



Duality in Fuzzy Linear Programming with Symmetric Trapezoidal Numbers

S.H. Nasser

Department of Mathematic
Mazandaran University
Babolasar, Iran
nasser@umz.ac.ir

A. Ebrahimnejad

Department of Mathematic
Ghaemshahr Branch
Islamic Azad University
Ghaemshahr, Iran
a.ebrahimnejad@srbiau.ac.ir

S. Mizuno

Industrial Engineering and Management Department
Tokyo Institute of Technology
Tokyo, Japan

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Abstract

Linear programming problems with trapezoidal fuzzy numbers have recently attracted much interest. Various methods have been developed for solving these types of problems. Here, following the work of Ganesan and Veeramani and using the recent approach of Mahdavi-Amiri and Nasser, we introduce the dual of the linear programming problem with symmetric trapezoidal fuzzy numbers and establish some duality results. The results will be useful for post optimality analysis.

Keywords: Duality, fuzzy arithmetic, fuzzy linear programming, symmetric trapezoidal fuzzy number

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1. Introduction

Fuzzy set theory has been applied to many disciplines such as control theory and operations research, mathematical modeling and industrial applications. The concept of fuzzy mathematical programming on general level was first proposed by Tanaka et al. (1984) in the framework of the fuzzy decision of Bellman and Zadeh (1970). Afterwards, many authors considered various types of fuzzy linear programming problems and proposed several approaches for solving them, for instance see Chanas (1983), Maleki et al. (2000), Mahdavi-Amiri and Nasserri (2006, 2007), Ebrahimnejad et al. (2010a, 2010b) and Nasserri and Ebrahimnejad (2010a, 2010b). Chanas (1983) showed an application of parametric programming techniques in fuzzy linear programming and obtained the set of solutions maximizing the objective function, being analytically dependent on a parameter. Vasant (2003) investigated an industrial application of interactive fuzzy linear programming through the modified S-curve membership function using a set of real life data collected from a Chocolate Manufacturing Company. Safi et al. (2007) introduced some definitions in the geometry of two-dimensional fuzzy linear programming.

After defining the optimal solution based on these definitions, they used the geometric approach for obtaining optimal solution(s) and showed that the algebraic solutions obtained by Zimmermann method and our geometric solutions are the same. Vijay et al. (2007) introduced a generalized model for a two person zero sum matrix game with fuzzy goals and fuzzy payoffs via fuzzy relation approach and then showed that it was equivalent to two semi-infinite optimization problems.

Some authors used the concept of comparison of fuzzy numbers for solving fuzzy linear programming problems; see Chanas (1983), Maleki et al. (2000) and also Mahdavi-Amiri and Nasserri (2006, 2007). Nevertheless, usually in such methods authors define a crisp model which is equivalent to the fuzzy linear programming problem and then use optimal solution of the model as the optimal solution of the fuzzy linear programming problem. Some authors considered types of linear programming problems in which the variables and the right-hand-sides of the constraints are fuzzy parameters, [Maleki et al. (2000), Ganesan and Veeramani (2006) and also Mahdavi-Amiri and Nasserri (2006, 2007)].

Maleki et al. (2000) defined an auxiliary problem called the fuzzy number linear programming for solving linear programming with fuzzy variables based on linear ranking functions. Ebrahimnejad and Nasserri (2009) used the complementary slackness for solving both fuzzy number linear programming problem and linear programming problems with fuzzy variables. Also, Ebrahimnejad et al. (In press) used the bounded primal simplex method for solving bounded linear programming with fuzzy cost coefficients.

The study of duality theory for fuzzy parameter linear programming problems has attracted researchers in fuzzy decision theory. The duality of fuzzy parameter linear programming was first studied by Rodder and Zimmermann (1980). Verdegay (1984) defined the fuzzy dual problem with the help of parametric linear programming and showed that the fuzzy primal and dual problems both have the same fuzzy solution under some suitable conditions. The fuzzy primal and dual linear programming problems with fuzzy coefficients were formulated by using

the fuzzy scalar product proposed in Wu (2003). Bector and Chandra (2002) discussed duality in fuzzy linear programming based on a modification of the dual formulation stated by Rodder and Zimmermann (1980). Inuiguchi et al. (2003) studied fuzzy linear programming duality in the setting of fuzzy relations. Ramik (2005) discussed a class of fuzzy linear programming problems based on fuzzy relations and a new concept of duality and deduced the weak and strong duality theorems. Ganesan (2006) established the concept of duality in fuzzy linear programming problems. Mahdavi et al. (2006, 2007) have established duality results and developed a primal simplex algorithm for solving linear programming problems with trapezoidal fuzzy variables by use of ranking functions for comparison of fuzzy numbers. Nasseri and Ebrahimnejad (2009) generalized the dual simplex algorithm for solving linear programming with fuzzy numbers. Also, Ebrahimnejad et al. (2010a) extend the primal-dual algorithm for solving linear programming with fuzzy variables. Ebrahimnejad and Nasseri (2010) developed the bounded dual simplex method for solving fuzzy number linear programming problems.

In addition, a new method for solving linear programming with symmetric trapezoidal fuzzy numbers (FLP) has been proposed by Ganesan and Veeramani (2006). The proposed method can solve the FLP problem without converting it to a crisp linear programming problem. Ebrahimnejad et al. (2010b) generalized their method for solving bounded FLP problems. Nasseri and Mahdavi-Amiri (2009), extended their results and proved the optimality theorem and then define the dual problem. In that study, they gave some duality results as a natural extension of duality results for linear programming problems with crisp data given by Bazaraa (2005). Here, we derive some another important results. In particular, our main contributions here are the establishment of duality and complementary slackness.

The rest of paper is organized as follows. In Section 2, we first give some necessary notations and definitions of fuzzy set theory. Then we provide a discussion of fuzzy numbers. The definition of the fuzzy linear programming problem is given in Section 3. Section 4 explains the notion of fuzzy basic feasible solution. We establish duality for the fuzzy linear programming problem in Section 5 and deduce the duality results. We conclude in Section 6.

