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On the Global Existence and Boundedness of Solutions of Nonlinear Vector Differential Equations of Third Order

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Abstract

In this paper, we give some criteria to ensure the global existence and boundedness of solutions to a kind of third order nonlinear vector differential equations. By using the Lyapunov's direct method, we obtain a new result on the topic and give an example for the illustrations. Our result includes, completes and improves some earlier results in the literature.

Keywords: Vector differential equation; third order; global existence; boundedness; the Lyapunov's direct method

MSC 2010 No.: 34C11; 74G25

1. Introduction

This paper studies the global existence and boundedness of solutions to the third order nonlinear vector differential equations of the form

$$(q(t)(r(t)X')')' + F(X,X')X'' + G(X')X' + cX = P(t) \tag{1}$$

in which $t \in R^+$, $R^+ = [0,\infty)$, $X \in R^n$; c is a positive constant, r and q are positive and continuously differentiable functions on R^+; F and G are $n \times n$-symmetric continuous and differentiable matrix functions; $P : R^+ \to R^n$ is a continuous function with respect to t.

We can write Equation (1) in the differential system form as

$$X' = \frac{Y}{r(t)},$$

$$Y' = \frac{Z}{q(t)},$$

$$Z' = -A(t)Z - B(t)Y - cX + P(t), \qquad (2)$$

where

$$A(t) = \frac{1}{r(t)q(t)} F(X, \frac{Y}{r(t)})$$

and

$$B(t) = \frac{1}{r(t)}(G(\frac{Y}{r(t)}) - \frac{r'(t)}{r(t)} F(X, \frac{Y}{r(t)}))$$

are $n \times n$-symmetric continuous and differentiable matrix functions.

Let

$$B'(t) = \frac{d}{dt}(b_{ij}(t)), \ (i, j = 1, 2, ..., n),$$

where $b_{ij}(t)$ are the components of $B(t)$. On the other hand $X(t)$, $Y(t)$ and $Z(t)$ are, respectively, abbreviated as X, Y and Z throughout the paper. Additionally, throughout this paper, the symbol $\langle X, Y \rangle$ corresponding to any pair X and Y in R^n stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is,

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Thus, $\langle X, X \rangle = \|X\|^2$, and also $\lambda_i(D)$, $(i = 1, 2, ..., n)$, are the eigenvalues of the $n \times n$ -matrix D.

To the best of our knowledge, from the literature the qualitative behaviors of solutions, boundedness of solutions, stability of solutions, existence of periodic solutions, except the global existence of solutions, to certain third order nonlinear scalar and vector differential equations have been discussed by many authors; see, for example, the book of Reissig et al. (1974) as a survey and the papers of Abou-El-Ela (1985), Afuwape (1985), Afuwape and Ukpera (2001), Afuwape and Omeike (2004), Ezeilo and Tejumola (1966), Ezeilo and Tejumola (1975), Feng (1995), Meng (1993), Omeike (2014), Tiryaki (1999), Tunc (1999), Tunc (2006), Tunc (2009), Tunc and Tunc (2006), Tunc and Ates (2006), Tunc and Ates (2006) and the references therein. However, the global existence and boundedness of solutions to Equation (1) have not been discussed in the literature yet. The basic reason may be the difficulty of finding a suitable Lyapunov function for differential systems of higher order.

It is now worth mentioning some related papers on the subject. Abou-El-Ela (1985) established sufficient conditions which guarantee that all solutions of vector differential equation

$$X''' + F(X, X')X'' + G(X') + H(X) = P(t, X, X', X'')$$

are ultimately bounded.

Later, Tunc and Tunc (2006) obtained some sufficient conditions under which all solutions of the third order vector differential equation

$$X''' + F(X, X', X'')X'' + G(X') + H(X) = P(t, X, X', X'')$$

are ultimately bounded, and the authors also established some sufficient conditions which ensure that there exists at least one periodic solution of that equation.

