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Modelling the Dynamics of a Renewable Resource under Harvesting with Taxation as a Control Variable

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Abstract

The present paper describes a model of resource biomass and population with a non-linear catch rate function on resource biomass. The harvesting effort is assumed to be a dynamical variable. Tax on per unit harvested resource biomass is used as a tool to control exploitation of the resource. Pontryagin's Maximum Principle is used to find the optimal control to maintain the resource biomass and population at an optimal level. A numerical simulation is also carried out to support the analytical results.

Keywords: Optimal tax; harvesting effort; stability, bionomic equilibrium

MSC 2010 No.: 34D23, 49K15, 93D05, 93D20

1. Introduction

Conservation of renewable resources is a very important task for ecologists to maintain the ecological balance. The first work on renewable resource using mathematical modelling was done by Clark (1976, 1990). The capital for the optimal exploitation of a renewable resource stock of restricted malleability has been discussed by Clark and De Pee (1976). Many researchers have worked on renewable resources. Chaudhuri (1986) described a model of competing logistically growing fish species which is being harvested according to catch-per-unit-effort hypothesis. He described the optimal harvesting policy using Pontryagin's Maximum Principle. Dubey et al. (2003a) investigated a model in an aquatic environment that consists of two zones: i) free fishing zone and ii) reserved zone by taking the fishing effort as a control variable. They also proved that if fishing is done continuously in the unreserved zone, fish population can be maintained at an appropriate equilibrium level in the habitat. The model proposed by Dubey et al. (2003a) was extended by Kar and Misra (2006).

A ratio-dependent model with selective harvesting of prey species has been discussed by Kar (2004a). Again, Kar (2004b) proposed a model of prey-predator system with delay and harvesting. He showed that both the delay and harvesting effort play important roles on the stability of the system. Dhar et al. (2008) proposed and discussed a phytoplankton-fishery model, where fish depends upon plankton which grows logistically and the revenue is generated from fishing. Then they converted this model with delay for the digestion of plankton by fish. They found a threshold of conversional parameter for Hopf-bifurcation. Misra and Dubey (2010) analyzed a prey-predator model with discrete delay and the predator is harvested. Then they discussed stability analysis of equilibrium points and Hopf bifurcation taking the delay as a bifurcation parameter. Ji and Wu (2010) proposed a prey-predator model with Holling type II functional response incorporating a constant prey refuge and a constant rate of prey harvesting. They also discussed instability, global stability, and existence and uniqueness of limit cycles of the model.

A ratio dependent eco-epidemiological system where prey population is harvested has been discussed by Chakraborty et al. (2010). They also obtained a suitable condition for non-existence of a periodic solution around the interior equilibrium. Yunfei et al. (2010) described a phytoplankton-zooplankton model, in which both species are harvested for food. They found stability conditions of equilibria and conditions for the existence of Hopf-bifurcation. They also discussed the existence of bionomic equilibria and the optimal harvesting policy. Sadhukhan et al. (2010) described a three competing species model: i) prey, ii) predator and iii) super predator. These three species are harvested in this model. They studied the global stability, bionomic equilibrium and optimal harvesting policy. A prey-predator harvested model with non-monotonic functional response has also been studied by Kar et al. (2010). In this model, they introduced scaled harvesting efforts for both the species. Chakraborty et al. (2011a) discussed a prey-predator fishery model with stage structure for prey, where the adult prey and predator populations are harvested. They also observed singularity induced bifurcation phenomena when variation of the economic interest of harvesting was taken into account. Olivares and Arcos (2011) investigated a model of a renewable resource in an aquatic environment composed of two different patches. They also discussed the optimal harvesting policy using Pontryagin's maximum principle.

Increasing industrialization and population are important factors for the depletion of renewable resources. Shukla et al. (2011) proposed and analyzed the depletion of a renewable resource by population and industrialization with resource dependent migration. The resource biomass, which grows logistically, is depleted by population and industrialization but it is conserved by technological effort. The growth rate of the technological effort depends on the difference between carrying capacity and the current density of the resource biomass. They proved that the resource never becomes extinct by population and industrialization, if technological effort is applied appropriately for its conservation. Dubey and Patra (2013a) discussed a resource-based on population model where both are growing logistically and the resource is harvested according to the catch-per-unit effort hypothesis. Using the optimal harvesting policy they proved that the harvesting effort should always be kept less than the effort to maintain the resource and the population at an optimal equilibrium level. Bischi et al. (2014) proposed a fishery model with a discontinuous on-off harvesting policy. The basic assumption in their modeling is that harvesting must be stopped whenever the fish stock goes below a threshold level. They investigated the effect of different time scales (from continuous to discrete) on the dynamics of the model. In all of the previous cases the harvesting effort is taken to be a control variable.

In addition to the harvesting effort, there are some other tools which have been used as control variables. Taxation, lease of property rights, seasonal harvesting, license fees, creating reserve zones, fishing period are all seen as a control instrument. Among all of these, taxation is assumed to be the most efficient because of its flexibility, and many of the advantages of a competitive economic system can be better maintained by taking taxation as a control instrument. A dynamical model of single-species fishery is described by Dubey et al. (2002) using taxation as a control instrument to protect the fish population from over- exploitation. The dynamics of inshore-offshore fishery under variable harvesting was discussed by Dubey et al. (2003b). They proved that by increasing tax and discount rates, the overexploitation of fishery resources can be protected. The dynamics of two competing prey and one predator species was proposed and discussed by Kar et al. (2009). Here both the prey species are harvested according to the catch-per-unit effort hypothesis. In the above model, tax on per-unit harvested biomass has been used to control the over-exploitation of the resource biomass. The work of Dubey et al. (2003a) was further extended by Huo et al. (2012) taking into account the harvesting effort as a dynamical variable and taxation is a control variable. They also examined the optimal harvesting policy using Ponyryagin’s Maximum Principle. Dubey and Patra (2013b) described model of resource biomass and population in which the crowding effect is taken into account. The harvesting effort is assumed to be a dynamical variable and taxation as a control variable. Guo and Zou (2015) considered a stock-effort fishing model with discontinuous harvesting strategies. They also proved that discontinuous harvesting strategies are superior to continuous harvesting strategies.

The nonlinear harvesting rate function has also been used by some researchers. Ganguly and Chaudhuri (1995) proposed and analyzed a single species fishery model with realistic catch rate function instead of usual catch-per-unit-effort hypothesis. The fishing effort is assumed to be a dynamical variable in their model. They also discussed the stability analysis and optimal harvesting policy. Pradhan and Chaudhuri (1999a) investigated a mathematical model for growth and exploitation of a schooling fish species by taking into account a realistic catch rate function and taxation as a control instrument. Peng (2008) discussed a mathematical model involving continuous harvesting of a single species fishery. He assumed a reasonable catch-rate function and a suitable tax per unit biomass of landed fish imposed by some external energy. A prey-predator fishery model incorporating prey refuge, where prey is harvested, is proposed and analyzed by Chakraborty et al. (2011b). They also discussed Hopf-bifurcation by considering a density dependent mortality for the predator as bifurcation parameter. Then, they found the optimal tax with the help of Pontrygin's Maximum Principle.

Recently, Gupta et al. (2012) proposed and analyzed a bi-dimensional system of prey-predator model with non-linear harvesting rate of the prey population. They showed that the model can have two, one or no interior equilibrium point in the first quadrant and the system shows complex dynamical behavior (such as Saddle-node and Hopf-bifurcation) considering the rate of harvesting as bifurcation parameter. Using Pontrygin's Maximum Principle they also discussed the optimal singular control. Gupta and Chandra (2013) discussed a modified Laslie-Gower prey-predator model considering the harvesting effort as a control variable. The prey population is harvested according to a non-linear harvesting rate. They observed the complex dynamical behaviour of the model system such as Saddle-node, Transicritical, Hopf- Andronov and Bogdanov-Takens bifurcation. Ghosh and Kar (2014) described a prey-predator system with harvesting of prey species in the presence of some alternative food to predator. They considered an alternative functional form as harvesting rate instead of using the catch-per-unit-effort (CPUE) hypothesis. They observed that alternative source of food to the predator has a negative effect on the growth of prey species. Using Pontrygin's Maximum Principle, they found an optimal tax policy. Gupta et al. (2014) considered a three dimensional prey-predator model with Holling type-II functional response and non-linear harvesting rate of prey population. Here they have considered the harvesting effort as a dynamical variable and tax as control variable. They proved that the system has periodic, quasi periodic and chaotic solutions. Using sensitivity analysis they have shown that the solutions are highly dependent on the initial conditions. Using Pontrygin's Maximum Principle they found optimal tax to maintain the resource at an optimal level.

In most of the harvesting models, the harvesting rate function follows the proportional to catch-per-unit effort hypothesis, i.e., $h(t) = qE(t)B(t)$, where q is catchability coefficient, E is the harvesting effort and B is the resource biomass. But in fishery models, this catch-rate-function has some unrealistic features such as

i) random search for the resource,
ii) equal likelihood of being captured for every resource,
iii) unbounded linear increase of h with E for fixed B, and
iv) unbounded linear increase of h with B for fixed E.