2. Preliminaries

We review the fundamental notions of fuzzy set theory, initiated by Bellman and Zadeh (1970).

Definition 2.1. A fuzzy number \tilde{a} on R (real line) is said to be a symmetric trapezoidal fuzzy number, if there exist real numbers a^L and a^U , $a^L \leq a^U$ and $\alpha > 0$, such that

$$\tilde{a}(x) = \begin{cases} \frac{x}{\alpha} + \frac{\alpha - a^L}{\alpha}, & x \in [a^L - \alpha, a^L], \\ 1, & x \in [a^L, a^U], \\ -\frac{x}{\alpha} + \frac{a^U + \alpha}{\alpha}, & x \in [a^U, a^U + \alpha], \\ 0, & \text{otherwise.} \end{cases}$$

We denote a symmetric trapezoidal fuzzy number \tilde{a} by $\tilde{a}=(a^L, a^U, \alpha)$, where $(a^L - \alpha, a^U + \alpha)$ is the support of \tilde{a} and $[a^L, a^U]$ its core, and the set of all symmetric trapezoidal fuzzy numbers by $F(\mathbb{R})$.

Let $\tilde{a}=(a^L, a^U, \alpha)$ and $\tilde{b}=(b^L, b^U, \beta)$ be two symmetric trapezoidal fuzzy numbers. Then the arithmetic operations on \tilde{a} and \tilde{b} are given by [Ganesan and Veeramani (2006)]:

$$\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \beta),$$

$$\tilde{a} - \tilde{b} = (a^L - b^U, a^U - b^L, \alpha + \beta),$$

$$\tilde{a} \tilde{b} = \left(\left(\frac{a^L + a^U}{2} \right) \left(\frac{b^L + b^U}{2} \right) - w, \left(\frac{a^L + a^U}{2} \right) \left(\frac{b^L + b^U}{2} \right) + w, |a^U \alpha + b^U \beta| \right),$$

where

$$w = \frac{t_2 - t_1}{2}, t_1 = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\} \text{ and } t_2 = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}.$$

From the above definition, it can be seen that

$$\lambda \geq 0, \lambda \in \mathbb{R}; \lambda \tilde{a} = (\lambda a^L, \lambda a^U, \lambda \alpha)$$

and

$$\lambda < 0, \lambda \in \mathbb{R}; \lambda \tilde{a} = (\lambda a^U, \lambda a^L, -\lambda \alpha).$$

Note that depending upon the need; one can also use a smaller w in the definition of multiplication involving symmetric trapezoidal fuzzy numbers.

Definition 2.2. Let $\tilde{a}=(a^L, a^U, \alpha)$ and $\tilde{b}=(b^L, b^U, \beta)$ be two symmetric trapezoidal fuzzy numbers. Define the relations \lesssim and \approx as given below:

$\tilde{a} \lesssim \tilde{b}$ if and only if

i) $\frac{(a^L - \alpha) + (a^U + \alpha)}{2} < \frac{(b^L - \beta) + (b^U + \beta)}{2}$, in this case we may write $\tilde{a} \prec \tilde{b}$,

ii) or $\frac{a^L + a^U}{2} = \frac{b^L + b^U}{2}, b^L < a^L$ and $a^U < b^U$,

iii) or $\frac{a^L+a^U}{2}=\frac{b^L+b^U}{2}, b^L=a^L, a^U=b^U$ and $\alpha\leq\beta$.

Note that in cases (ii) and (iii), we also write $\tilde{a}\approx\tilde{b}$ and say that \tilde{a} and \tilde{b} are equivalent.

Remark 2.1. Two symmetric trapezoidal fuzzy numbers $\tilde{a}=(a^L,a^U,\alpha)$, $\tilde{b}=(b^L,b^U,\beta)$ are equivalent if and only if $\frac{a^L+a^U}{2}=\frac{b^L+b^U}{2}$.

For any trapezoidal fuzzy number \tilde{a} we define $\tilde{a}\succeq\tilde{b}$ if there exist $\varepsilon\geq 0$ and $\alpha>0$ such that $\tilde{a}\succeq(-\varepsilon,\varepsilon,\alpha)$. We also denote $(-\varepsilon,\varepsilon,\alpha)$ by $\tilde{0}$. We note that $\tilde{0}$ is equivalent to $\tilde{0}=(0,0,0)$. Naturally, one may consider $\tilde{0}=(0,0,0)$ as the zero symmetric trapezoidal fuzzy number.

Remark 2.2. If $\tilde{x}\approx\tilde{0}$, then \tilde{x} is said to be a zero symmetric trapezoidal fuzzy number.

It is to be noted that if $\tilde{x}=\tilde{0}$, then $\tilde{x}\approx\tilde{0}$ but the converse need not be true. If $\tilde{x}\neq\tilde{0}$ (that is \tilde{x} is not equivalent to $\tilde{0}$), then it is said to be a non-zero symmetric trapezoidal fuzzy number.

It is, also, to be noted that if $\tilde{x}\neq\tilde{0}$, then $\tilde{x}\neq\tilde{0}$. But, the converse need not be true. If $\tilde{x}\succeq\tilde{0}$ ($\tilde{x}\preceq\tilde{0}$) and $\tilde{x}\neq\tilde{0}$, then \tilde{x} is said to be a positive (negative) symmetric trapezoidal fuzzy number and is denoted by $\tilde{x}\succ\tilde{0}$ ($\tilde{x}\prec\tilde{0}$).

Now if $\tilde{a},\tilde{b}\in F(R)$, it is easy to show that if $\tilde{a}\succeq\tilde{b}$, then $\tilde{a}-\tilde{b}\succeq\tilde{0}$.

The following lemma immediately follows from Definition 2.1.

Lemma 2.1. If $\tilde{a},\tilde{b}\in F(R)$ and $c\in R$ such that $c\neq 0$, then

$$\tilde{a}\tilde{b}\approx\tilde{b}\tilde{a},$$

and

$$c(\tilde{a}\tilde{b})\approx(c\tilde{a})\tilde{b}\approx\tilde{a}(c\tilde{b}).$$

The two following results are taken from Ganesan and Veeramani (2006) and we omit the proofs.

