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New Exact Solutions of the Perturbed Nonlinear Fractional Schrödinger Equation Using two Reliable Methods

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Abstract

In this paper, the fractional derivatives in the sense of the modified Riemann-Liouville derivative and the first integral method and the Bernoulli sub-ODE method are employed for constructing the exact complex solutions of the perturbed nonlinear fractional Schrödinger equation and comparing the solutions.

Keywords: Bernoulli sub-ODE method; first integral method; fractional calculus; new exact solutions; perturbed nonlinear fractional Schrödinger equation

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1. Introduction

The investigation of exact solutions to nonlinear fractional differential equations plays an important role in various applications in physics, fluid flow, engineering, signal processing, control theory, systems identification, biology, finance and fractional dynamics (Kilbas et al., 2006; Mille and Ross, 1993; Podlubny, 1999). Recently, a good deal of literature has been provided to construct the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical problems.

The fractional Schrödinger equation is a basic equation of fractional quantum mechanics. It was first proposed by Nick Laskin as a result of extending the Feynman path integral, from the Brownian-like to Lvy-like quantum mechanical paths (Laskin, 2002).

Some numerical methods have been proposed to obtain approximate solutions for the fractional Schrödinger equation, such as the Homotopy analysis method (Hemida et al., 2012; Zheng and Zhao, 2013), Adomian decomposition method (Herzallah and Gepreel, 2012; Zheng and Zhao, 2013), and so on. Using the the first integral method (Feng, 2002) and the Bernoulli sub-ODE method (Zheng, 2012) we are able to find exact solutions of the perturbed nonlinear fractional Schrödinger equation.

The rest of this paper is organized as follows. First in Sec. , we give some definitions and properties of the modified Riemann-Liouville derivative. In Sec. , we give a description of the two methods. Then in Sec. -B, we apply both methods to establish exact solutions for the perturbed nonlinear fractional Schrödinger equation. Conclusions are presented at the end of the paper.

2. The Modified Riemann-Liouville Derivative

(Jumarie, 2006) proposed a modified Riemann-Liouville derivative. With this kind of fractional derivative and some useful formulas, we can convert fractional differential equations into integro-order differential equations by variable transformation.

Assume that $f : R \longrightarrow R, \quad x \longrightarrow f(x)$ denote a continuous function. The Jumarie modified Riemann-Liouville derivative of order α is defined by the expression

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi & 0<\alpha<1, \\ (f^n(t))^{\alpha-n} & n \leq \alpha < n+1, \quad n \geq 1. \end{cases}$$

We list some important properties for the modified Riemann- Liouville derivative as follows:

$$\begin{aligned} D_t^\alpha(f(t)g(t)) &= g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \\ D_t^\alpha t^r &= \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \\ D_t^\alpha f(t)[g(t)] &= f(t)'_{g(t)}[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g(t)')^\alpha, \end{aligned}$$

direct results of the equality $D^\alpha x(t) = \Gamma(1+a)\ Dx(t)$, which holds for non-differentiable functions.

3. The Two Methods

Let us consider the fractional differential equation with independent variables $x = (x_1, x_2, \ldots, x_n, t)$

and a dependent variable u

$$F(u, D_t^{\alpha}u, u_{x_1}, u_{x_2}, u_{x_3}, \ldots, D_t^{2\alpha}u, u_{x_1x_1}, u_{x_2x_2}, u_{x_3x_3}, \ldots) = 0. \tag{1}$$

Using the variable transformation

$$u(x_1, x_2, \ldots, x_n, t) = u(\xi), \quad \xi = l_1x_1 + l_2x_2 + \ldots + l_nx_n + \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)},$$

where l_j $(j = 1, 2, \ldots, n)$ and λ are constants to be determined later, the fractional differential equation (1) is reduced to the nonlinear ordinary differential equation

$$F(u(\xi), \lambda u'(\xi), l_1u'(\xi), \ldots, l_nu'(\xi), \lambda^2u''(\xi), \ldots) = 0, \tag{2}$$

where $''' = \frac{d}{d\xi}$.