Further, Tunc (2009) proved two results, for the cases $P = 0$ and $P \neq 0$, respectively, on the stability and boundedness of solutions to the vector differential equation of third order

$$X''' + \psi(X')X'' + BX' + cX = P(t).$$

For the same cases, Omeike (2014) discussed the global asymptotic stability and boundedness of solutions to nonlinear vector differential equation of third order

$$X''' + \psi(X')X'' + \phi(X)X' + cX = P(t).$$

In addition to the mentioned papers, the motivation of this paper comes from the books or the papers of Ahmad and Rama Mohana Rao (1999), Baxley (1997), Burton (1985), Changian et al. (2012), Constantin (1995), Fujimoto and Yamaoka (2014), Graef and Tunç (2015), Mustafa and Rogovchenko (2003), Napoles Valdes (2001), Oudjedi et al. (2014), Tidke and Dhakne (2010), Tidke (2010), Tiryaki and Zafer (2013), Wu et al. (2012) and Yin (2004). Through all the mentioned papers and the book of Reissig et al. (1974), the Lyapunov's direct method, Lyapunov (1966), is used as a basic tool to prove the results in there. The aim of this paper is to extend and improve the results obtained on the global existence and boundedness of solutions to scalar differential equations of second order, (see Fujimoto and Yamaoka (2014), Mustafa and Rogovchenko (2003), Tidke (2010), Tiryaki and Zafer (2013), Wu et al. (2012) and Yin (2004)), to the same topics for a certain third order non-linear vector differential equation, Equation (1). This is the novelty and originality of this paper.

We suppose that there exist positive constants a, b, c, δ, m, L and M such that the following assumptions hold

$(A1)$ $b - c^2 \geq 0,$

$(A2)$ $0 < m \leq q(t) \leq r(t) \leq M,$ $r'(t) \leq q'(t) \leq 0,$ $r''(t) \geq 0,$

$(A3)$ the matrices A and B are symmetric, $\lambda_i(A(t)) \geq a,\ \lambda_i(B(t)) \geq b$ and

$$\delta = \max\left|\lambda_i(B'(t))\right|,\ (i = 1, 2, ..., n),$$

$(A4)\ \|P(t)\| \leq \theta(t)$ for all $t \geq 0,\ \max\theta(t) < \infty$ and $\theta(t) \in L^1(0, \infty),$ where $L^1(0, \infty)$ is the space of the Lebesque-integrable functions.

2. Main result

Before stating our main result, we give two well known algebraic results which will be needed in the proof.

Lemma 1. [Bellman (1997)]

Let A be a real symmetric $n \times n$ -matrix and

$$\bar{a} \geq \lambda_i(A) \geq a > 0\ (i = 1, 2, ..., n),$$

where $\bar{a}$ and a are constants. Then

$$\bar{a}\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle$$

and

$$\bar{a}^2\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle.$$

Lemma 2. [Afuwape (1983)]

Let $Q,\ D$ be any two real $n \times n$ commuting symmetric matrices. Then,

(i) The eigenvalues $\lambda_i(QD)\ (i = 1, 2, ...,)$ of the product matrix QD are real and satisfy

$$\max_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D) \geq \lambda_i(QD) \geq \min_{1 \leq j,k \leq n} \lambda_j(Q)\lambda_k(D).$$

(ii) The eigenvalues $\lambda_i(Q + D)\ (i = 1, 2, ...,)$ of the sum matrices Q and D are real and satisfy

$$\left\{\max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D)\right\} \geq \lambda_i(Q + D) \geq \left\{\min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D)\right\},$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are, respectively, the eigenvalues Q and D.

Theorem

Suppose that assumptions $(A1) - (A4)$ hold. Then all solutions of system (2) are continuable and bounded.