Lemma 2.2. For any trapezoidal fuzzy number \tilde{a},\tilde{b} and $\tilde{c}\in F(R)$, we have

$$\tilde{c}(\tilde{a}+\tilde{b})\approx(\tilde{c}\tilde{a}+\tilde{c}\tilde{b}),$$

and

$$\tilde{c}(\tilde{a} - \tilde{b}) \approx (\tilde{c}\tilde{a} - \tilde{c}\tilde{b}).$$

Lemma 2.3. If $\tilde{a}, \tilde{b} \in F(R)$, then

The relation \preceq is a partial order relation on the set of symmetric trapezoidal fuzzy numbers. The relation \preceq is a linear order relation on the set of symmetric trapezoidal fuzzy numbers. For any trapezoidal fuzzy number \tilde{a} and \tilde{b} , if $\tilde{a} \preceq \tilde{b}$, then $\tilde{a} \preceq (1-\lambda)\tilde{a} + \lambda\tilde{b} \preceq \tilde{b}$, for all $\lambda, 0 \leq \lambda \leq 1$.

Here, we give some new results.

Lemma 2.4. If $\tilde{a}, \tilde{b} \in F(R)$, then

$$\begin{aligned} \tilde{a}\tilde{b} \preceq \tilde{0}, & \text{ if and only if } \tilde{a} \succeq \tilde{0} \text{ and } \tilde{b} \preceq \tilde{0} \text{ or } \tilde{a} \preceq \tilde{0} \text{ and } \tilde{b} \succeq \tilde{0}. \\ \tilde{a}\tilde{b} \preceq \tilde{0}, & \text{ if and only if } \tilde{a} \succeq \tilde{0} \text{ and } \tilde{b} \succeq \tilde{0} \text{ or } \tilde{a} \preceq \tilde{0} \text{ and } \tilde{b} \preceq \tilde{0}. \\ \tilde{a} \preceq \tilde{b}, & \text{ if and only if } \lambda\tilde{a} \succeq \lambda\tilde{b}, \text{ for any } \lambda < 0, \lambda \in R. \end{aligned}$$

Proof:

It is straightforward.

Lemma 2.5. If $\tilde{a}, \tilde{b}, \tilde{c} \in F(R)$ such that $\tilde{a} \preceq \tilde{b}$, then

$$\begin{aligned} \text{If } \tilde{c} \succeq \tilde{0}, & \text{ then } \tilde{c}\tilde{a} \preceq \tilde{c}\tilde{b}, \\ \text{and} \\ \text{If } \tilde{c} \preceq \tilde{0}, & \text{ then } \tilde{c}\tilde{a} \succeq \tilde{c}\tilde{b}. \end{aligned}$$

Proof:

From $\tilde{a} \preceq \tilde{b}$, we have $\tilde{b} - \tilde{a} \succeq \tilde{0}$. Hence, from Lemma 2.4, we have $\tilde{c}(\tilde{b} - \tilde{a}) \succeq \tilde{0}$, if $\tilde{c} \succeq \tilde{0}$ and $\tilde{c}(\tilde{b} - \tilde{a}) \preceq \tilde{0}$, if $\tilde{c} \preceq \tilde{0}$. Now results follow from Lemma 2.2.

Lemma 2.6. If $\tilde{a}, \tilde{b}, \tilde{c} \in F(R)$ such that $\tilde{a} \approx \tilde{b}$ and $\tilde{c} \neq \tilde{0}$, then we have $\tilde{c}\tilde{a} \approx \tilde{c}\tilde{b}$.

Proof:

Let $\tilde{a}=(a^L, a^U, \alpha)$, $\tilde{b}=(b^L, b^U, \beta)$ and $\tilde{c}=(c^L, c^U, \gamma)$. Since $\tilde{a} \approx \tilde{b}$ and $\tilde{c} \neq \tilde{0}$, $\frac{a^L + a^U}{2} = \frac{b^L + b^U}{2}$ and $\frac{c^L + c^U}{2} \neq 0$. So, it follows that $(\frac{c^L + c^U}{2})(\frac{a^L + a^U}{2}) = (\frac{c^L + c^U}{2})(\frac{b^L + b^U}{2})$. Therefore, from Remark 2.1, we have $\tilde{c}\tilde{a} \approx \tilde{c}\tilde{b}$.

3. Fuzzy Linear Programming

Consider the following fuzzy linear programming problem.

Example 3.1. Assume that a company makes two products. Product P_1 has a profit of around \$40 per unit and product P_2 has a profit of around \$30 per unit. Each unit of P_1 requires twice as many labor hours as each available labor hours are somewhat close to 500 hours per day, and may possibly be changed due to special arrangements for overtime work. The supply of material is almost 400 units of both products, P_1 and P_2 , per day, but may possibly be changed according to past experience. The problem is, how many units of products P_1 and P_2 should be made per day to maximize the total profit?

Let \tilde{x}_1, \tilde{x}_2 denote the number of units of products P_1, P_2 made in one day, respectively. Then, the problem can be formulated as the following fuzzy linear programming problem:

$$\begin{aligned} \max \quad & \tilde{z} \simeq 40\tilde{x}_1 + 30\tilde{x}_2 \\ \text{s. t.} \quad & \tilde{x}_1 + \tilde{x}_2 \leq 400 \\ & 2\tilde{x}_1 + \tilde{x}_2 \leq 500 \\ & \tilde{x}_1, \tilde{x}_2 \geq \tilde{0} \end{aligned}$$