A. The First Integral Method(FIM)

This method introduces a new independent variable

$$X(\xi) = f(\xi), \qquad Y = \frac{\partial f(\xi)}{\partial \xi}, \tag{3}$$

which leads to a system of nonlinear ordinary differential equations

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \tag{4}$$

$$\frac{\partial Y(\xi)}{\partial \xi} = F_1(X(\xi), Y(\xi)).$$

By the qualitative theory of ordinary differential equations (Ding and Li, 1996), if we can find the integrals to equation (4) under the same conditions, then the general solutions to equation (4) can be obtained directly. In general, however, it is really difficult to realize this even for the first integral, since for a given plane autonomous system, there is no systematic theory that can tell how to find the first integrals, nor is there a logical way for telling what these first integrals are. We will apply the Division Theorem to obtain one first integral to equation (4) which reduces equation (2) to a first order integrable ordinary differential equation. An exact solution to equation (1) is then obtained by solving this equation. Now, we recall the Division Theorem:

Division Theorem. Suppose that $P(w, z)$ and $Q(w, z)$ are polynomials in $C[w, z]$; and $P(w, z)$ is irreducible in $C[w, z]$; If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C[w, z]$ such that

$$Q(w, z) = P(w, z)G(w, z).$$

B. The Bernoulli Sub-ODE Method

In this method, we assume that equation (2) has a solution in the form

$$u(\xi) = \sum_{i=0}^{m} k_i \phi^i(\xi), \tag{5}$$

where k_i $(i = 0, 1, 2, \ldots, m)$ are real constants to be determined later. $\phi(\xi)$ expresses the solution of the auxiliary ordinary differential equation

$$\phi'(\xi) = \alpha\phi(\xi) + \beta\phi^2(\xi). \tag{6}$$

When $\beta \neq 0$, equation (6) is the type of Bernoulli equation, and we can obtain the solution as

$$\phi = \frac{e^{\alpha\xi}}{c_0 - \frac{\beta}{\alpha} e^{\alpha\xi}}.$$

Integer m in (5) can be determined by considering the homogeneous balance between the nonlinear terms and the highest derivatives of $u(\xi)$ in equation (2). Substituting (5) into (2) with (6) then the left hand side of equation (2) is converted into a polynomial in $\phi(\xi)$ equating each coefficient of the polynomial to zero yields a set of algebraic equation for $k_i, l_j, \alpha, \beta, \lambda$. Solving the algebraic equation obtained and substituting the results into (5) then we obtain the exact travelling wave solutions for equation (1).

4. Perturbed Nonlinear Fractional Schrödinger Equation

Now we seek the perturbed nonlinear fractional Schrödinger equation with the kerr law nonlinearity,

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + u_{xx} + \gamma u|u|^2 + i[\gamma_1 u_{xxx} + \gamma_2 |u|^2 u_x + \gamma_3(|u|^2)_x u] = 0, \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{7}$$

where γ_1 is third order dispersion, γ_2 is the non-linear dispersion, while γ_3 is a also a version of nonlinear dispersion. We use the transformation

$$u(x,t) = u(\xi), \qquad \xi = ax + \frac{bt^\alpha}{\Gamma(\alpha+1)}, \tag{8}$$

where a and b are constants. Substituting (8) into equation (7) we can show that equation (7) is reduced into an ordinary differential equation:

$$ibu_\xi + a^2 u_{\xi\xi} + \gamma u|u|^2 + i[\gamma_1 a^3 u_{\xi\xi\xi} + \gamma_2 a|u|^2 u_\xi + \gamma_3 a(|u|^2)_\xi u] = 0. \tag{9}$$

Function u is a complex function so we can write

$$u(\xi) = e^{is\xi} w(\xi),$$

where s is a constant and $w(\xi)$ is a real function. Then (9) reduced to

$$\begin{aligned} &[(-bs - a^2 s^2 + \gamma_1 a^3 s^3)w + (\gamma - \gamma_2 as)w^3 + (a^2 - 3\gamma_1 a^3 s)w_{\xi\xi}] \\ &+ i[(b + 2a^2 s - 3\gamma_1 a^3 s^2)w_\xi + \gamma a^3 w_{\xi\xi\xi} + (\gamma_2 a + 2\gamma_3 a)w^2 w_\xi] = 0. \end{aligned}$$

Then we have two equation as follows:

$$(-bs - a^2s^2 + \gamma_1 a^3 s^3)w + (\gamma - \gamma_2 as)w^3 + (a^2 - 3\gamma_1 a^3 s)w_{\xi\xi} = 0, \tag{10}$$

$$(b + 2a^2 s - 3\gamma_1 a^3 s^2)w_\xi + (\gamma_2 a + 2\gamma_3 a)w^2 w_\xi + \gamma_1 a^3 w_{\xi\xi\xi} = 0. \tag{11}$$