In order to avoid the above circumstances, the following non-linear harvesting rate function has been used by some researches (Ganguly and Chaudhuri (1995), Pradhan and Chaudhuri (1999a), Peng (2008), Chakraborty et al. (2011b), Gupta *et al*. (2012), Gupta and Chandra (2013), Ghosh and Kar (2014)):

$$h = \frac{qEB}{mE + nB},$$

where m and n are positive constant. This catch rate function is always saturated with respect to effort level and stock abundance. The parameter m is proportional to the ratio of the stock-level to the catch rate at higher level of effort and n is proportional to the ratio of the effort level to the catch rate at higher stock levels.

In this catch rate function we observe the following:

a) $h \to \dfrac{qB}{m}$ as $E \to \infty$ for fixed value of B,

b) $h \to \dfrac{qE}{n}$ as $B \to \infty$ for fixed value of E, and

c) h has singularity at $B = 0$ and $E = 0$.

In order to remove the singularity of *h*, we modify the harvesting rate *h* in the following form

$$h(t) = \frac{qE(t)B(t)}{1 + mE(t) + nB(t)}. \tag{1}$$

Keeping the above in view, we formulate a dynamical model of resource biomass and population, both growing logistically. The resource biomass, which is of commercial importance, is harvested according to the harvesting rate function *h(t)* defined in equation (1). The harvesting effort is taken as a dynamical variable and taxation as a control variable. Then we analyze the existence of non-negative equilibria and their local and global stability. We also discuss the maximum sustainable yield (MSY) and optimal harvesting policy.

2. Mathematical Model

Let us consider a resource biomass of density *B(t)* and a population of density *N(t)*, both growing logistically in absence of each other. Following Dubey and Patra (2013b), the dynamics of resource biomass and population may be governed by the following system of ordinary differential equations:

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right) - \alpha_1 NB - \alpha_2 NB^2 - \frac{qEB}{1 + mE + nB},$$
$$\frac{dN}{dt} = sN\left(1 - \frac{N}{L}\right) + \beta_1 NB + \beta_2 NB^2.$$

In the above model, r and s are the intrinsic growth rates of resource and population respectively, K and L are their respective carrying capacities, α_1 and α_2 are the depletion rates of resource biomass due to the population, β_1 and β_2 are the growth rates of population due to the presence of resource biomass. Now we assume that the resource biomass is being harvested according to the modified harvesting rate function $h(t)$ given in equation (1) and a regulatory agency imposes a tax $\tau\ (>0)$ per unit resource biomass to protect the over-exploitation of the resource.

Thus, the net economic revenue is

$$R(t) = E\left\{\frac{(p - \tau)qB}{1 + mE + nB} - c\right\},$$

where p is the fixed selling price per unit biomass and c is the fixed cost of harvesting per unit of effort.

Taking the harvesting effort E as a dynamic variable, its dynamics may be governed by the following differential equation

$$\frac{dE}{dt} = \alpha_0 E \left\{ \frac{(p-\tau)qB}{1+mE+nB} - c \right\},$$

where α_0 is the stiffness parameter measuring the intensity of reaction between the effort and the perceived rent. Therefore, we consider the following dynamical system of equations

$$\frac{dB}{dt} = rB\left(1-\frac{B}{K}\right) - \alpha_1 NB - \alpha_2 NB^2 - \frac{qEB}{1+mE+nB}, \tag{2a}$$

$$\frac{dN}{dt} = sN\left(1-\frac{N}{L}\right) + \beta_1 NB + \beta_2 NB^2, \tag{2b}$$

$$\frac{dE}{dt} = \alpha_0 E \left\{ \frac{qB(p-\tau)}{1+mE+nB} - c \right\}, \tag{2c}$$

$$B(0) \geq 0,\ N(0) \geq 0,\ E(0) \geq 0.$$

The units of variables and parameters are given in Table 1. The parameter δ is defined in equation (11).

Table 1. Units of variables and parameters

Variables/Parameters	**Units**
B, N	number per unit area (tons)
E	total number of vessel per day
r, s	per day
K, L	number per unit area (tons)
$\alpha_i, \beta_i, i = 1, 2$	per day
q	per day
m, n	constants
α_0	per day
p	dollar per individual per day
τ	dollar per individual per day
c	dollar per individual per day
δ	dollar per individual per day

In the next section, we shall discuss the stability analysis of the model system (2a)-(2c).

3. Stability Analysis

First of all, we state the following lemma which establishes a region of attraction of the model system (2a)-(2b).

Lemma 1.

The set

$$\Omega = \{(B, N, E):\ 0 \le B(t) \le K,\ 0 \le \mathrm{N(t)} \le L_0,\ 0 \le B(t) + E(t) \le \theta\}$$

is a region of attraction for all solutions initiating in the interior of the positive orthant, where

$$L_0 = \frac{L}{s}(s + \beta_1 K + \beta_2 K^2),\ \ \theta = \frac{2r\alpha_0(p-\tau)K}{\delta},\ p > \tau.$$

The above lemma shows that all solutions of model (2a)-(2c) are non-negative and bounded, so our model is biologically well-behaved. The proof of this lemma is the same as the Freedman and So (1985), Shukla and Dubey (1997), and hence omitted.

To discuss the biological equilibrium of the system (2a)-(2c), we note that the model system has six non-negative equilibrium points, *viz*,

$$P_0(0,0,0),\ P_1(K,0,0),\ P_2(0,L,0),\ P_3(\tilde{B},\tilde{N},0),\ P_4(\hat{B},0,\hat{E}),\ P^*(B^*,N^*,E^*).$$

The equilibrium points P_0, P_1 and P_2 always exist. We show the existence of other equilibrium points as follows:

Existence of $P_3(\tilde{B},\tilde{N},0)$: $\tilde{B}$ and $\tilde{N}$ are the positive solutions of the following two equations

$$r\left(1-\frac{B}{K}\right) - \alpha_1 N - \alpha_2 NB = 0, \tag{3a}$$

$$s\left(1-\frac{N}{L}\right) + \beta_1 B + \beta_2 B^2 = 0. \tag{3b}$$

From equation (3b), we have

$$N = \frac{L}{s}\left(s + \beta_1 B + \beta_2 B^2\right). \tag{3c}$$

Putting the value of N in equation (3a), we get a cubic equation in B, i.e.,

$$a_1 B^3 + a_2 B^2 + a_3 B + a_4 = 0\,, \tag{3d}$$

where

$$a_1 = \alpha_2 \beta_2 LK,$$
$$a_2 = (\alpha_2 \beta_1 + \alpha_1 \beta_2) LK,$$
$$a_3 = rs + \alpha_1 \beta_1 LK + s\alpha_2 LK,$$
$$a_4 = (\alpha_1 L - r)sK.$$

Equation (3d) has a unique positive real root if $r > \alpha_1 L$.

Putting the value of $\tilde{B}$, we can calculate $\tilde{N}$ from (3c), and, thus, we can state the following result.

Theorem 1.

The equilibrium point $P_3(\tilde{B}, \tilde{N}, 0)$ exists if $r > \alpha_1 L$.

The above theorem shows that for the co-existence of resource biomass and population, the intrinsic growth rate of resource biomass should be larger than a threshold value which depends on the carrying capacity L of the population and the bilinear depletion rate coefficient α_1 of the resource biomass.

Existence of the point $P_4(\hat{B}, 0, \hat{E})$:

In this case, $\hat{B}$ and $\hat{E}$ are the positive solutions of the following equations:

$$r\left(1 - \frac{B}{K}\right) - \frac{qE}{1 + mE + nB} = 0, \tag{4a}$$

$$\frac{qB(p-\tau)}{1 + mE + nB} - c = 0. \tag{4b}$$

The above two equations yield

$$E = \frac{r}{c}\left(1 - \frac{B}{K}\right)(p - \tau)B. \tag{4c}$$

From (4c), we note that $E > 0$, as $B < K$ and $p > \tau$.

Then putting the value of E from (4c) in equation (4a), we get a quadratic equation in B, i.e.,

$$b_1 B^2 + b_2 B - 1 = 0, \tag{4d}$$

where

$$b_1 = \frac{mr(p-\tau)}{cK}, \; b_2 = \left\{\frac{q(p-\tau)}{c} - \frac{(p-\tau)mr}{c} - n\right\}.$$

Thus equation (4d) has always a positive real root given by

$$B = \frac{-b_2 + \sqrt{b_2^2 + 4b_1}}{2b_1},$$

and we can state the following result.

Theorem 2.

The equilibrium $P_4(\hat{B}, 0, \hat{E})$ always exists.