Proof:

The proof of this theorem depends on a scalar differentiable function $V(t)=V(t,X,Y,Z)$. We impose some conditions on the function $V(t)$ and its time derivative, both of which imply the global existence and boundedness of solutions of Equation (1). We define the function $V(t)$ by

$$V(t)=\frac{r(t)}{2}\langle X,X\rangle+cq(t)\langle X,Y\rangle+\frac{q(t)}{2}\langle B(t)Y,Y\rangle+\frac{1}{2}\langle Z,Z\rangle. \tag{3}$$

It is clear from Equation (3) that $V(t,0,0,0)=0.$ From the definition of $V(t)$ in Equation (3), we observe that

$$V(t)=\frac{q(t)}{2}(\frac{r(t)}{q(t)}\langle X,X\rangle+2c\langle X,Y\rangle+\langle B(t)Y,Y\rangle)+\frac{1}{2}\langle Z,Z\rangle.$$

In view of the assumptions of Theorem and Lemma 1, respectively, it follows that

$$\begin{aligned}V(t)&\geq\frac{q(t)}{2}(\langle X,X\rangle+2c\langle X,Y\rangle+b\langle Y,Y\rangle)+\frac{1}{2}\|Z\|^2\\&\geq\frac{m}{2}(\|X+cY\|^2+(b-c^2)\|Y\|^2)+\frac{1}{2}\|Z\|^2.\end{aligned} \tag{4}$$

Thus, it is evident from the terms contained in Equation (4) that there exists a sufficiently small positive constant k such that

$$V(t)\geq k(\|X\|^2+\|Y\|^2+\|Z\|^2). \tag{5}$$

Calculating the time derivative of the function $V(t)$ along any solution $(X(t),Y(t),Z(t))$ of system (2), we have

$$\begin{aligned}V'(t)=&\frac{q'(t)}{2}(\frac{r'(t)}{q'(t)}\langle X,X\rangle+2c\langle X,Y\rangle+\langle B(t)Y,Y\rangle)+\langle X,Y\rangle+c\frac{q(t)}{r(t)}\langle Y,Y\rangle\\&+\frac{q(t)}{2}\langle B'(t)Y,Y\rangle-\langle Z,A(t)Z\rangle+\langle Z,P(t)\rangle.\end{aligned}$$

From the benefits of assumptions $(A1)-(A4)$, Lemma 1 and the inequalities:

$$\langle U,V\rangle\leq\|UV\|\leq\|U\|\|V\|,\ 2\|UV\|\leq\|U\|^2+\|V\|^2,$$

the following estimates can be derived:

$$\begin{aligned}&\frac{q'(t)}{2}(\frac{r'(t)}{q'(t)}\langle X,X\rangle+2c\langle X,Y\rangle+\langle B(t)Y,Y\rangle)\\&\qquad\leq\frac{q'(t)}{2}(\langle X,X\rangle+2c\langle X,Y\rangle+b\langle Y,Y\rangle)\\&\qquad=\frac{q'(t)}{2}(\|X+cY\|^2+(b-c^2)\|Y\|^2)\leq 0,\end{aligned}$$

$$\frac{q(t)}{r(t)} c\langle Y, Y\rangle \le c\|Y\|^2,$$

$$\frac{q(t)}{2}\langle B'(t)Y, Y\rangle \le \frac{M\delta}{2}\|Y\|^2,$$

$$\langle Z, A(t)Z\rangle \ge a\|Z\|^2$$

and

$$\langle Z, P(t)\rangle \le \|Z\|\theta(t) \le (1+\|Z\|^2)\theta(t).$$

From these estimates and (5), we have

$$V'(t) \le (c+\frac{M\delta}{2})\|Y\|^2 + \|X\|^2 + \|Y\|^2 + (1+\|Z\|^2)\theta(t)$$

$$\le \theta(t) + \frac{1}{k}(1+c+\frac{M\delta}{2}+\theta(t))V(t). \tag{6}$$

Integrating both sides of inequality (6), from 0 to t $(t \ge 0)$, we get

$$V(t) - V(0) \le \int_0^t \theta(s)ds + \frac{1}{k}\int_0^t ((1+c+\frac{M\delta}{2}+\theta(s))V(s)ds.$$

