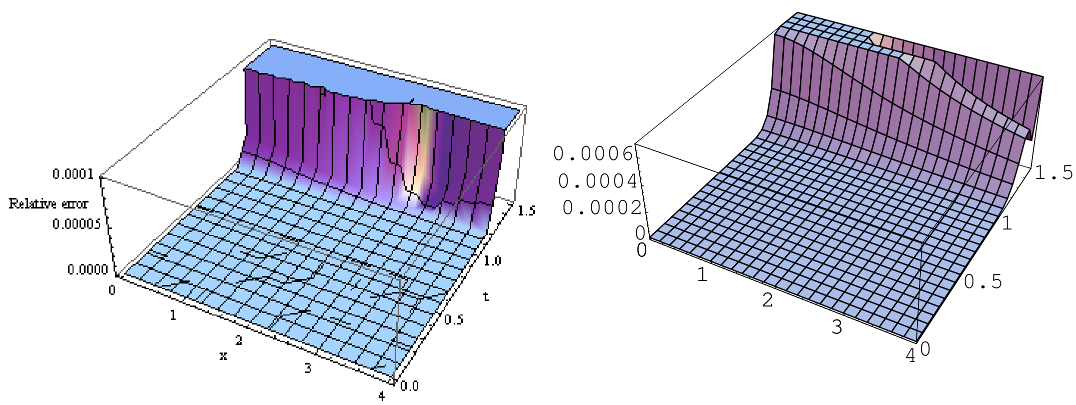

Fig. 2: Comparison between the relative errors of the approoimate solutions obtained by using The RDTM and The HAM.

4. Conclusion

In this paper, the reduced differential transform method (RDTM) has been successfully applied for reaction-diffusion-convection equations with given initial condition which gives rapidly converging series solutions. The obtained solution was compared with the exact solution.

In the first example the exact solution of problem was obtained. In the last example the approximate solution was compared with the solution obtained by the homotopy analysis method. The results show that the RDTM is more accurate and faster than the homotopy analysis method. It can be concluded that, RDTM is a very powerful and efficient technique for finding exact solutions for wide classes of problems and can be applied to many complicated linear and nonlinear problems and does not require linearization, discretization or perturbation. Computations in this paper were performed using Mathematica 7.

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