Existence of $P^*(B^*, N^*, E^*)$:

Here B^*, N^* and E^* are the positive solutions of the following equations:

$$r\left(1 - \frac{B}{K}\right) - \alpha_1 N - \alpha_2 NB - \frac{qE}{1 + mE + nB} = 0, \tag{5a}$$

$$s\left(1 - \frac{N}{L}\right) + \beta_1 B + \beta_2 B^2 = 0, \tag{5b}$$

$$\frac{qB(p-\tau)}{1 + mE + nB} - c = 0. \tag{5c}$$

From equations (5b) and (5c), we get

$$N = \frac{L}{s}\left(s + \beta_1 B + \beta_2 B^2\right), \text{ and } E = \frac{q(p-\tau)B}{mc} - \frac{nB}{m} - \frac{1}{m}. \tag{5d}$$

Putting these values in equation (5a), we get a biquadratic equation in B, namely,

$$c_1 B^4 + c_2 B^3 + c_3 B^2 + c_4 B + c_5 = 0, \tag{5e}$$

where

$$c_1 = \frac{L\alpha_2\beta_2}{s}, \quad c_2 = \frac{L}{s}\left(\alpha_2\beta_1 + \alpha_1\beta_2\right), \quad c_3 = \left(\frac{L}{s}\alpha_1\beta_1 + \frac{r}{K} + L\alpha_2\right),$$
$$c_4 = L\alpha_1 + \frac{q(p-\tau) - nc}{m(p-\tau)} - r, \quad c_5 = -\frac{c}{m(p-\tau)}.$$

Using Descartes' rule of sign, we note that equation (5e) has always a unique positive real root, $B = B^*$. Using the value of B^*, we can get the values of N^* and E^* from (5d) .

Now we can state the following result.

Theorem 3.

The equilibrium $P^*(B^*, N^*, E^*)$ exists if

$$0 < \tau < p - \frac{c(nB^* + 1)}{qB^*}. \tag{6}$$

Now we analyze the local as well as global stability behavior of these non-negative equilibrium points. For local stability behavior, first we find the variational matrices at each equilibrium point and then using Eigenvalue method and Routh-Hurwitz criteria, we can conclude the following results:

i) The point P_0 is a saddle point with unstable manifold in the *B*-*N* plane and stable manifold in the *E*-direction.

ii) a) The point P_1 is always a saddle point with stable manifold in the *B*-direction and unstable manifold in the *N*-*E* plane if $0 < \tau < p - \frac{c}{qK}(1 + nK)$.

b) If $\tau > p - \frac{c}{qK}(1 + nK)$, then P_1 is again a saddle point with stable manifold in the *B*-*E* plane and unstable manifold in the *N*-direction.

iii) a) The point P_2 is locally asymptotically stable if $r < \alpha_1 L$.

b) If $r > \alpha_1 L$, then P_2 is a saddle point with unstable manifold in the *B*-direction and stable manifold in the *N*-*E* plane.

iv) a) If $\tau > p - \frac{c(1 + n\tilde{B})}{q\tilde{B}}$, then the point P_3 is locally asymptotically stable.

b) If $\tau < p - \frac{c(1 + n\tilde{B})}{q\tilde{B}}$, then P_3 is a saddle point with stable manifold in the *B*-*N* plane and unstable manifold in the *E*-direction.

v) The point P_4 is always a saddle point with unstable manifold in the *N*-direction and stable manifold in the *B*-*E* plane if

$$\frac{r}{K} > \frac{qn\hat{E}}{(1 + m\hat{E} + m\hat{B})^2}.$$

Let λ be an eigenvalue of the variational matrix M^* evaluated at the interior equilibrium point $P^*(B^*, N^*, E^*)$. Then the characteristic equation is given by

$$\lambda^3 + A_1\lambda^2 + B_1\lambda + C_1 = 0\,, \tag{7}$$

where

$$A_1 = \left(\frac{rB^*}{K} + \alpha_2 N^* B^* - \frac{qnE^* B^*}{(1+mE^*+nB^*)^2}\right) + \frac{sN^*}{L} + \frac{\alpha_0 qm(p-\tau)B^* E^*}{(1+mE^*+nB^*)^2}\,,$$

$$\begin{aligned} B_1 = {} & \left(\frac{rB^*}{K} + \alpha_2 N^* B^* - \frac{qnE^* B^*}{(1+mE^*+nB^*)^2}\right)\left(\frac{sN^*}{L} + \frac{\alpha_0 qm(p-\tau)B^* E^*}{(1+mE^*+nB^*)^2}\right) \\ & + \frac{\alpha_0 q(p-\tau)B^* E^*}{(1+mE^*+nB^*)^2}\left(\frac{smN^*}{L} + \frac{q(1+nB^*)(1+mE^*)}{(1+mE^*+nB^*)^2}\right) \\ & + (\alpha_1 B^* + \alpha_2 B^{*2})(\beta_1 N^* + 2\beta_2 N^* B^*), \end{aligned}$$

$$\begin{aligned} C_1 = {} & \left(\frac{rB^*}{K} + \alpha_2 N^* B^* - \frac{qnE^* B^*}{(1+mE^*+nB^*)^2}\right)\left(\frac{sN^*}{L} \times \frac{\alpha_0 qm(p-\tau)B^* E^*}{(1+mE^*+nB^*)^2}\right) \\ & + \frac{qB^*(1+nB^*)}{(1+mE^*+nB^*)^2}\left(\frac{sN^*}{L} \times \frac{\alpha_0 q(p-\tau)E^*(1+mE^*)}{(1+mE^*+nB^*)^2}\right) \\ & + (\alpha_1 B^* + \alpha_2 B^{*2})(\beta_1 N^* + 2\beta_2 N^* B^*)\frac{\alpha_0 qm(p-\tau)B^* E^*}{(1+mE^*+nB^*)^2}. \end{aligned}$$

Using the Routh-Hurwitz criteria, we note that all roots of equation (7) have negative real parts iff

$$A_1 > 0, \quad C_1 > 0 \text{ and } A_1 B_1 - C_1 > 0. \tag{8}$$

Thus, we are now able to state the following results.

Theorem 4.

The interior equilibrium point P^* is locally asymptotically stable iff inequalities in equation (8) hold.

In the following theorem, we state sufficient conditions under which P^* is globally asymptotically stable.

Theorem 5.

The interior equilibrium point P^* is globally asymptotically stable in the region Ω if the following conditions hold:

$$\left(\frac{r}{K} + \alpha_2 N^*\right)(1 + mE^* + nB^*) > nqE^*\,, \tag{9a}$$

$$\left[-\alpha_1+\alpha_2K+k_1\left\{\beta_1+\beta_2(K+B^*)\right\}\right]^2<\left[\frac{r}{K}+\alpha_2N^*-\frac{nqE^*}{(1+mE^*+nB^*)}\right]\frac{k_1s}{L}. \tag{9b}$$

Proof:

Proof of this theorem is given in Appendix A.

4. Maximum Sustainable Yield

The maximum rate of harvesting any biological resource biomass is called the maximum sustainable yield (MSY) and any larger harvest rate will lead to the depletion of resource eventually to zero. In absence of any population, the value of MSY is given by [Clark (1976)]

$$h^0_{MSY}=\frac{rK}{4}.$$

If the resource biomass is subjected to the harvesting by a population, the sustainable yield is given by

$$h=\frac{qE^*B^*}{1+mE^*+nB^*}=rB^*(1-\frac{B^*}{K})-\alpha_1N^*B^*-\alpha_2N^*B^{*2}.$$

We note that

$$\frac{\partial h}{\partial B^*}=0 \quad \text{yields} \quad B^*=\frac{K(r-\alpha_1N^*)}{2(r+\alpha_2KN^*)} \text{ and } \frac{\partial^2 h}{\partial B^{*2}}<0.$$

Thus,

$$h_{MSY}=\frac{K(r-\alpha_1N^*)^2}{4(r+\alpha_2N^*K)}, \text{ when } B^*=\frac{K(r-\alpha_1N^*)}{2(r+\alpha_2KN^*)}.$$

From the above equations, it is interesting to note that, when $N^*=0,$ then $B^*=\frac{K}{2}$ and

$$h_{MSY}=\frac{rK}{4}=h^0_{MSY}.$$

This result matches the result of Clark (1976).

If $h>h_{MSY}$, then it denotes the overexploitation of the resource and consequently the resource biomass decreases. If $h<h_{MSY}$, then the resource biomass is under exploitation and the resource biomass may be maintained at an appropriate level.

5. Bionomical Equilibrium

The bionomic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass is equal to the total cost of harvested biomass, i.e. the economic rent is completely dissipated.

The economic revenue at time t is given by

$$\Pi = \left(\frac{pqB}{1+mE+nB} - c\right)E. \tag{10a}$$

The bionomic equilibrium is $P_\infty(B_\infty, N_\infty, E_\infty)$, where B_∞, N_∞ and E_∞ are the positive solutions of

$$\dot{B} = \dot{N} = \dot{E} = \Pi = 0.$$

From $\Pi = 0$, we get $E = E_\infty = \dfrac{1}{m}\left[\left(\dfrac{pq}{c} - n\right)B - 1\right],$ (10b)

$$\dot{N} = 0 \text{ gives us } N_\infty = \frac{L}{s}(s + \beta_1 B + \beta_2 B^2), \tag{10c}$$

Using the values of $E = E_\infty$ and $N = N_\infty$, then $\dot{B} = 0$ gives a biquadratic equation in B which is given by

$$d_1 B^4 + d_2 B^3 + d_3 B^2 + d_4 B + d_5 = 0, \tag{10d}$$

where

$$d_1 = \frac{L\alpha_2\beta_2}{s},\ d_2 = (\alpha_2\beta_1 + \alpha_1\beta_2)\frac{L}{s},\ d_3 = \left(\frac{r}{K} + L\alpha_2 + \frac{L\alpha_1\beta_1}{s}\right),$$

$$d_4 = (L\alpha_1 + \frac{pq - cn}{mp} - r),\ d_5 = -\frac{c}{mp}.$$

We note that equation (10d) has always a unique positive real root $B = B_\infty$. Putting the value of B_∞ in (10b) and (10c), we can find the value of N_∞ and E_∞. It may be noted that E_∞ exists if

$$\left(\frac{pq}{c} - n)\right)B_\infty > 1.$$

6. Optimal Harvesting Policy

The net revenue to the society,

$\Pi(B,N,E,\tau,t)$ = the net economic revenue to the harvesting agency + the net economic revenue to the regulatory agency

$$= \left(\frac{pqB}{1+mE+nB}-c\right)E .$$

Thus, our aim is to solve the maximization problem

$$J=\int_0^\infty e^{-\delta t}\left(\frac{pqB}{1+mE+nB}-c\right)E(t)dt , \tag{11}$$

subject to the state equation (5a)-(5c) and to the control constraints

$$\tau_{\min}<\tau<\tau_{\max} . \tag{12}$$

In equation (11), δ is the instantaneous rate of the annual discount.