The supply of material and the available labor hours are close to 400 and 500, and hence are modeled as (400, 410, 8) and (495, 515, 5), respectively. Also the profits for P_1 and P_2 which are close to \$40 and \$30 respectively are modeled as (38, 42, 2) and (29, 32, 1). The corresponding fuzzy linear programming problem may then be modeled as follows:

$$\begin{aligned} \max \quad & \tilde{z} \simeq (38, 42, 2)\tilde{x}_1 + (29, 32, 1)\tilde{x}_2 \\ \text{s. t.} \quad & \tilde{x}_1 + \tilde{x}_2 \leq (400, 410, 8) \\ & 2\tilde{x}_1 + \tilde{x}_2 \leq (495, 515, 5) \\ & \tilde{x}_1, \tilde{x}_2 \geq \tilde{0} \end{aligned}$$

Now, in general, a fuzzy linear programming (FLP) problem is defined as [see Ganesan and Veeramani (2006)]:

$$\begin{aligned} \max \quad & \tilde{z} \simeq \tilde{c}\tilde{x} \\ \text{s. t.} \quad & A\tilde{x} \leq \tilde{b} \\ & \tilde{x} \geq \tilde{0}, \end{aligned} \tag{1}$$

where $\tilde{b} \in (F(\mathbb{R}))^m$, $\tilde{c} \in (F(\mathbb{R}))^n$ and $A \in R^{m \times n}$ ($rank(A) = m$) are given and $\tilde{x} \in F(\mathbb{R})^n$ is to be determined.

Definition 3.1. We say that a fuzzy vector $\tilde{x} \in F(\mathbb{R})^n$ is a fuzzy feasible solution to the problem (1) if \tilde{x} satisfies the constraints of the problem.

Definition 3.2. A fuzzy feasible solution $\tilde{x}_* \in F(\mathbb{R})^n$ is a fuzzy optimal solution for (1), if for all fuzzy feasible solution \tilde{x} for (1), we have $\tilde{c}\tilde{x} \preceq \tilde{c}\tilde{x}_*$.

A method for solving the FLP problem has been given by Ganesan and Veeramani (2006). They stated some results and proposed a new method for solving the FLP problem without converting it to a crisp linear programming problem. Here we extend their results by introducing the dual of the FLP problem, and establish the duality theory on fuzzy linear programming with symmetric trapezoidal fuzzy numbers.

4. Fuzzy Basic Feasible Solution

Here, we explore the concept of fuzzy basic feasible solution for FLP problems. Consider the FLP problem,

$$\begin{aligned} \max \quad & \tilde{z} \approx \tilde{c}\tilde{x} \\ \text{s. t.} \quad & A\tilde{x} \approx \tilde{b} \\ & \tilde{x} \succcurlyeq \tilde{0}, \end{aligned} \tag{2}$$

where the parameters of the problem are as defined in (1).

Let $A = [a_{ij}]_{m \times n}$. Assume $rank(A) = m$. Partition A as $[B \ N]$ where $B, m \times m$, is nonsingular. It is obvious that $rank(B)=m$. Let y_j be the solution to $By = a_j$. It is apparent that the basic solution

$$\tilde{x}_B = (\tilde{x}_{B_1}, \dots, \tilde{x}_{B_m})^T \approx B^{-1}\tilde{b}, \tilde{x}_N \approx \tilde{0} \tag{3}$$

is a solution of $A\tilde{x} = \tilde{b}$. In fact the basic solution is $\tilde{x} = (\tilde{x}_B^T \ \tilde{x}_N^T)^T$. In this case, if $\tilde{x}_B \succcurlyeq \tilde{0}$, then this basic solution will be feasible and the corresponding fuzzy objective value will be $\tilde{z} \approx \tilde{c}_B\tilde{x}_B$, in which $\tilde{c}_B = (\tilde{c}_{B_1}, \dots, \tilde{c}_{B_m})$.

Now, corresponding to every index $j, 1 \leq j \leq n$, define

$$\tilde{z}_j \approx \tilde{c}_B y_j \approx \tilde{c}_B B^{-1} a_j \tag{4}$$

Observe that for any basic index $j, j = B_i, (i = 1, \dots, m)$, we have $B^{-1}a_j = e_i$ where $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T$ is the i th unit vector, since $Be_i = [a_{B_1}, \dots, a_{B_i}, \dots, a_{B_m}]e_i = a_{B_i} = a_j$. Thus, $\forall j = B_i, (i = 1, \dots, m)$, we have

$$\tilde{z}_j - \tilde{c}_j \approx \tilde{c}_B B^{-1} a_j - \tilde{c}_j \approx \tilde{c}_B e_j - \tilde{c}_j \approx \tilde{c}_j - \tilde{c}_j \approx \tilde{0}. \quad (5)$$

The following theorem characterizes optimal solutions. The converse part of the result needs the nondegeneracy assumption of the problem, where all fuzzy basic variables corresponding to every basis B are nonzero (and hence positive).

Theorem 4.1. (Optimality conditions). Assume the fuzzy variable linear programming problem (2) is nondegenerate and B is a feasible basis. A fuzzy basic feasible solution $\tilde{x}_B^* = B^{-1}\tilde{b} \succcurlyeq \tilde{0}$, $\tilde{x}_N^* \approx \tilde{0}$ is optimal to (2), if and only if $\tilde{z}_j^* = \tilde{c}_B B^{-1} a_j \succcurlyeq \tilde{c}_j$ for all $j, 1 \leq j \leq n$.