Integrating (11) and taking zero as the integration constant, we have :

$$(b + 2a^2 s - 3\gamma_1 a^3 s^2)w + \frac{1}{3}(\gamma_2 a + 2\gamma_3 a)w^3 + \gamma_1 a^3 w_{\xi\xi} = 0. \tag{12}$$

By (10) and (12) we have the same solutions. So, we have the following equation:

$$\frac{-bs - a^2s^2 + \gamma_1 a^3 s^3}{b + 2a^2 s - 3\gamma_1 a^3 s^2} = \frac{\gamma - \gamma_2 as}{\frac{1}{3}(\gamma_2 a + 2\gamma_3 a)} = \frac{a^2 - 3\gamma_1 a^3 s}{\gamma_1 a^3}, \tag{13}$$

from (13) we obtain

$$s = \frac{-b}{2a^2} + \frac{3}{2}\gamma_1 a s^2 + \frac{A}{2C}\gamma_1 a,$$

where we assume that

$$A = -bs - a^2 s^2 + \gamma_1 a^3 s^3, \quad B = \gamma - \gamma_2 as, \quad C = a^2 - 3\gamma_1 a^3 s.$$

So (10) is transformed into the following form:

$$Aw + Bw^3 + Cw_{\xi\xi} = 0. \tag{14}$$

C. Exact Solutions by Using the First Integral Method

In this case, by using (3) and (4), we can get

$$\dot{X}(\xi) = Y(\xi), \tag{15}$$

$$\dot{Y}(\xi) = \frac{-A}{C}X - \frac{B}{C}X^3. \tag{16}$$

According to the first integral method, we suppose the $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (15) and (16) and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$

is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$Q(X(\xi), Y(\xi)) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0, \tag{17}$$

where $a_i(X)(i = 0, 1, ..., m)$, are polynomials of X and $a_m(X) \neq 0$. Equation (17) is called the first integral of (15) and (16). By the Division Theorem, there exists a polynomial $g(X)+h(X)Y$, in the complex domain $C[X,Y]$ such that

$$\frac{dQ}{d\xi} = \frac{dQ}{dX}\frac{dX}{d\xi} + \frac{dQ}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{18}$$

In this example, suppose that $m = 1$, by comparing the coefficients of $Y^i (i = 0, 1, 2)$ on both sides of (18), we have

$$\dot{a}_1(X) = h(X)a_1(X), \tag{19}$$

$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X), \tag{20}$$

$$a_1(X)[\frac{-A}{C}X - \frac{B}{C}X^3] = g(X)a_0(X). \tag{21}$$

Since $a_i(X)$ $(i = 0, 1)$ are polynomials, then from (19) we deduce that $a_1(X)$ is constant and $h(X) = 0$. For simplicity, take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = k_0X + k_1$, then we find $a_0(X)$,

$$a_0(X) = \frac{1}{2}k_0X^2 + k_1X + k_2,$$

where k_2 is an arbitrary integration constant.

Substituting $a_0(X)$ and $g(X)$ into (21) and setting all the coefficients of powers X to be zero, we then obtain a system of nonlinear algebraic equations which solves to yield the following results.

Case ***I***:

$$k_0 = \sqrt{\frac{-2B}{C}}, \quad k_1 = 0, \quad k_2 = -\frac{A}{C}\sqrt{\frac{-C}{2B}}. \tag{22}$$

Using conditions (22) in (17), we obtain

$$Y_1(\xi) = -\sqrt{\frac{-B}{2C}}X^2(\xi) + \frac{A}{C}\sqrt{\frac{-C}{2B}}. \tag{23}$$

Combining (23) with (15), we have the exact solution to equations (15) and (16) and also the exact solution of the perturbed nonlinear fractional Schrödinger equation can be written as

$$u_1(x,t) = \sqrt{\frac{-A}{B}}\ e^{is(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\ \tanh(\sqrt{\frac{A}{2C}}(ax + \frac{bt^\alpha}{\Gamma(\alpha+1)} + \xi_0)), \tag{24}$$

where $A = -bs - a^2s^2 + \gamma_1a^3s^3, \quad B = \gamma - \gamma_2as$ and $C = a^2 - 3\gamma_1a^3s$.