To solve the maximization problem, we adopt Pontryagin's Maximum Principle. The Hamiltonian function H is given by

$$\begin{aligned} H=e^{-\delta t}\left(\frac{pqB}{1+mE+nB}-c\right)E+\lambda_1(t)\left[rB\left(1-\frac{B}{K}\right)-\alpha_1NB-\alpha_2NB^2-\frac{qEB}{1+mE+nB}\right] \\ +\lambda_2(t)\left[sN\left(1-\frac{N}{L}\right)+\beta_1NB+\beta_2NB^2\right]+\lambda_3(t)\left[\alpha_0E\left\{\frac{qB(p-\tau)}{1+mE+nB}-c\right\}\right], \end{aligned} \tag{13}$$

where $\lambda_i,\ i=1,2,3$ are adjoint variables.

The optimal control will be a combination of bang-bang control and singular control as H is linear in the control variable τ in equation (13).

H will be maximized under the control set (12), if

$$\frac{\partial H}{\partial \tau}=0\Rightarrow\lambda_3=0. \tag{14}$$

This is a necessary condition for the singular control to be optimal. Using the Maximum Principle, we get

$$\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial B},\quad \frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial N},\quad \frac{d\lambda_3}{dt}=-\frac{\partial H}{\partial E}. \tag{15}$$

From the previous equations we get

$$\frac{d\lambda_1}{dt} = -pqe^{-\delta t}\frac{(1+mE)E}{(1+mE+nB)^2} \\ -\lambda_1\left[r\left(1-\frac{2B}{K}\right)-\alpha_1 N-2\alpha_2 NB-\frac{qE(1+mE)}{(1+mE+nB)^2}\right]-\lambda_2\left[\beta_1 N+2\beta_2 NB\right], \tag{16a}$$

$$\frac{d\lambda_2}{dt} = \lambda_1\left[\alpha_1 B+\alpha_2 B^2\right]-\lambda_2\left[s\left(1-\frac{2N}{L}\right)+\beta_1 B+\beta_2 B^2\right], \tag{16b}$$

$$\frac{d\lambda_3}{dt} = e^{-\delta t}\left[\frac{pqB(1+nB)}{(1+mE+nB)^2}-c\right]-\lambda_1\frac{qB(1+nB)}{(1+mE+nB)^2} = 0\,. \tag{16c}$$

Using equations (5a)-(5c), we can re-write these previous equations as follows:

$$\frac{d\lambda_1}{dt} = -pqe^{-\delta t}\frac{(1+mE)E}{(1+mE+nB)^2} \\ +\lambda_1\left[\frac{rB}{K}+\alpha_2 NB-\frac{qnEB}{(1+mE+nB)^2}\right]-\lambda_2\left[\beta_1 N+2\beta_2 NB\right], \tag{17a}$$

$$\frac{d\lambda_2}{dt} = \lambda_1\left[\alpha_1 B+\alpha_2 B^2\right]+\lambda_2\left[\frac{sN}{L}\right], \tag{17b}$$

$$\lambda_1 = e^{-\delta t}\left[p-\frac{c(1+mE+nB)^2}{qB(1+nB)}\right]. \tag{17c}$$

The shadow price along the singular path is $\mu(t)=\lambda_1(t)e^{\delta t}=\left(p-\frac{c(1+mE+nB)^2}{qB(1+nB)}\right)$.

Putting the value of λ_1 in equation (17b), we get an equation

$$\frac{d\lambda_2}{dt}-A_2\lambda_2 = B_2 e^{-\delta t}, \tag{18a}$$

where

$$A_2=\frac{sN}{L},\quad B_2=\left[p-\frac{c(1+mE+nB)^2}{qB(1+nB)}\right](\alpha_1 B+\alpha_2 B^2)\,.$$

The solution of this equation is

$$\lambda_2 = K_0 e^{A_2 t}-\frac{e^{-\delta t}}{(\delta+\frac{sN}{L})}\left[p-\frac{c(1+mE+nB)^2}{qB(1+nB)}\right](\alpha_1 B+\alpha_2 B^2)\,.$$

Now, when $t \to \infty$, then $\lambda_2 e^{\delta t}$ is bounded if $K_0 = 0$. Thus,

$$\lambda_2 = -\frac{e^{-\delta t}}{(\delta + \frac{sN}{L})}\left[p - \frac{c(1+mE+nB)^2}{qB(1+nB)} \right](\alpha_1 B + \alpha_2 B^2)\,. \tag{18b}$$

Putting the value of λ_2 in equation (17a), we get

$$\frac{d\lambda_1}{dt} - A_3\lambda_1 = B_3 e^{-\delta t}\,, \tag{19a}$$

where

$$A_3 = \left[\frac{rB}{K} + \alpha_2 NB - \frac{qnEB}{(1+mE+nB)^2} \right],$$

$$B_3 = \left[-\frac{pqE(1+mE)}{(1+mE+nB)^2} + \frac{\left\{ p - \frac{c(1+mE+nB)^2}{qB(1+nB)} \right\}(\alpha_1 B + \alpha_2 B^2)(\beta_1 N + 2\beta_2 NB)}{\left(\delta + \frac{sN}{L} \right)} \right].$$

The solution of equation (19a) can be written as

$$\lambda_1 = K_1 e^{A_4 t} - \frac{B_3 e^{-\delta t}}{A_3 + \delta}\,,$$

Now, when $t \to \infty$, then $\lambda_1 e^{\delta t}$ is bounded if $K_1 = 0$. Thus,

$$\lambda_1 = -\frac{B_3 e^{-\delta t}}{A_3 + \delta}\ . \tag{19b}$$

From equation (17c) and (19b), we get

$$\left[p - \frac{c(1+mE+nB)^2}{qB(1+nB)} \right] + \frac{B_3}{A_3 + \delta} = 0. \tag{20}$$

This equation gives us the optimal level of resource biomass and population, i.e., $B = B_\delta, N = N_\delta$ and $E = E_\delta$. Then tax τ_δ is given by

$$\tau = \tau_\delta = p - \frac{c}{qB_\delta}(1 + mE_\delta + nB_\delta)\,. \tag{21}$$

7. Hopf-Bifurcation

Mathematical analysis of the present model suggests that the system has six equilibrium points out of which some are stable and some unstable. Since the model is highly non-linear, it is interesting to explore the non-linear behaviour of the system in the form of existence of limit cycle and Hopf bifurcation. The characteristic polynomial for the system at P^* is given in equation (7).

If τ is the bifurcation parameter, then for some $\tau = \tau_{cr}$, the necessary and sufficient conditions for Hopf Bifurcation to occur are

$$\begin{aligned}
&1.\ A_3(\tau_{cr}) > 0, \quad C_3(\tau_{cr}) > 0 \\
&2.\ f(\tau_{cr}) \equiv A_3(\tau_{cr})B_3(\tau_{cr}) - C_3(\tau_{cr}) = 0 \text{ and} \\
&3.\ \mathrm{Re}\left[\frac{d\lambda_j}{d\tau}\right]_{\tau=\tau_{cr}} \neq 0, \quad j = 1, 2, 3
\end{aligned} \tag{22}$$

The condition $f \equiv A_3B_3 - C_3 = 0$ results in the equation in τ having one of the root as τ_{cr}. Since we have $B_3 > 0$ at $\tau = \tau_{cr}$, there exists an interval containing τ_{cr}, say $(\tau_{cr} - \varepsilon, \tau_{cr} + \varepsilon)$ for some $\varepsilon > 0$ for which $\tau_{cr} - \varepsilon > 0$ such that B_3 remains positive for $\tau \in (\tau_{cr} - \varepsilon, \tau_{cr} + \varepsilon)$. Thus, for $\tau \in (\tau_{cr} - \varepsilon, \tau_{cr} + \varepsilon)$ the characteristic polynomial of P^* cannot have real positive roots. For $\tau = \tau_{cr}$ we get

$$(\lambda^2 + B_3)(\lambda + A_3) = 0. \tag{23}$$

This has three roots $\lambda_1 = +i\sqrt{B_3}$, $\lambda_2 = -i\sqrt{B_3}$, $\lambda_3 = -A_3$.