Proof:

Suppose $\tilde{x}_* = (\tilde{x}_B^* \quad \tilde{x}_N^*)^T$ is the fuzzy basic feasible solution of (1) corresponding to the basis B , where $\tilde{x}_B^* = B^{-1}\tilde{b}$, $\tilde{x}_N^* \approx \tilde{0}$. Then, the corresponding fuzzy objective value is:

$$\tilde{z}^* \approx \tilde{c}\tilde{x}_* \approx \tilde{c}_B \tilde{x}_B^* \approx \tilde{c}_B B^{-1}\tilde{b}. \quad (6)$$

On the other hand, for any fuzzy basic feasible solution \tilde{x} to (2), using the general solution corresponding to basis B , we have $\tilde{x}_B \approx B^{-1}\tilde{b} - B^{-1}N\tilde{x}_N$ for the appropriate \tilde{x}_N . Thus, for any fuzzy basic feasible solution to (2), we have

$$\begin{aligned} \tilde{z} &\approx \tilde{c}\tilde{x} \approx \tilde{c}_B \tilde{x}_B + \tilde{c}_N \tilde{x}_N \approx \tilde{c}_B (B^{-1}\tilde{b} - B^{-1}N\tilde{x}_N) \\ &\approx \tilde{c}_B B^{-1}\tilde{b} - \sum_{j=1}^n (\tilde{c}_B B^{-1} a_j - \tilde{c}_j) \tilde{x}_j \approx \tilde{c}_B B^{-1}\tilde{b} - \sum_{j=1}^n (\tilde{z}_j^* - \tilde{c}_j) \tilde{x}_j. \end{aligned}$$

Hence, using (5) and (6) we have

$$\tilde{z} \approx \tilde{z}^* - \sum_{j \neq B_i} (\tilde{z}_j^* - \tilde{c}_j) \tilde{x}_j. \quad (7)$$

Now, if for all $j, 1 \leq j \leq n$, $\tilde{z}_j^* \succcurlyeq \tilde{c}_j$, then for all \tilde{x} we have $(\tilde{z}_j^* - \tilde{c}_j) \tilde{x}_j$ and so we obtain $\sum_{j \neq B_i} (\tilde{z}_j^* - \tilde{c}_j) \tilde{x}_j \succcurlyeq \tilde{0}$. Therefore it follows from (7) that $\tilde{z} \preccurlyeq \tilde{z}^*$ and thus \tilde{x}_* is optimal solution.

For “only if” part, let \tilde{x}_* be a fuzzy optimal basic feasible solution to (2). For $j = B_i$, ($i = 1, \dots, m$), from (5) we know that $\tilde{z}_j^* - \tilde{c}_j \approx \tilde{0}$. From (7) it is obvious that if corresponding to any nonbasic variable \tilde{x}_j we have $\tilde{z}_j^* \preccurlyeq \tilde{c}_j$, then we can enter \tilde{x}_j into the basis and obtain an objective value bigger than \tilde{z}^* (because the problem is nondegenerate and $\tilde{x}_j^* > \tilde{0}$ in the new basis). This is a contradiction to \tilde{z}^* being optimal. Hence, we must have $\tilde{z}_j^* \succcurlyeq \tilde{c}_j, 1 \leq j \leq n$.

Now we state some results proven in Ganesan and Veeramani (2006). Below, suppose $Y = B^{-1}N$ corresponding to a basis B .

Theorem 4.2. Let $\tilde{x}_B = B^{-1}\tilde{b}$ be a fuzzy basic feasible solution of (2). If for any column a_j in A which is not in B , the condition $\tilde{z}_j - \tilde{c}_j < \tilde{0}$ holds and $y_{ij} > 0$ for some $i, (i = 1, \dots, m)$, then it is possible to obtain a new fuzzy basic feasible solution by replacing one of the columns in B by a_j .

Corollary 4.1. If $\tilde{x}_B = B^{-1}\tilde{b}$ is a fuzzy basic feasible solution of (2) with $\tilde{z}_0 \approx \tilde{c}_B\tilde{x}_B$ as the fuzzy value of the objective function and if \bar{x}_B be another fuzzy basic feasible solution with $\bar{z} \approx \bar{c}_B\bar{x}_B$ obtained by admitting a nonbasic column vector a_j into the basic for which and for $\tilde{z}_j - \tilde{c}_j < \tilde{0}$ holds and $y_{ij} > 0$ for some $i, (i = 1, \dots, m)$, then $\bar{z} \succcurlyeq \tilde{z}_0$.

Theorem 4.3. Let $\tilde{x}_B = B^{-1}\tilde{b}$ be a fuzzy basic feasible solution of (2). If there exists an a_j in A which is not in B such that $\tilde{z}_j - \tilde{c}_j < \tilde{0}$ and $y_{ij} < 0$, for all $i, (i = 1, \dots, m)$, then the fuzzy linear programming problem (2) has an unbounded solution.

5. Duality

5.1. Formulation of the Dual Problem

Definition 5.1. For the primal FLP problem

$$\begin{aligned} \text{FLP:} \quad & \max \quad \tilde{z} \approx \tilde{c}\tilde{x} \\ & \text{s. t.} \quad A\tilde{x} \leq \tilde{b} \\ & \quad \quad \tilde{x} \geq \tilde{0} \end{aligned} \tag{8}$$

define the dual problem (DFLP problem) as follows:

$$\begin{aligned} \text{DFLP:} \quad & \min \quad \tilde{u} \approx \tilde{w}\tilde{b} \\ & \text{s. t.} \quad \tilde{w}A \geq \tilde{c} \\ & \quad \quad \tilde{w} \geq \tilde{0}. \end{aligned} \tag{9}$$

Relationships between FLP and DFLP problems. We shall discuss here the relationships between the FLP problem and its corresponding dual.

Lemma 5.1. The dual of the DFLP problem is the FLP problem.

Proof:

Since the DFLP problem is a fuzzy linear programming problem, we may consider Definition 5.1 for its dual. We write the DFLP problem as follows:

$$\begin{aligned} \max \quad & (-\tilde{b}^T)\tilde{w}^T \\ \text{s. t.} \quad & (-A^T)\tilde{w}^T \leq -\tilde{c}^T \\ & \tilde{w}^T \geq \tilde{0}. \end{aligned} \tag{10}$$

Now, using the column vector \tilde{x} as the fuzzy dual variable, the dual of (10) is:

$$\begin{aligned} \min \quad & \tilde{x}^T (-\tilde{c})^T \approx -\tilde{c}\tilde{x} \\ \text{s. t.} \quad & \tilde{x}^T (-A)^T \succcurlyeq -\tilde{b}^T \\ & \tilde{x}^T \succcurlyeq \tilde{0}. \end{aligned} \tag{11}$$

This is the same as problem (8).

