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Some Results on Renewal process with Erlang Interarrival Times

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Abstract

This paper develops the probability functions of a renewal process, whose interarrival times are independent and identically distributed (i.i.d.) random variables with Erlang distribution. The results are obtained and proved through relation between Poisson and Erlang and between Beta and Binomial distributions. The distribution of the number of renewals in $A = [a,b)$, $0 \le a < b$, and its expectation and their numerical values are given in the form of tables. An example is presented, to show the application.

Keywords: Renewal Process; Independent Increment; Stationary Increment; Erlang Distribution

MSC (2000) No.: 60G55, 60G99

1. Introduction

A renewal process is a point process characterized by the fact that successive interarrival times $T_1, T_2, \ldots$ are i.i.d. non-negative random variables. Let $S_n = \sum_{i=1}^{n} T_i$ for $n \ge 1$, $S_0 = 0$.

Suppose A is a bounded Borel subset of $[0,\infty)$. Let N_A be the number of the elements of the set $\{n > 0 : S_n \in A\}$. In the case $A = [0,t)$ we denote N_A by N_t.

It is well known that if interarrival times in a renewal process are i.i.d. $\exp(\lambda)$, random variables, then N_A has Poisson distribution with parameter $\lambda m(A)$, m denoting the Lebesgue measure. In particular for $A = [a,b)$, $a < b$, N_A has Poisson distribution with parameter $\lambda(b-a)$ and hence N_t has Poisson distribution with parameter λt [Kingsman (1993), Varsei (2007)]. In fact, it is a Poisson renewal process. This process and its generalization have been studied by several authors both from mathematical and physical points of view [Mainardi et al. (2007), Barkai (2002)]. Cuff and Fridman (2006) obtained the exact distribution of N_t, when interarrival times are sums of two independent exponential random variables with likely unequal parameters Cuffe and Friedman (2006). It can be shown that if interarrival times in a renewal process are i.i.d. Erlang random variables with parameters r and λ, then p.d.f., c.d.f, and the expectation of N_t is

$$p(N_t = k) = \sum_{i=kr}^{(k+1)r-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} , \quad k = 0,1,\cdots, \tag{1}$$

$$p(N_t \le k) = \sum_{i=0}^{(k+1)r-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!}, \quad k = 0,1,\cdots, \tag{2}$$

$$E(N_t) = \frac{\lambda t}{r} + \frac{1}{r}\sum_{k=1}^{r-1} \frac{z^k}{1-z^k}(1-\exp(-\lambda t(1-z^k))), \tag{3}$$

where $z = e^{\frac{2\pi i}{r}}$, Parzen (1999).

In this paper, the distribution of N_A, $A = [a, b)$, $0 < a < b$, and its expectation is obtained. It is shown that, as it is expected, this process may have neither independent nor stationary increments. The tables of p.d.f. and expectation of N_A, $A = [a, b)$, $0 \le a < b$, for different values of a, b, r, and λ are given. We also consider the case where r is random.

2. Distribution of N_A

In the renewal process let $T_1, T_2, \ldots$ be i.i.d. Erlang random variables with parameters r and λ and let $S_n = \sum_{i=1}^{n} T_i$ for $n \ge 1$, $S_0 = 0$. Let $A = [a,b)$, $0 < a < b$. To obtain the distribution of N_A, we first calculate the joint distribution of (N_a, N_A).

Theorem 1.

In the renewal process the joint distribution of (N_a, N_A) is equal to

$$p(N_a = k, N_A = s) = \sum_{i=kr}^{(k+1)r-1} \sum_{j=\max(i,(s+k)r)}^{(s+k+1)r-1} \frac{e^{-\lambda b}(\lambda b)^j}{j!} \binom{j}{i} (\frac{a}{b})^i (1-\frac{a}{b})^{j-i}, \ k,s = 0,1,2,\cdots. \qquad (4)$$

Proof:

For different values of k and s we have

$$p(N_a = 0, N_A = 0) = p(N_b = 0), \qquad (5)$$
$$p(N_a = 0, N_A = 1) = p(a < S_1 < b < S_2), \qquad (6)$$
$$p(N_a = k, N_A = 0) = p(S_k < a < b < S_{k+1}) \quad k \geq 1, \qquad (7)$$
$$p(N_a = k, N_A = 1) = p(S_k < a < S_{k+1} < b < S_{k+2}) \quad k \geq 1, \qquad (8)$$
$$p(N_a = k, N_A = s) = p(S_k < a < S_{k+1} < S_{k+s} < b < S_{k+s+1}) \quad k \geq 1,\ s \geq 2. \qquad (9)$$