Case ***II***:

$$k_0 = -\sqrt{\frac{-2B}{C}}, \quad k_1 = 0, \quad k_2 = \frac{A}{C}\sqrt{\frac{-C}{2B}}. \tag{25}$$

Using conditions (25) in (17), we obtain

$$Y_2(\xi) = \sqrt{\frac{-B}{2C}}X^2(\xi) - \frac{A}{C}\sqrt{\frac{-C}{2B}}. \tag{26}$$

Combining (26) with (15), we obtain the exact solution to equations (15) and (16) and then the exact solution of the perturbed nonlinear fractional Schrödinger equation to be:

$$u_2(x,t) = -\sqrt{\frac{-A}{B}}\ e^{is(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\ \tanh(\sqrt{\frac{A}{2C}}(ax + \frac{bt^\alpha}{\Gamma(\alpha+1)} + \xi_0)), \tag{27}$$

where $A = -bs - a^2s^2 + \gamma_1a^3s^3, \quad B = \gamma - \gamma_2as$ and $C = a^2 - 3\gamma_1a^3s$.

Remark 1:
In the perturbed nonlinear fractional Schrödinger equation, where $\gamma_1 = \gamma_2 = \gamma_3 = 0$, equation (7) degenerates to the nonlinear fractional Schrödinger equation,

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + u_{xx} + \gamma u|u|^2 = 0, \qquad t > 0, \quad 0 < \alpha \leq 1. \tag{28}$$

The exact solution of equation (28) has the following forms:

$$u_1(x,t) = \frac{b}{2a\sqrt{-\gamma}}\ e^{\frac{-ib}{2a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\ \tanh(\frac{b}{2\sqrt{2}a^2}(ax + \frac{bt^\alpha}{\Gamma(\alpha+1)} + \xi_0)),$$

and

$$u_2(x,t) = \frac{-b}{2a\sqrt{-\gamma}}\ e^{\frac{-ib}{2a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\ \tanh(\frac{b}{2\sqrt{2}a^2}(ax + \frac{bt^\alpha}{\Gamma(\alpha+1)} + \xi_0)),$$

where a, b and ξ_0 are arbitrary constants and γ is a negative constant.

D. Exact Solutions by Using the Bernoulli Sub-ODE Method

In this case, the Bernoulli sub-ODE method admits to determine the parameter m by balancing $w_{\xi\xi}$ and w^3 in equation (14), so we have $m = 1$.

Therefore, the Bernoulli sub-ODE method in the form (5) admits the use of the finite expansion

$$w(\xi) = k_1\phi + k_0. \tag{29}$$

Substituting (29) into equation (14) and using (6), collecting the coefficients of ϕ, we obtain:

$$\begin{aligned}
\phi^3 &: \quad Bk_1^3 + 2Ck_1\beta^2 = 0,\\
\phi^2 &: \quad 3Bk_0k_1^2 + 3Ck_1\alpha\beta = 0,\\
\phi^1 &: \quad Ak_1 + 3Bk_0^2k_1 + Ck_1\alpha^2 = 0,\\
\phi^0 &: \quad Ak_0 + Bk_0^3 = 0.
\end{aligned}$$

Solving this system, using Maple, gives

Case ***I***:

$$k_0 = \sqrt{\frac{-A}{B}}, \quad k_1 = \beta\sqrt{\frac{-2C}{B}}, \quad \alpha = \sqrt{\frac{2A}{C}}.$$

Then the exact solution of the equation (7) has the following form:

$$u_1(x,t) = -e^{is(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\ \sqrt{\frac{-A}{B}}\ [1 - \frac{2c_0}{c_0 - \beta\sqrt{\frac{C}{2A}}e^{\sqrt{\frac{2A}{C}}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}], \tag{30}$$

where $A = -bs - a^2s^2 + \gamma_1 a^3 s^3, \quad B = \gamma - \gamma_2 as$ and $C = a^2 - 3\gamma_1 a^3 s$.

Case ***II***:

$$k_0 = \sqrt{\frac{-A}{B}}, \quad k_1 = -\beta\sqrt{\frac{-2C}{B}}, \quad \alpha = -\sqrt{\frac{2A}{C}}.$$

Then the exact solution of the equation (7) has the following form:

$$u_2(x,t) = -e^{is(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-A}{B}}\,[1-\frac{2c_0}{c_0+\beta\sqrt{\frac{C}{2A}}e^{-\sqrt{\frac{2A}{C}}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}], \tag{31}$$

where $A=-bs-a^2s^2+\gamma_1a^3s^3, \quad B=\gamma-\gamma_2as$ and $C=a^2-3\gamma_1a^3s$.