Remark 5.1. Lemma 5.1 indicates that the duality results can be applied to any one of the primal or dual problem posed as the primal problem.

Theorem 5.1 (Weak duality.) If \tilde{x}_0 and \tilde{w}_0 are feasible solutions to FLP and DFLP problems, respectively, then $\tilde{c}\tilde{x}_0 \preceq \tilde{b}\tilde{w}_0$.

Proof:

Multiplying $A\tilde{x}_0 \preceq \tilde{b}$ on the left by $\tilde{w}_0 \succcurlyeq \tilde{0}$ and $\tilde{w}_0 A \succcurlyeq \tilde{c}$ on the right by $\tilde{x}_0 \succcurlyeq \tilde{0}$ and using Lemma 2.5 (i), we get then $\tilde{c}\tilde{x}_0 \preceq \tilde{w}_0 A \tilde{x}_0 \preceq \tilde{b}\tilde{w}_0$.

Corollary 5.1. If \tilde{x}_0 and \tilde{w}_0 are feasible solutions to FLP and DFLP problems, respectively, and then $\tilde{c}\tilde{x}_0 \approx \tilde{b}\tilde{w}_0$, then \tilde{x}_0 and \tilde{w}_0 are optimal solutions to their respective problems.

Proof:

It is straightforward, using Theorem 5.1.

The following corollary relates unboundeness of one problem to infeasibility of the other. We use the definition below.

Definition 5.2. We say FLP problem (or DFLP problem) is unbounded if feasible solutions exist with arbitrary large (or small) fuzzy objective value.

Corollary 5.2. If any one of the FLP or DFLP problem is unbounded, then the other problem has no feasible solution.

Proof:

It is straightforward, using Theorem 5.1 [see Ganesan and Veeramani (2006)].

We are now ready to present the strong duality result.

Theorem 5.2. (Strong duality). If any one of the FLP or DFLP problem has an optimal solution, then the other problem has an optimal solution and the two optimal fuzzy objective values are

equal. (In fact, if \tilde{x}_* is an optimal solution of the primal problem then the fuzzy vector $\tilde{w}_* \approx \tilde{c}_B B^{-1}$, where B is the optimal basis corresponding to \tilde{x}_* , is an optimal solution of the dual problem.)

Proof:

Assume that the FLP problem has a fuzzy optimal solution, and $rank(A) = m$. Let $\tilde{y} \geq \tilde{0}$, an $m \times 1$ vector, be the fuzzy slack variables for the constraints $A\tilde{x} \leq \tilde{b}$. The new equivalent problem to the FLP problem is:

$$\begin{aligned} \max \quad & \tilde{z} \approx \tilde{c}\tilde{x} \\ \text{s. t.} \quad & A\tilde{x} + \tilde{y} \approx \tilde{b} \\ & \tilde{x}, \tilde{y} \geq \tilde{0}. \end{aligned} \tag{12}$$

Assume B is the optimal basis matrix and $\tilde{x}_* = (\tilde{x}_B^* \quad \tilde{0}^T)^T = (\tilde{b}^T B^{-T}, \tilde{0})^T$ is the fuzzy basic optimal solution corresponding to the FLP problem. From Theorem 4.1 we have:

$$\tilde{c}_B B^{-1} a_j - \tilde{c}_j \geq \tilde{0} \quad j = 1, \dots, n, n + 1, \dots, n + m$$

or equivalently,

$$\tilde{c}_B B^{-1} a_j \geq \tilde{c}_j \quad j = 1, \dots, n$$

$$\tilde{c}_B B^{-1} e_i \geq \tilde{0} \quad i = 1, \dots, m.$$

Hence, we have:

$$\tilde{c}_B B^{-1} A \geq \tilde{c}$$

$$\tilde{c}_B B^{-1} \geq \tilde{0}.$$

Now, let $\tilde{w}_* \approx \tilde{c}_B B^{-1}$. Using the above inequalities, we can write, $\tilde{w}_* A \geq \tilde{c}$, $\tilde{w}_* \geq \tilde{0}$. Thus, \tilde{w}_* is a feasible solution to the DFLP problem and

$$\tilde{w}_* \tilde{b} \approx \tilde{c}_B B^{-1} \tilde{b} \approx \tilde{c}_B \tilde{x}_B^* \approx \tilde{c} \tilde{x}_*.$$

Hence, $\tilde{w}_* \tilde{b} \approx \tilde{c} \tilde{x}_*$. Therefore, the result follows immediately from Theorem 5.1.

We can now state the fundamental theorem of fuzzy linear programming for the FLP and DFLP problems.

Theorem 5.3. (Fundamental Theorem). For any FLP problem and its corresponding DFLP problem, exactly one of the following statements is true:

Both have optimal solutions \tilde{x}_* and \tilde{w}_* with $\tilde{w}_* \tilde{b} \approx \tilde{c} \tilde{x}_*$
 One problem is unbounded and the other is infeasible.
 Both problems are infeasible.

Proof:

Without loss of the generality, suppose the FLP has an optimal solution such as \tilde{x}_* . Thus, based on strong duality theorem, the DFLP problem has also optimal solution such as \tilde{w}_* with $\tilde{w}_* \tilde{b} \approx \tilde{c} \tilde{x}_*$.