For $\tau \in (\tau_{cr} - \varepsilon, \tau_{cr} + \varepsilon)$, the roots of characteristic polynomial are of the general forms

$$\lambda_1 = \beta_1(\tau) + i\gamma_1(\tau),\ \lambda_2 = \beta_2(\tau) - i\gamma_2(\tau),\ \lambda_3 = -A_3(\tau).$$

The third condition can be verified as follows:
Substituting $\lambda_j = \beta_1(\tau) + i\gamma_1(\tau)$ into (23) and taking its derivative, we have

$$\begin{aligned}
&R(\tau)\beta_1'(\tau) - S(\tau)\gamma_1'(\tau) + T(\tau) = 0, \\
&S(\tau)\beta_1'(\tau) + R(\tau)\gamma_1'(\tau) + U(\tau) = 0,
\end{aligned}$$

where

$$\begin{aligned}
R(\tau) &= 3\beta_1^2(\tau) + 2d_1(\tau)\beta_1(\tau) + d_2(\tau) - 3\gamma_1^2(\tau),\\
S(\tau) &= 6\beta_1(\tau)\gamma_1(\tau) + 2d_1(\tau)\gamma_1(\tau),\\
T(\tau) &= \beta_1^2(\tau)d_1'(\tau) + d_2'(\tau)\beta_1(\tau) + d_3'(\tau) - d_1'(\tau)\gamma_1^2(\tau),\\
U(\tau) &= 2\beta_1(\tau)\gamma_1(\tau)d_1'(\tau) + d_2'(\tau)\gamma_1(\tau).
\end{aligned}$$

Hence,

$$\operatorname{Re}\left[\frac{d\lambda_j}{d\tau}\right]_{\tau=\tau_{cr}} = \left.\frac{SU+RT}{R^2+S^2}\right|_{\tau=\tau_{cr}} \neq 0,$$

as $S(\tau_{cr})U(\tau_{cr}) + R(\tau_{cr})T(\tau_{cr}) \neq 0$ and also $\lambda_3(\tau_{cr}) = -A_3(\tau_{cr}) \neq 0$. Thus, we can state the following theorem.

Theorem 6.

Under the assumptions given in equation (22), there is a simple Hopf bifurcation at equilibrium point P^* at some critical value of the parameter τ given by the equation $f(\tau_{cr}) = 0$.

8. Numerical Simulations

In this section, we present numerical simulation results. For the model system (2a)-(2c), we choose the following set of values for the parameters

$$\begin{aligned}
&r = 1.6,\ s = 1.2,\ K = 100,\ L = 100,\ p = 25,\ q = 1,\ \alpha_0 = 1,\ \alpha_1 = 0.001,\\
&\alpha_2 = 0.0001,\ \beta_1 = 0.01,\ \beta_2 = 0.0001,\ c = 7,\ \tau = 0.1,\ m = 4,\ n = 1,
\end{aligned} \tag{24}$$

with initial conditions $B(0) = 5, N(0) = 25, E(0) = 10.$

For the above set of values of the parameters, condition (6) for the existence of the interior equilibrium is satisfied. Thus, the positive equilibrium point $P^*(B^*, N^*, E^*)$ is given by

$$B^* = 41.2666,\ N^* = 148.5799,\ E^* = 26.1311.$$

We also note that all the conditions of Theorem 4 are satisfied for the set of parameters chosen in (24). Thus the equilibrium point $P^*(B^*, N^*, E^*)$ is locally asymptotically stable.

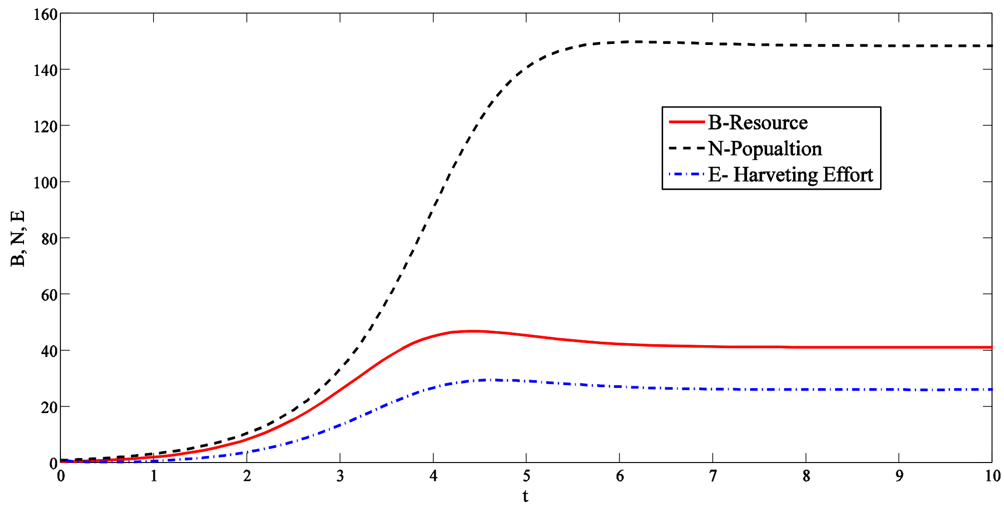


Figure 1. Time series for *B, N* and *E* for the set of values of parameters given in equation (24)

The time series of *B, N* and *E* are presented in Figure 1. This Figure shows that *B, N* and *E* increase as time increases and finally settle down at their steady states. It is also observed here that the increase in the population density is much more in comparison to the increase in the density of *B* and *E*.

It may be pointed out that values of parameters chosen in (24) satisfy local stability conditions but they do not satisfy global stability conditions.

Now we choose the following set of values for the parameters:

$$\begin{aligned} &r=1.6,\ s=3,\ K=100,\ L=100,\ p=0.5,\ q=0.01,\ \alpha_0=0.1,\ \alpha_1=0.001,\\ &\alpha_2=0.0001,\ \beta_1=0.01,\ \beta_2=0.0001,\ c=0.001,\ \tau=0.1,\ m=4,\ n=1,\ c_1=1, \end{aligned} \tag{25}$$

with different initial conditions.

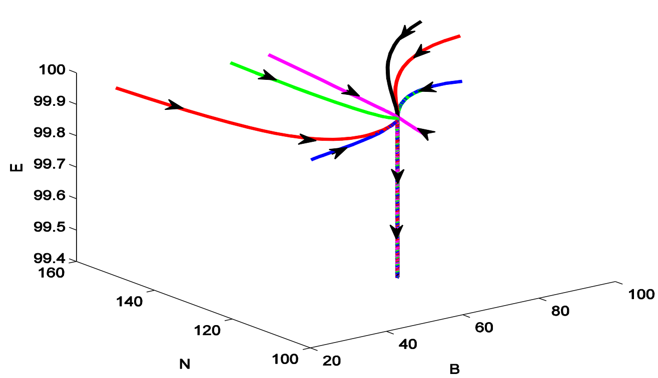


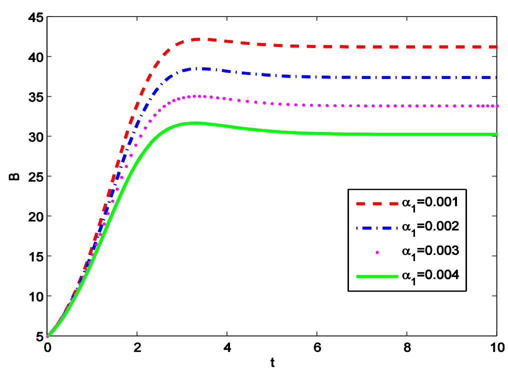
Figure 2. Stable solution in *BNE*- plane for the set of values of parameters given in equation (25)

These values of parameters satisfy the global stability conditions of Theorem 5. The trajectories of *B*, *N* and *E* with various initial values are plotted in Figure 2. From this Figure, we note that all the trajectories starting from the various initial conditions converge to the equilibrium point $P^*(54.6889, 111.7918, 40.7652)$. This shows that P^* is asymptotically stable.

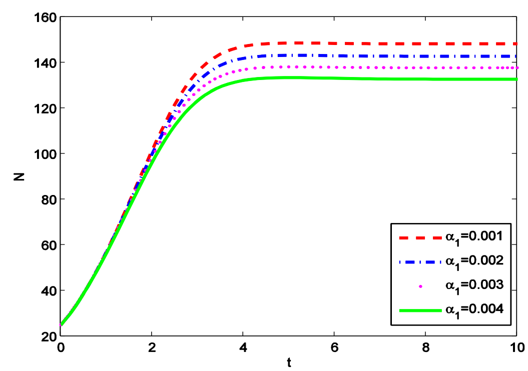
In this model we observe that $\alpha_1, \alpha_2, \beta_1$ and β_2 are important parameters governing the dynamics of the system.

In Figures (3a), (3b) and (3c), we have plotted the trajectories of *B*, *N* and *E* respectively for different values of α_1. Figure (3a) shows that *B* increases with time and after little decrease it settles down at its equilibrium level. It may also be noted that *B* decreases as α_1 increases due to which *N* and *E* also decrease as α_1 increases (see Figures (3b) and (3c)). We also observe that *N* increases with time and finally stabilize at its steady state level. *E* first decreases with time, then increases and settle down at its equilibrium level.