The equality (5) is obtained by (1), i.e.,

$$p(N_b = 0) = \sum_{j=0}^{r-1} \frac{e^{-\lambda b}(\lambda b)^j}{j!}, \qquad (10)$$

which equals

$$\sum_{j=0}^{r-1} \frac{e^{-\lambda b}(\lambda b)^j}{j!} \sum_{i=0}^{j} \binom{j}{i} (\frac{a}{b})^i (1-\frac{a}{b})^{j-i} = \sum_{i=0}^{r-1} \sum_{j=i}^{r-1} \frac{e^{-\lambda b}(\lambda b)^j}{j!} \binom{j}{i} (\frac{a}{b})^i (1-\frac{a}{b})^{j-i}.$$

Hence, in the case $k = s = 0,$ the theorem holds. Now we prove the theorem in the case $k \geq 1$ and $s \geq 2$. Other cases are obtained similarly.

Let

$$X_1 = S_k = Y_1,$$
$$X_2 = S_{k+1} = Y_1 + Y_2,$$
$$X_3 = S_{k+s} = Y_1 + Y_2 + Y_3,$$
$$X_4 = S_{k+s+1} = Y_1 + Y_2 + Y_3 + Y_4, \qquad (11)$$

where

$$Y_1 \sim Erlang(kr, \lambda),\ Y_2 \sim Erlang(r, \lambda),\ Y_3 \sim Erlang((s-1)r, \lambda), Y_4 \sim Erlang(r, \lambda)$$

and $Y_1, Y_2, Y_3,$ and Y_4 are independent.

The value of (9) equals

$$\int_b^\infty \int_a^b \int_a^{x_3} \int_0^a g(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4 \,, \tag{12}$$

where g is the joint density function of (X_1, X_2, X_3, X_4) which is obtained by applying transformation (11), Casella and Berger (1990). In fact, we have

$$g(x_1,\ldots,x_4) = \begin{cases} \dfrac{\lambda^{(k+s+1)r} e^{-\lambda x_4}}{\Gamma(r)\Gamma(s)\Gamma((s-1)r)\Gamma(kr)} x_1^{kr-1}(x_2-x_1)^{r-1}(x_3-x_2)^{(s-1)r}(x_4-x_3)^{(r-1)}, \\ \hfill 0 < x_1 < x_2 < x_3 < x_4, \\ 0, \hfill \textit{otherwise.} \end{cases} \tag{13}$$

We can calculate the integral (12) and obtain (4) for $k \geq 1, s \geq 2$, by using the relation between Beta and Binomial distribution, i.e.,

$$\int_0^c \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1}(1-x)^{n-1} dx = \sum_{i=m}^{m+n-1} \binom{m+n-1}{i} c^i (1-c)^{m+n-i-1}, \qquad 0 < c < 1 \quad m, n = 1, 2, \ldots,$$

and the relation between Erlang and Poisson distribution, i.e.,

$$\int_d^\infty \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} dx = \sum_{i=0}^{r-1} \frac{e^{-\lambda d}(\lambda d)^i}{i!}, \qquad \lambda > 0,\ d > 0,\ r = 1, 2, \cdots.$$

Remark 1.

Let X and Y be two random variables whose joint distribution is

$$p(X = i, Y = j) = \frac{e^{-\lambda b}(\lambda b)^j}{j!} \binom{j}{i} \left(\frac{a}{b}\right)^i \left(1 - \frac{a}{b}\right)^{j-i}, \qquad i = 0, 1, \cdots, j, \tag{14}$$

i.e., $Y \sim$ Poisson(λb) and $X|Y = y \sim$ Binomial$(y, \frac{a}{b})$. The joint distribution of (N_a, N_A) is

$$p(N_a = k, N_A = s) = p(kr \leq X < (k+1)r,\ (s+k)r \leq Y < (s+k+1)r). \tag{15}$$

Corollary 1.

The distribution of N_a in the renewal process is

$$p(N_A = s) = \sum_{k=0}^{\infty} \sum_{i=kr}^{(k+1)r-1} \sum_{j=\max(i,(s+k)r)}^{(s+k+1)r-1} \frac{e^{-\lambda b}(\lambda b)^j}{j!} \binom{j}{i} (\frac{a}{b})^i (1-\frac{a}{b})^{j-i}, \tag{16}$$

or in terms of the notation of Remark 1

$$p(N_A = s) = \sum_{k=0}^{\infty} p(kr \le X < (k+1)r,\ (s+k)r \le Y < (s+k+1)r). \tag{17}$$

Table 2 shows the p.d.f. of N_A, $A=[0,1)$ for some values of r and λ and table 3 shows the p.d.f. of N_A, $A=[1,2)$ for the same values.