Now suppose the FLP problem is unbounded, while the DFLP problem is feasible. Let \tilde{x}_* and \tilde{w}_* be the feasible solutions for FLP and DFLP problems, respectively. Thus, by Theorem 5.1, we have $\tilde{c} \tilde{x}_* \geq \tilde{w}_* \tilde{b}$. Since the DFLP is unbounded, so we can make the right hand side of this inequality sufficiently small, this is a contradiction. Finally, the following FLP problem and its corresponding DFLP problem show that both FLP and DFLP problems may be infeasible:

$$\begin{aligned} \min \quad & \tilde{z} \approx \left(\frac{1}{2}, \frac{3}{4}, \frac{1}{4}\right) \tilde{x}_1 + \left(\frac{-19}{8}, \frac{-9}{4}, \frac{1}{8}\right) \tilde{x}_2 \\ \text{s. t.} \quad & \tilde{x}_1 - \tilde{x}_2 \geq \left(1, 2, \frac{1}{2}\right) \\ & -\tilde{x}_1 + \tilde{x}_2 \geq \left(-4, -3, \frac{1}{4}\right) \\ & \tilde{x}_1, \tilde{x}_2 \geq \tilde{0} \\ \\ \max \quad & \tilde{u} \approx \left(1, 2, \frac{1}{2}\right) \tilde{w}_1 + \left(-4, -3, \frac{1}{4}\right) \tilde{w}_2 \\ \text{s. t.} \quad & \tilde{w}_1 - \tilde{w}_2 \leq \left(\frac{1}{2}, \frac{3}{4}, \frac{1}{4}\right) \\ & -\tilde{w}_1 + \tilde{w}_2 \leq \left(\frac{-19}{8}, \frac{-9}{4}, \frac{1}{8}\right) \\ & \tilde{w}_1, \tilde{w}_2 \geq \tilde{0}. \end{aligned}$$

It can easily be checked that both problems are infeasible.

Theorem 5.4. (Complementary Slackness). Suppose \tilde{x}_* and \tilde{w}_* are feasible solutions of the FLP problem and its corresponding dual, the DFLP problem, respectively. Then, \tilde{x}_* and \tilde{w}_* are respectively optimal if and only if

$$(\tilde{w}_* A - \tilde{c}) \tilde{x}_* \approx \tilde{0}, (\tilde{w}_* (\tilde{b} - A \tilde{x}_*)) \approx \tilde{0} \quad (13)$$

Proof:

\tilde{x}_* and \tilde{w}_* being feasible solutions of the FLP and DFLP problems, respectively, we have $A\tilde{x}_* \leq \tilde{b}$ and $\tilde{w}_*A \geq \tilde{c}$. Multiplying $A\tilde{x}_* \leq \tilde{b}$ on the left by $\tilde{w}_* \geq \tilde{0}$ yields $\tilde{w}_*A\tilde{x}_* \leq \tilde{w}_*\tilde{b}$. Also, multiplying $\tilde{w}_*A \geq \tilde{c}$ on the right by $\tilde{x}_* \geq \tilde{0}$, yields $\tilde{w}_*A\tilde{x}_* \geq \tilde{c}\tilde{x}_*$. Therefore, we will have:

$$\tilde{w}_*\tilde{b} \geq \tilde{w}_*A\tilde{x}_* \geq \tilde{c}\tilde{x}_* \tag{14}$$

On the other hand, since \tilde{x}_* and \tilde{w}_* are respectively optimal solutions to the primal and dual problems, then by Theorem 5.2 we have $\tilde{w}_*\tilde{b} \approx \tilde{w}_*A\tilde{x}_* \approx \tilde{c}\tilde{x}_*$ and (14) must be written as:

$$(\tilde{w}_*A - \tilde{c}) \tilde{x}_* \approx \tilde{0}, \tilde{w}_*(\tilde{b} - A\tilde{x}_*) \approx \tilde{0}.$$

The converse of the theorem follows from the fact that $(\tilde{w}_*A - \tilde{c}) \tilde{x}_* \approx \tilde{0}$ and $\tilde{w}_*(\tilde{b} - A\tilde{x}_*) \approx \tilde{0}$ imply that $\tilde{w}_*\tilde{b} \approx \tilde{c}\tilde{x}_*$. Therefore, optimality of \tilde{x}_* and \tilde{w}_* follows from Corollary 5.1.

Remark 5.2. The complimentary slackness condition (13) is equivalent to (below, \bar{a}_i refers to the i^{th} row and \bar{a}_j refers to the j^{th} column of A):

$$\begin{aligned} \tilde{w}_*a_j > \tilde{c}_j &\implies \tilde{x}_{*j} \approx \tilde{0} \quad \text{or} \quad \tilde{x}_{*j} > \tilde{0} \implies \tilde{w}_*a_j \approx \tilde{c}_j \quad j = 1, \dots, n, \\ a_i\tilde{x}_* < \tilde{b}_i &\implies \tilde{w}_{*i} \approx \tilde{0} \quad \text{or} \quad \tilde{w}_{*i} > \tilde{0} \implies a_i\tilde{x}_* \approx \tilde{b}_i \quad i = 1, \dots, m. \end{aligned}$$

For an illustration of the above discussion, we give an example.

Example 5.2. Consider the following FLP problem:

$$\begin{aligned} \max \quad & \tilde{z} \approx (5, 7, 1)\tilde{x}_1 + (7, 9, 1)\tilde{x}_2 \\ \text{s. t.} \quad & 14\tilde{x}_1 + 21\tilde{x}_2 \leq (31, 46, 2) \\ & 35\tilde{x}_1 + 28\tilde{x}_2 \leq (53, 80, 3) \\ & \tilde{x}_1, \tilde{x}_2 \geq \tilde{0} \\ \min \quad & \tilde{u} \approx (31, 46, 2)\tilde{w}_1 + (53, 80, 3)\tilde{w}_2 \\ \text{s. t.} \quad & \tilde{w}_1 - \tilde{w}_2 \geq (5, 7, 1) \\ & -\tilde{w}_1 + \tilde{w}_2 \geq (7, 9, 1) \\ & \tilde{w}_1, \tilde{w}_2 \geq \tilde{0}. \end{aligned}$$

One can find the fuzzy optimal solution of the FLP problem as $\tilde{x}_1 \approx (\frac{5}{7}, \frac{8}{7}, \frac{1}{49})$, $\tilde{x}_2 \approx (1, \frac{10}{7}, \frac{4}{49})$ with the optimal objective value $\tilde{z} \approx (\frac{76}{7}, \frac{138}{7}, \frac{169}{49})$. Hence, if we let $\tilde{w} \approx \tilde{c}_B B^{-1}$, we obtain $\tilde{w}_1 \approx (\frac{1}{7}, \frac{25}{49}, \frac{9}{49})$, $\tilde{w}_2 \approx (\frac{-3}{49}, \frac{1}{7}, \frac{5}{49})$. This is the optimal solution of the DFLP problem with its optimal objective value as $\tilde{u} \approx (\frac{-77}{49}, \frac{1574}{49}, \frac{421}{49})$.