Taking

$$V(0) + \int_0^t \theta(s)ds = N \text{ and } 1+c+\frac{M\delta}{2} = \varepsilon,$$

it follows that

$$V(t) \le N + \frac{1}{k}\int_0^t (\varepsilon + \theta(s))V(s)ds.$$

By using Gronwall-Bellman inequality [see Ahmad and Rama Mohana Rao (1999)], we conclude that

$$V(t) \le N\exp(\frac{1}{k}\int_0^t (\varepsilon+\theta(s))ds). \tag{7}$$

Since all the functions appearing in Equation (1) are continuous, then there exists a solution defined on $[t_0, t_0+\delta)$ for some $\delta > 0$. We now need to show that the solution can be extended to the entire interval $[t_0, \infty)$. Suppose on the contrary that there is a first time $T < \infty$ such that the solution exists on $[t_0, T)$ and

$$\lim_{t\to T^-}(\|X\|+\|Y\|+\|Z\|) = \infty.$$

Let $(X(t),Y(t),Z(t))$ be such a solution of system (2) with initial condition (X_0,Y_0,Z_0). In view of inequalities (5) and (7), we get

$$\|X(T)\|^2+\|Y(T)\|^2+\|Z(T)\|^2 \le \frac{N}{k}\exp(\frac{1}{k}\int_0^T(\varepsilon+\theta(s))ds)=K.$$

This inequality implies that $\|X(t)\|$, $\|Y(t)\|$ and $\|Z(t)\|$ are bounded as $t\to T^-$. Hence, we conclude that $T<\infty$ is not possible. Therefore, we must have $T=\infty$. This completes the proof of Theorem.

Example

Let $n=2$ in Equation (1) and choose the functions F,G,P,q and r as the following:

$$r(t)=1+\frac{1}{1+t},\ q(t)=1+\frac{1}{2+t},$$

$$F(X,X')=\begin{bmatrix} 1+\dfrac{1}{1+x^2+y^2} & 0 \\ 0 & 1+\dfrac{1}{1+x^2+y^2}\end{bmatrix},$$

$$G(X')=\begin{bmatrix} 4+\dfrac{1}{1+y^2} & 0 \\ 0 & 4+\dfrac{1}{1+y^2}\end{bmatrix},$$

$$c=1$$

and

$$P(t)=\begin{bmatrix} \dfrac{\sin t}{1+t^2} \\ \dfrac{\cos t}{1+t^2}\end{bmatrix}.$$

It is obvious that F and G are symmetric matrices and

$$m=1\le q(t)\le r(t)\le 2=M,\ -1\le r'(t)\le q'(t)\le 0,\ r''(t)\ge 0.$$

We obtain eigenvalues of the matrices F and G as

$$\lambda_1(F)=\lambda_2(F)=1+\frac{1}{1+x^2+y^2},$$

$$\lambda_1(\mathrm{G}) = \lambda_2(\mathrm{G}) = 4 + \frac{1}{1+y^2}.$$

Next, it is also clear that

$$1 \le \lambda_i(F) \le 2 \text{ and } 4 \le \lambda_i(\mathrm{G}) \le 5,$$

$$\lambda_i(A(t)) \ge \frac{1}{3} = a,\ \lambda_i(B(t)) \ge 2 = b,\ \max\left|\lambda_i(B'(t))\right| = \frac{5}{4} = \delta,\ i = (1,2),$$

$$\|P(t)\| \le \frac{2}{1+t^2} = \theta(t),\ \max\theta(t) = 2 < \infty$$

and

$$\int_0^\infty \theta(\mathrm{t})\,\mathrm{dt} = \int_0^\infty \frac{2}{1+t^2}\,\mathrm{dt} = \pi,\ \text{that is, } \theta(\mathrm{t}) \in L^1(0,\infty).$$

Thus, all the assumptions of the Theorem hold. Hence, we can conclude that all solutions are continuable and bounded for the special case chosen.

3. Conclusion

A kind of vector differential equations of the third order was considered. The global existence and boundedness of solutions of this equation were discussed by using the Lyapunov's second method. The obtained result includes, completes, and improves some earlier results in the literature and makes a contribution to the subject.

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