The expectation of N_A can be obtained by the p.d.f. of N_A and also by using (3) observing that $E(N_A)=E(N_b)-E(N_a)$. Hence, we have

$$E(N_A) = \frac{\lambda(b-a)}{r} + \frac{1}{r}\sum_{k=1}^{r-1} \frac{z^k}{1-z^k}(\exp(-\lambda a(1-z^k)) - \exp(-\lambda b(1-z^k))), \tag{18}$$

where $z = e^{\frac{2\pi i}{r}}$.

Table 4 shows the expectation of N_A, $A=[0,1)$ and A=$[1,2)$ for some values r and λ. The comparison between them is of interest.

Corollary 2.

If a is sufficiently large, then the expectation of N_A, $A=[a,b)$ is approximately $\frac{\lambda(b-a)}{r}$.

Corollary 3.

The conditional distribution of T_1 given $N_t = 1$ is

$$p(T_1 < x | N_t = 1) = \frac{p(r \le X < 2r, r \le Y < 2r)}{p(r \le Y < 2r)},$$

where X and Y are random variables whose joint distribution is (14) with parameters a=x, b=t.

The conclusion is easily obtained in view of

$$p(T_1 < x,\ N_t = 1) = p(T_1 < x < t < T_1 + T_2),$$

and equations (1) and (7).

The conditional distribution of $N_{[c,d)}$ given $N_{[a,b)}$, $0 \le a \le c < d \le b$, is obtained similarly.

Remark 2.

The renewal process may have neither independent nor stationary increment. For example let $a = 1,\ b = 2,\ r = 2,\ \lambda = 2.$ Using equations (1) and (16) we have

$$p(N_a = 0) = 3e^{-2},\ \ p(N_A = 0) = 2e^{-2} + e^{-6},\ \ p(N_a = 0,\ N_A = 0) = p(N_b = 0) = 5e^{-4}.$$

Hence obviously,

$$p(N_a = 0) \ne p(N_A = 0),$$

and also

$$p(N_a = 0,\ N_A = 0) \ne p(N_a = 0)\,p(N_A = 0).$$

3. The Case where *r* is a Random Variable

In this section, we let r be a random variable which henceforth is denoted by R. More exactly, suppose the interarrival times of a renewal process given $R=r$ are i.i.d. random variables with $Erlang(r,\ \lambda)$ distribution. Some interesting results can be obtained when R is negative binomial and in particular has geometric distribution.

It is easy to show that the following lemma, Casella and Berger (1990).

Lemma.

Suppose R has negative binomial distribution with parameters (n, p), $n \in \mathrm{N}$, $0 < p < 1$ and $Y|R=r$ has *Erlang* distribution with parameters $(r,\ \lambda)$, $\lambda > 0.$ The marginal distribution of Y is $Erlang(n,\ \lambda p),$ and hence if R has geometric distribution with parameter p then the distribution of Y is $Exp(\lambda p)$.

Now, the following result can be easily obtained.

Theorem 2.

Let the interarrival times of a renewal process be i.i.d. random variables whose distributions of the renewal process have $Erlang(R,\ \lambda),$ where $R \sim NB(n,\ p).$ The interarrival times of the renewal process have $Erlang(n,\ \lambda p).$ In particular if R has geometric distribution with parameter p, then the distribution of interarrival times is $Exp(\lambda p)$ and hence the renewal process is Poisson with rate λp.

4. Example: Total Claims on a Life-Insurance Company for the Youth

Let $W_1, W_2, \ldots$ denote the occurrence times of the deaths of the policy holders of a certain Life-Insurance company. Treating these times of insurance claims, the number of deaths can be treated as a Poisson process with rate λ. The company has to pay too much for deaths of youngsters which are considered particular deaths. So the company wants to know the total number of such deaths during the time period $[a, b)$. Suppose after r-1 ordinary deaths, one particular death occurs, i.e. the r^{th}, $2r^{th}$,… deaths are particular, where r can be estimated by the observations. More exactly let

$$T_1 = \sum_{i=1}^{r} W_i, \;\; T_2 = \sum_{i=r+1}^{2r} W_i, \ldots,$$

where $T_1, T_2, \ldots$ are i.i.d. random variables with *Erlang*(r, λ) distribution. Interarriaval times $T_1, T_2, \ldots$ denote the occurrence times and N_A is the total number of particular deaths during the time period $A = [a, b)$. We can obtain the distribution and expectation of NA, by the formulas (16) and (18). For example when λ = 1 (i.e., in average, one death occurs in each day) for r = 5, 10, 15 the expectation of the number of particular deaths in the first few months of the first year are given in Table 1. As it can be seen, in view of corollary 2, the expectation remains constant from a time onwards.