Therefore, using Definition 2.2(ii), both problems have optimal solutions and the two optimal fuzzy objective values are equal.

6. Conclusion

We established the dual of a linear programming problem with symmetric trapezoidal fuzzy numbers and developed some duality results for the fuzzy primal and fuzzy dual problems. The duality results include weak and strong duality, and complementary slackness. These results would be useful for post optimality analysis.

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REFERENCES

- Bazaraa, M.S., Jarvis, J.J., and Sherali, H.D. (2005) Linear Programming and Network Flows, 3rd ed., New York: John Wiley and Sons.
- Bector, C.R. and Chandra, S. (2002). On duality in linear programming under fuzzy environment, *Fuzzy Sets and Systems*, 125, 317–325.
- Bellman, R.E., and Zadeh, L.A. (1970). Decision making in a fuzzy environment, *Management Science*, 17, 141–164.
- Chanas, S. (1983). The use of parametric programming in fuzzy linear programming, *Fuzzy Sets and Systems*, 11, 243–251.
- Ebrahimnejad, A., and Nasseri, S.H. (2009). Using complementary slackness property to solve linear programming with fuzzy parameters. *Fuzzy Information and Engineering*, 3, 233-245.
- Ebrahimnejad, A., Nasseri, S.H., Hosseinzadeh Lotfi, F., and Soltanifar, M. (2010a). A primal-dual method for linear programming problems with fuzzy variables. *European Journal of Industrial Engineering*, 4(2), 189-209.
- Ebrahimnejad, A., Nasseri, S.H., and Hosseinzadeh Lotfi, F., (2010b). Bounded linear programs with trapezoidal fuzzy numbers. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 18(3), 269-286.
- Ebrahimnejad, A., Nasseri, S.H., and Mansourzadeh, S.M., (In press). Bounded primal simplex algorithm for bounded linear programming with fuzzy cost coefficients, *International Journal of Operations Research and Information Systems*.
- Ebrahimnejad, A., and Nasseri, S.H. (2010). A dual simplex method for bounded linear programmes with fuzzy numbers, *International Journal of Mathematics in Operational Research*, 2 (6), 762-779.
- Ganesan, K., and Veeramani, P. (2006). Fuzzy linear programming with trapezoidal fuzzy numbers. *Ann. Oper. Res.*, 143, 305-315.
- Ganesan, K., (2006). Duality in fuzzy linear programming problems, *Journal of Computational Mathematics and Optimization*, 2(1), 35 – 49.

- Inuiguchi, M., J. Ramik, Tanino, T., and Vlach, M. (2003). Satisficing solutions and duality in interval and fuzzy linear programming, *Fuzzy Sets and Systems*, 135, 151–177.
- Mahdavi-Amiri, N., and Nasseri, S.H. (2006). Duality in fuzzy number linear programming by use of a certain linear ranking function, *Applied Mathematics and Computation*, 180, 206-216.
- Mahdavi-Amiri, N., and Nasseri, S.H. (2007). Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables, *Fuzzy Sets and Systems*, 158, 1961-1978.
- Mahdavi-Amiri, N., Nasseri, S.H., and Yazdani, A. (2009). Fuzzy primal simplex algorithms for solving fuzzy linear programming problems, *Iranian Journal of Operational Research*, 1, 68–84.
- Maleki, H.R., Tata, M., & Mashinchi, M. (2009). Linear programming with fuzzy variables, *Fuzzy Sets and Systems*, 109, 21-33.
- Nasseri, S.H., and Ebrahimnejad, A. (2010a). A fuzzy dual simplex method for fuzzy number linear programming problem. *Advances in Fuzzy Sets and Systems*, 5(2), 81-95.
- Nasseri, S.H., and Ebrahimnejad, A. (2010b). A fuzzy primal simplex algorithm and its application for solving flexible linear programming problems. *European Journal of Industrial Engineering*, 4(3), 372-389.
- Nasseri, S.H., and Mahdavi-Amiri, N. (2009). Some duality results on linear programming problems with symmetric fuzzy numbers. *Fuzzy Information and Engineering*, 1, 59–66.
- Ramik, J., (2005). Duality in fuzzy linear programming: some new concepts and results, *Fuzzy Optimization and Decision Making*, 4, 25–39.
- Rodder, W. and Zimmermann, H.J. (1980). Duality in fuzzy linear programming. In *Internat. Symp. On Extremal Methods and Systems Analysis*. University of Texas at Austin, 415–427. (Proceedings, Fiacco, A.V. and Kortanek, K.O. (1980)(eds.). Berlin, Heidelberg, New York.)
- Safi, Z. M. R., Maleki, H. R., and Zaeimazad, E. (2007). A geometric approach for solving fuzzy linear programming problems, *Fuzzy Optimization and Decision Making*, 6, 315-336.
- Tanaka, H., Okuda, T., and Asai, K. (1984). Fuzzy linear programming problems with fuzzy numbers, *Fuzzy Sets and Systems*, 13, 1–10.
- Vasant, P.M. (2003). Application of Fuzzy Linear Programming in Production Planning, *Fuzzy optimization and Decision Making*, 3, 229-241.
- Verdegay, J.L. (1984). A dual approach to solve the fuzzy linear programming problems, *Fuzzy Sets and Systems*, 14, 131–141.
- Vijay, V., Mehra, A., Chandra, S., and Bector, C. R. (2007). Fuzzy matrix games via a fuzzy relation approach, *Fuzzy Optimization and Decision Making*, 6, 299-314.
- Wu, H.C. (2003). Duality theory in fuzzy linear programming problems with fuzzy coefficients, *Fuzzy Optimization and Decision Making*, 2, 61–73.