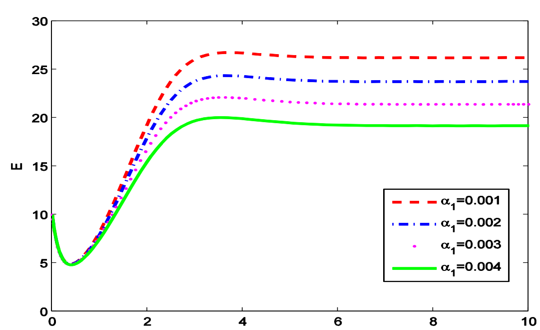
The effect of α_2 on *B*, *N* and *E* are shown in Figures (4a), (4b) and (4c) respectively. We notice that B, *N* and *E* all decrease as α_2 increases. By comparing Figure (3a) with Figure (4a), Figure (3b) with Figure (4b) and Figure (3c) with Figure (4c), we note that α_2 is a very sensitive parameter in comparison to α_1.



(a)

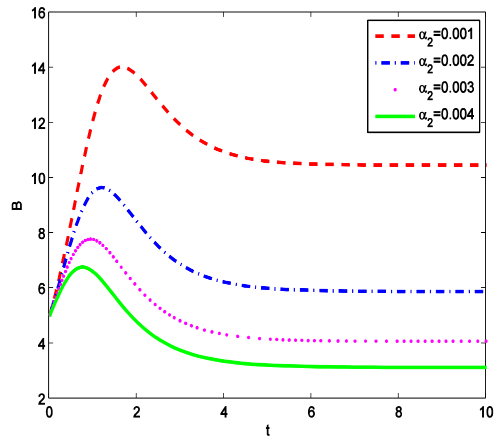


(b)

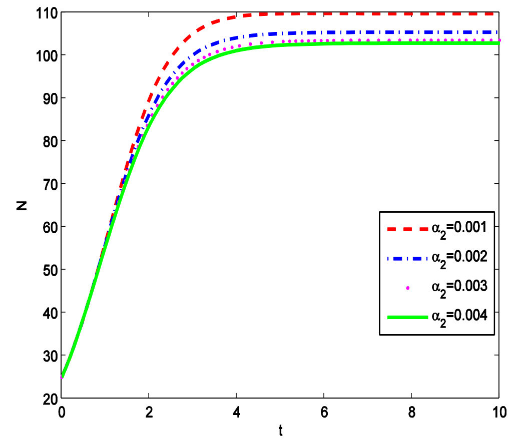


(c)

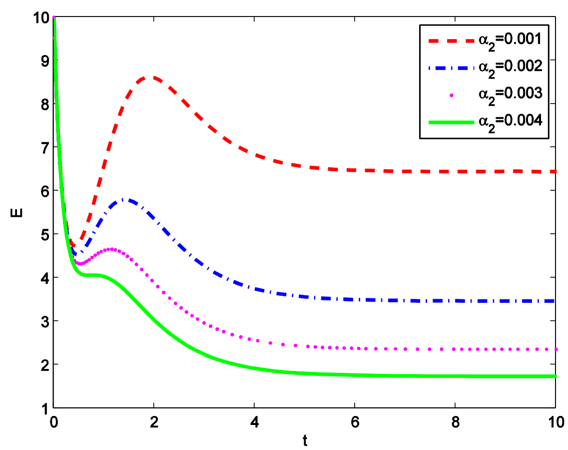
Figures 3(a-c). Behavior of *B*, *N* and *E* with time *t* for different values of α_1 and others values are same as in equation (25)



(a)



(b)



(c)

Figures 4(a-c). Behaviour of *B, N* and *E* with time *t* for different values of α_2 and others values are same as given in equation (25)

The behavior of *B*, *N* and *E* with respect to time *t* for different values of β_1 are shown in Figures (5a), (5b) and (5c) respectively. If β_1 increases, then the population *N* increases and after that it settles down at its equilibrium level. We know that the population utilizes the resource for its own growth and development. So, if the population increases, then, obviously, the resource biomass decreases. Thus if β_1 increases, then *B* and *E* decrease and then settle down at its lower equilibrium levels. The effect of β_2 on *B*, *N* and *E* are shown in Figures (6a), (6b) and (6c) respectively. From Figure (6b), we note that if β_2 increases then *N* also increases. *N* increases very quickly with respect to time *t* and attains its peak; after that it decreases very quickly and settle down at its equilibrium level. Again, *B* and *E* decrease as β_2 increases and then attain their respective equilibrium levels. It is also observed here that the dynamics of the system is highly sensitive with respect to the parameter β_2.

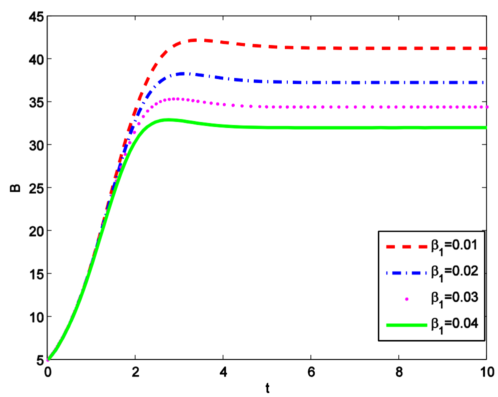
For the optimal harvesting part, we choose the following set of parameters:

$$\begin{aligned} &r = 1.6,\ s = 1.2,\ K = 100,\ L = 100,\ p = 25,\ q = 1,\ \alpha_0 = 1,\ \alpha_1 = 0.001, \\ &\alpha_2 = 0.0001,\ \beta_1 = 0.01,\ \beta_2 = 0.0001,\ c = 7,\ m = 4,\ n = 1,\ \delta = 0.1. \end{aligned} \tag{26}$$

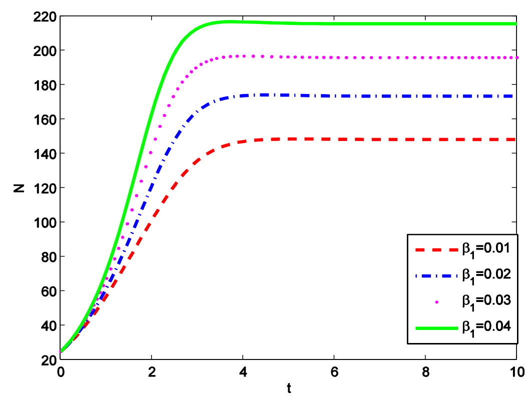
Solving (20) and (21) with the help of equations (5a)-(5c), we get the optimal values

$$B_\delta = 42.908,\ N_\delta = 151.099,\ E_\delta = 9.20654 \text{ and } \tau_\delta = 11.829.$$

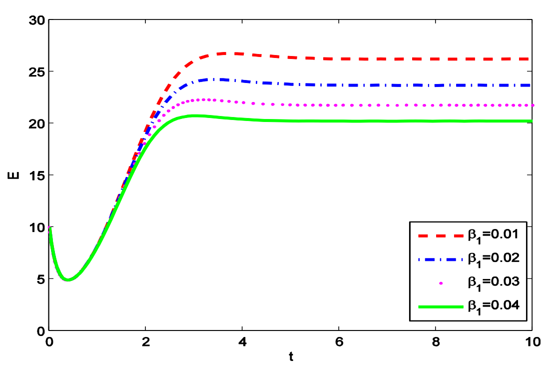
The behavior of B, N and E with respect to time t for the differing values of τ are given in Figures (7a), (7b) and (7c), respectively. We observe that when τ increases, B and N initially increase, attain the peak and after a slight decrease settle down at their equilibrium level. We also note that E decreases as τ increases (see Figure (7c)). If $\tau > \tau_\delta$ ($\tau = 20$), then E decreases and tends to the zero level. This shows that high level of tax will discourage the fisherman and fishery will be closed.



(a)



(b)



(c)