Case ***III***:

$$k_0=-\sqrt{\frac{-A}{B}}, \quad k_1=-\beta\sqrt{\frac{-2C}{B}}, \quad \alpha=\sqrt{\frac{2A}{C}}.$$

Then the exact solution of the equation (7) is of the following form:

$$u_3(x,t) = e^{is(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-A}{B}}\,[1-\frac{2c_0}{c_0-\beta\sqrt{\frac{C}{2A}}e^{\sqrt{\frac{2A}{C}}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}], \tag{32}$$

where $A=-bs-a^2s^2+\gamma_1a^3s^3, \quad B=\gamma-\gamma_2as$ and $C=a^2-3\gamma_1a^3s$.

Case ***IV***:

$$k_0=-\sqrt{\frac{-A}{B}}, \quad k_1=\beta\sqrt{\frac{-2C}{B}}, \quad \alpha=-\sqrt{\frac{2A}{C}}.$$

Then the exact solution of the equation (7) has the form:

$$u_4(x,t) = e^{is(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-A}{B}}\,[1-\frac{2c_0}{c_0+\beta\sqrt{\frac{C}{2A}}e^{-\sqrt{\frac{2A}{C}}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}], \tag{33}$$

where $A=-bs-a^2s^2+\gamma_1a^3s^3, \quad B=\gamma-\gamma_2as$ and $C=a^2-3\gamma_1a^3s$.

Remark 2:

According to ***Remark 1***, the exact solution of equation (28), using the Bernoulli sub-ODE method, has the following form:

$$u_1(x,t) = -e^{\frac{-ib}{2a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-b^2}{4a^2\gamma}}\,[1-\frac{2c_0}{c_0-\frac{\sqrt{2}a^2\beta}{\sqrt{b^2}}e^{\frac{\sqrt{b^2}}{\sqrt{2}a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}],$$

and

$$u_2(x,t) = -e^{\frac{-ib}{2a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-b^2}{4a^2\gamma}}\,[1-\frac{2c_0}{c_0+\frac{\sqrt{2}a^2\beta}{\sqrt{b^2}}e^{-\frac{\sqrt{b^2}}{\sqrt{2}a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}],$$

and

$$u_3(x,t) = e^{\frac{-ib}{2a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-b^2}{4a^2\gamma}}\,[1-\frac{2c_0}{c_0-\frac{\sqrt{2}a^2\beta}{\sqrt{b^2}}e^{\frac{\sqrt{b^2}}{\sqrt{2}a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}],$$

and

$$u_4(x,t) = e^{\frac{-ib}{2a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}\sqrt{\frac{-b^2}{4a^2\gamma}}\,[1-\frac{2c_0}{c_0+\frac{\sqrt{2}a^2\beta}{\sqrt{b^2}}e^{-\frac{\sqrt{b^2}}{\sqrt{2}a^2}(ax+\frac{bt^\alpha}{\Gamma(\alpha+1)})}}],$$

where a, b and c_0 are arbitrary constants and γ is a negative constant.

5. Conclusion

This paper has successfully applied the first integral method and the Bernoulli sub-ODE method to find the new exact solutions for the perturbed nonlinear fractional Schrödinger equation. The two methods are powerful and applicable to many nonlinear fractional partial differential equations.

Comparing the first integral method with the Bernoulli sub-ODE method shows that the two methods in the general case, do not produce each other solutions. However, they are equivalent under certain conditions:

- In expression (30), if $\beta = -\sqrt{\frac{2A}{C}}$ and $c_0 = e^{-\sqrt{\frac{2A}{C}}\xi_0}$ then $u_1 = u_1(x,t)$ is equivalent to the expression (27).
- In expression (31), if $\beta = \sqrt{\frac{2A}{C}}$ and $c_0 = e^{\sqrt{\frac{2A}{C}}\xi_0}$ then $u_2 = u_2(x,t)$ is equivalent to the expression (24).
- In expression (32), if $\beta = -\sqrt{\frac{2A}{C}}$ and $c_0 = e^{-\sqrt{\frac{2A}{C}}\xi_0}$ then $u_3 = u_3(x,t)$ is equivalent to the expression (24).
- In expression (33), if $\beta = \sqrt{\frac{2A}{C}}$ and $c_0 = e^{\sqrt{\frac{2A}{C}}\xi_0}$ then $u_4 = u_4(x,t)$ is equivalent to the expression (27).

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