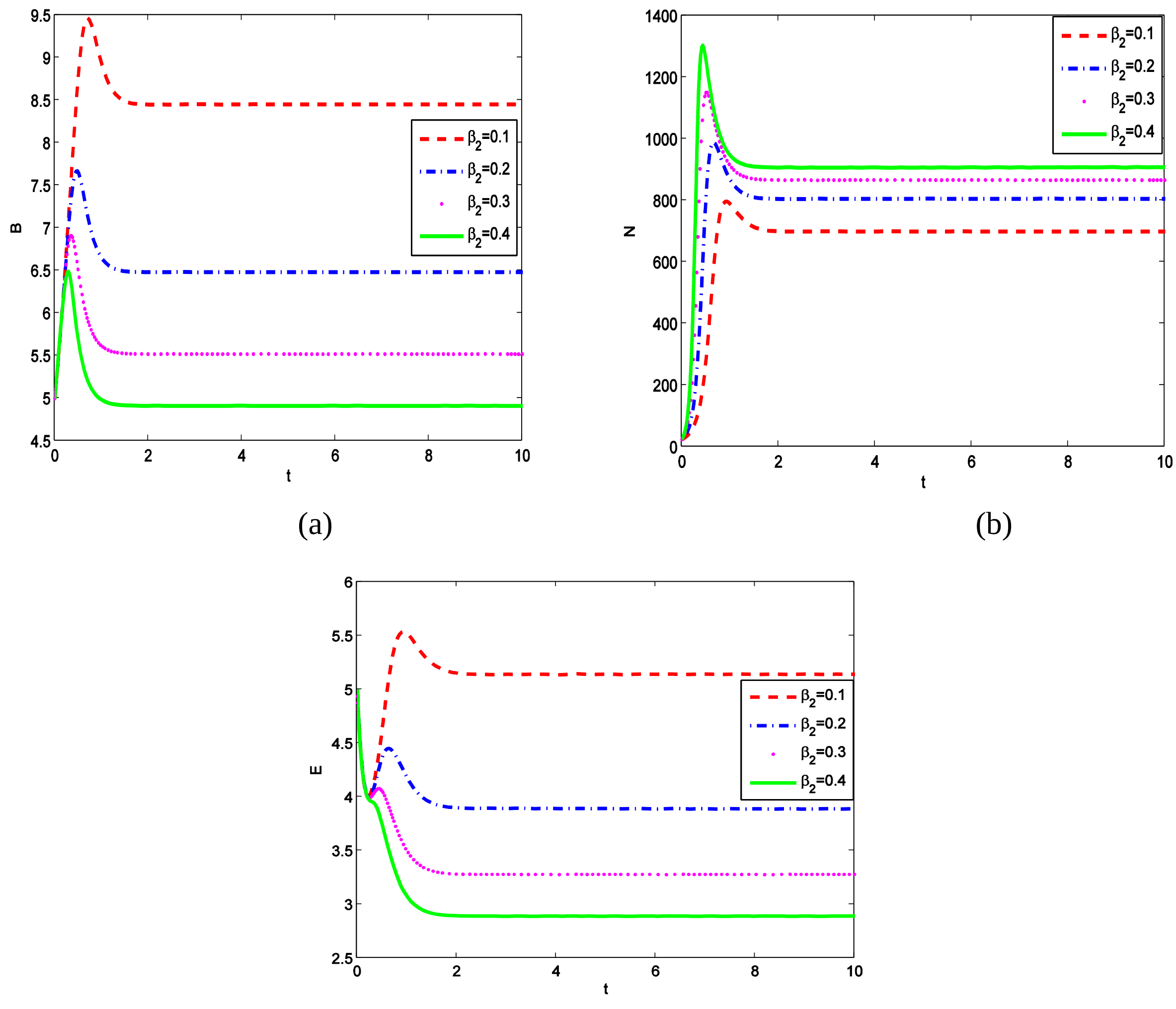
Figures 5(a-c). Behavior of B, N and E with time t for different values of β_1 and others values are same as in equation (25)

For the existence of periodic solution of the underlying equations (2a)-(2c), we consider the following set of values of the parameters for the system:

$$\begin{aligned} &r = 1.6,\ s = 0.00292969,\ K = 64,\ L = 0.5,\ p = 1,\ q = 1, \\ &\alpha_0 = 2.75,\ \alpha_1 = 1.16875, \alpha_2 = 0.03125,\ \beta_1 = 0.00146484, \\ &\beta_2 = 0.00146484,\ c = 0.1875,\ m = 0.5,\ n = 1.6667. \end{aligned} \tag{27}$$

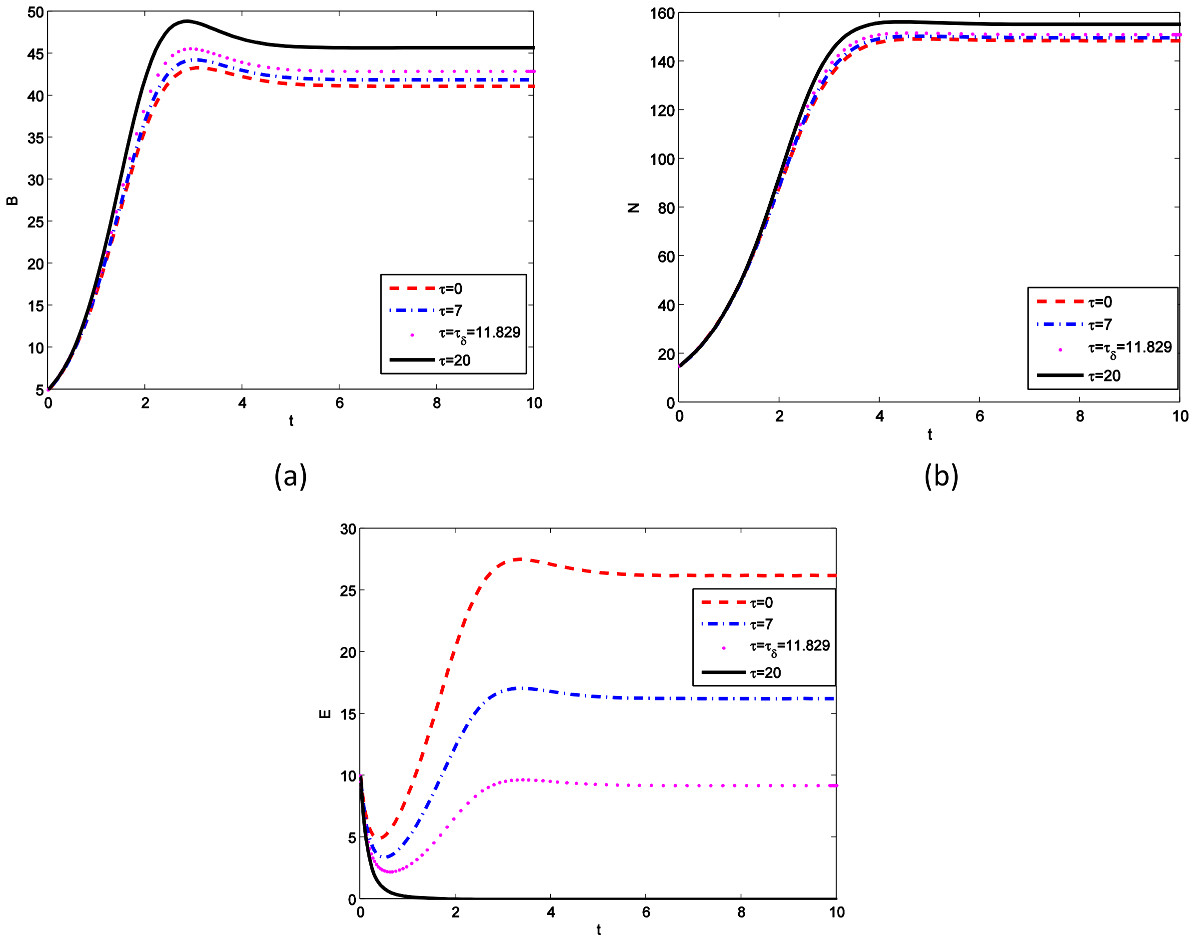
For this set of parameters and $\tau = 0.6$, the equilibrium point $P^*(B^*, N^*, E^*)$ is given by

$$B^* = 1.2264,\ N^* = 1.1826,\ E^* = 0.3711.$$



(a) (b) (c)

Figures 6(a-c). Behavior of *B*, *N* and *E* with time *t* for different values of β_2 and others values are same as in (25)

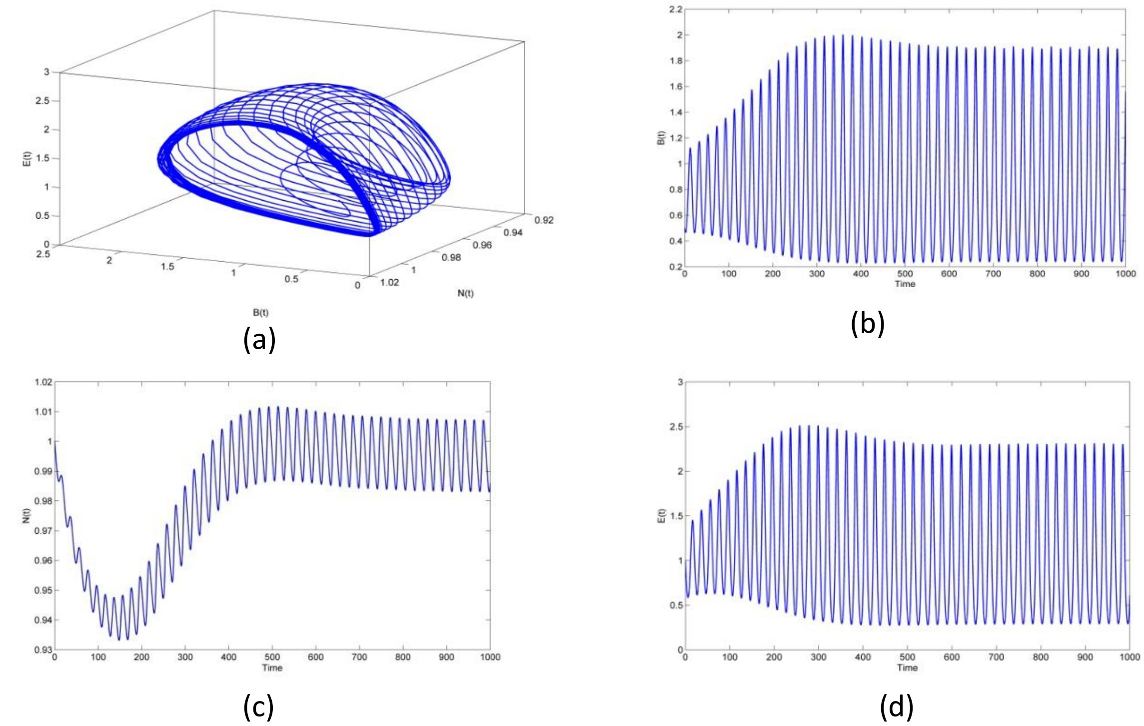


(a) (b)

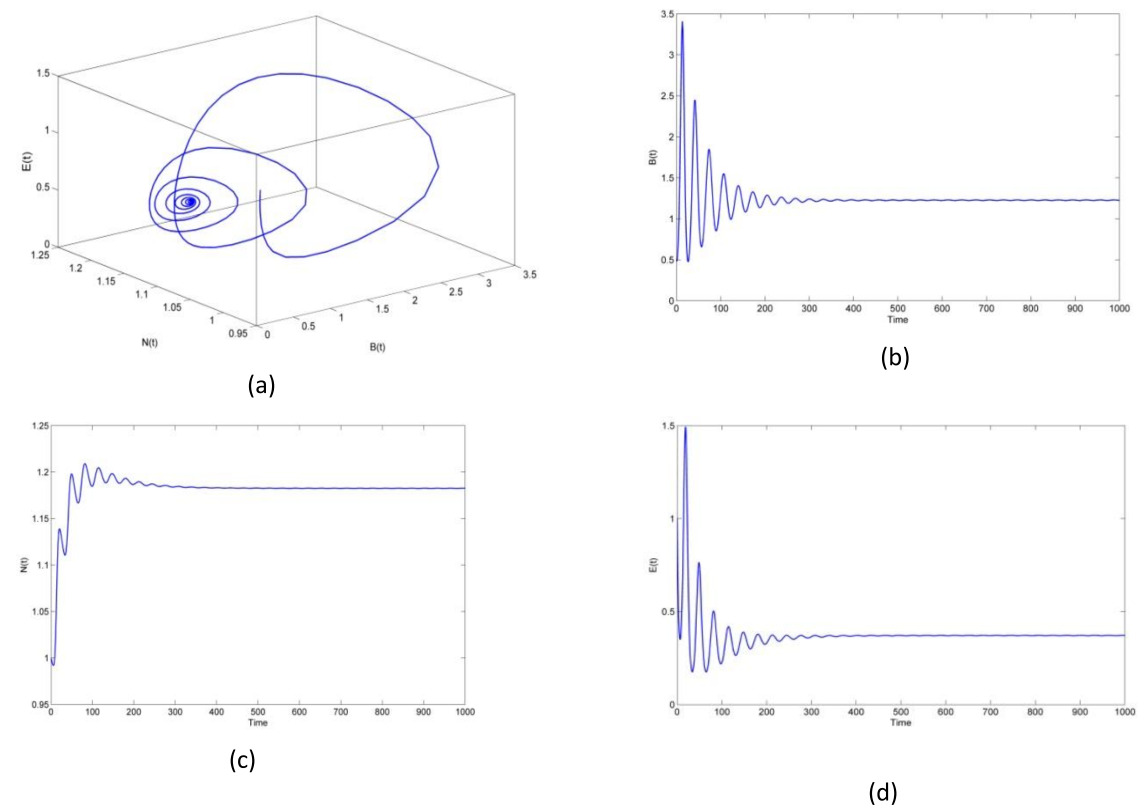
(c)

Figures 7(a-c). Behavior of B, N and E with time t for different values of τ and others values are same as in equation (26)

For parameter $\tau = 0.4$, the system has periodic solution (Limit cycle) and as τ increase to a value 0.6, the system converges to the stable equilibrium point $P^*(B^*, N^*, E^*)$. This is illustrated in Figure 8 and Figure 9, respectively.



Figures 8(a-d). Limit cycle behavior of *B*, *N* and *E* with time *t* for $\tau = 0.4$ and others values of parameters are same as in equation (27)



Figures 9(a-d). Limit cycle behavior of *B*, *N* and *E* with time *t* for $\tau = 0.6$ and others values of parameters are same as in equation (27)

9. Conclusions

This paper analyzed a model of resource biomass and population in which both are growing logistically. The resource biomass, which has commercial importance, is harvested according to a realistic non-linear catch-rate function. The population utilizes the resource for its own growth and development. The harvesting effort is taken to be a dynamical variable and taxation as a control variable. Here we discussed the existence of equilibria, local stability by Eigenvalue method and Routh-Hurwitz criteria and global stability using the Liapunov direct method.

It has been shown that the positive equilibrium point P^* exists if the tax on per unit harvested biomass is less than a threshold value. This threshold value depends on the selling price per unit biomass, the fixed cost of harvesting per unit of effort and the equilibrium value of resource biomass. This point P^* is locally and globally asymptotically stable under certain conditions. When the population utilizes the resource biomass for its growth and development and the resource is harvested, then equation (6) gives the range of the tax which may be used by a regulatory agency. The maximum sustainable yield (MSY), h_{MSY} has been computed for our model system. It has been found that $h^0_{MSY} = \frac{rK}{4}$ obtained by Clark (1976) is a special case of h_{MSY} which was proposed in this paper. Then, bionomic equilibrium has been obtained and we observed that for this model the bionomic equilibrium point exists under certain condition. Choosing an appropriate Hamiltonian function and using Pontryagin's Maximum Principle, we analyzed the optimal harvesting policy.

Finally, numerical simulation experiments were carried out with the help of MATLAB 7.1. It was observed that α_2 and β_2 are very sensitive parameters in comparison to α_1 and β_1. An optimal level of tax τ_δ to be imposed by the regulatory agency has been suggested. It has been shown that if $\tau > \tau_\delta$, then E decreases and goes to the zero level but B and N increase with respect to time t. Thus the regulatory agency should keep $\tau < \tau_\delta$, so that one can maintain the resource and population at an optimal level.

The present system can also become unstable under certain parametric values once the conditions of Routh-Hurwitz criteria the violated as shown in Figure 8. The Hopf-bifurcation analysis with respect to the parameter τ, suggests that under certain conditions, imposing a tax by the regulatory agency per unit resource biomass to protect the over-exploitation of the resource could decide the fate of the system in terms of stability and unstability.

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APPENDIX A

Proof of Theorem 5:

Consider a positive definite function about P^*:

$$W = \left(B - B^* - B^* \ln \frac{B}{B^*} \right) + k_1 \left(N - N^* - N^* \ln \frac{N}{N^*} \right) + k_2 \left(E - E^* - E^* \ln \frac{E}{E^*} \right), \qquad \text{(A1)}$$

where k_1 and k_2 are positive constants.

Differentiating W with respect to time t along the solutions of model (2), a little algebraic manipulation yields

$$\begin{aligned}\frac{dW}{dt} = & -\frac{1}{2} a_{11} (B - B^*)^2 + a_{12} (B - B^*)(N - N^*) - \frac{1}{2} a_{22} (N - N^*)^2 \\ & -\frac{1}{2} a_{11} (B - B^*)^2 + a_{13} (B - B^*)(E - E^*) - \frac{1}{2} a_{33} (E - E^*)^2 \qquad \text{(A2)} \\ & -\frac{1}{2} a_{22} (N - N^*)^2 + a_{23} (N - N^*)(E - E^*) - \frac{1}{2} a_{33} (E - E^*)^2,\end{aligned}$$

where

$$a_{11} = \left[\frac{r}{K} + \alpha_2 N^* - \frac{nqE^*}{(1 + mE^* + nB^*)(1 + mE + nB)} \right],$$

$$a_{22} = \frac{k_1 s}{L} > 0,$$

$$a_{33} = \frac{k_2 mq\alpha_0 (p - \tau) B^*}{(1 + mE^* + nB^*)(1 + mE + nB)} > 0,$$

$$a_{12} = \left[-\alpha_1 - \alpha_2 B + k_1 \beta_1 + k_1 \beta_2 (B + B^*) \right],$$

$$a_{23} = 0,$$

$$a_{13} = \left[-\frac{q(1+nB^*)}{(1+mE^*+nB^*)(1+mE+nB)} + \frac{k_2\alpha_0 q(p-\tau)(1+mE^*)}{(1+mE^*+nB^*)(1+mE+nB)} \right].$$

Sufficient conditions for $\frac{dW_1}{dt}$ to be negative definite are that the following inequalities hold:

$$a_{11} > 0, \tag{A3}$$

$$a_{12}^2 < a_{11}a_{22}, \tag{A4}$$

$$a_{13}^2 < a_{11}a_{33}, \tag{A5}$$

$$a_{23}^2 < a_{22}a_{33}. \tag{A6}$$

Clearly, (A6) holds as $a_{23} = 0$.

If we choose $k_2 = \frac{(1+nB^*)}{\alpha_0(p-\tau)(1+mE^*)}$, then condition (A5) is satisfied.

Again (9a) $\Rightarrow$ (A3) and (9b) $\Rightarrow$ (A4). Thus, W_1 is a Liapunov function for all solutions initiating in the interior of the positive orthant whose domain contains the region of attraction Ω, proving the theorem.