5. Tables

Table1: The Expectation of N_A, $\lambda = 1$

r	A = [0, 30)	A = [30, 60)	A = [60, 90)	A = [90, 120)	A = [120, 150)	
5	5.6	6	6	6	6	6
10	2.5492	3.0008	3	3	3	3
15	1.5296	2.028	2.0008	2.0001	2	2

Table2: The p.d.f. of N_A, $A = [0, 1)$

r	x	$\lambda = .5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 5$
2	0	.9098	.7358	.4060	.1991	.0916	.0404
	1	.0884	.1442	.4511	.4481	.3419	.2246
	2	.0018	.0184	.0263	.2689	.3516	.3510
	3	0	.0006	.0155	.0720	.1638	.2506
	4	0	0	.0011	.0118	.0430	.1016
	5	0	0	0	.0001	.0008	.0263
	6	0	0	0	0	.0001	.0048
	7	0	0	0	0	.0001	.0006
	8	0	0	0	0	0	.0001
3	0	.9856	.9197	.6767	.4282	.2381	.1247
	1	.0144	.0797	.3067	.5829	.5470	.4913
	2	0	.0006	.0064	.0801	.1935	.3159
	3	0	0	.0002	.0037	.0205	.0626
	4	0	0	0	.0001	.0001	.0053
	5	0	0	0	0	0	.0002
5	0	.9998	.9563	.9473	.8153	.6288	.4405
	1	.0002	.0037	.0527	.1836	.3621	.5297
	2	0	0	0	.0011	.0081	.0316
	3	0	0	0	0	0	.0002
	4	0	0	0	0	0	0
10	0	1.000	1.000	1.000	.9989	.9919	.9682
	1	0	0	0	.0011	.0081	.0318
	2	0	0	0	0	0	0
	3	0	0	0	0	0	0

Table3: The p.d.f. of N_A, $A = [1, 2)$

r	x	$\lambda = .5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 5$
2	0	.8139	.5767	.2731	.1243	.05050	.0236
	1	.1803	.3798	.4954	.4108	.2809	.1713
	2	.0057	.0436	.1972	.3303	.3712	.3334
	3	0	.0019	.0315	.1115	.2121	.2862
	4	0	0	.0028	.0202	.06161	.1356
	5	0	0	0	.0023	.0129	.0404
	6	0	0	0	0	.0017	.0082
	7	0	0	0	0	0	0.0012
	8	0	0	0	0	0	0
3	0	.9337	.7460	.4115	.2226	.1157	.0572
	1	.0661	.2488	.5199	.5696	.4996	.3832
	2	0	.0052	.0667	.1913	.3235	.4131
	3	0	0	.0019	.0160	.0572	.1295
	4	0	0	0	0	.0038	.0160
	5	0	0	0	0	0	0
5	0	.9965	.9510	.6761	.4116	.2646	.1719
	1	.0035	.0490	.3212	.5624	.6498	.6525
	2	0	0	.0027	.0259	.0848	.1709
	3	0	0	0	0	0	.0046
	4	0	0	0	0	0	0
10	0	1.000	1.000	.9919	.9172	.7246	.4875
	1	0	0	.0081	.0828	.2753	.5113
	2	0	0	0	0	0	.0012
	3	0	0	0	0	0	0

Table 4: The Expectation of N_A

r	A	$\lambda = .5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 5$
2	$[0, 1)$	.0920	.2838	.7546	12506	1.7501	2.2500
	$[1, 2)$	.1919	.4707	.9955	1.4994	1.9999	2.5000
3	$[0, 1)$	.0144	.0809	.3401	.6646	.9991	.1.3332
	$[1, 2)$	.0665	.2592	.6589	1.0021	1.3343	1.6668
5	$[0, 1)$	.0002	.0037	.0527	.1858	.3793	.5916
	$[1, 2)$	.0038	.0490	.3266	.6144	.8219	1.0082
10	$[0, 1)$	0	0	0	.0011	.0081	.0318
	$[1, 2)$	0	0	.0081	.0828	.2753	.5113

6. Discussion

This paper develops the distribution function of a renewal process whose interarrival times are independent and identically distributed random variables with Erlang distribution. The distribution of the number of renewals in $[a,b)$, $a > 0$ is much more challenging and interesting than when $a = 0$. This distribution is obtained using the joint distribution of $N_{[0,a)}$ and $N_{[a,b)}$.

An interesting result is obtained where the first parameter of Erlang distribution as interarrival times in the renewal process is random variable. In particular, if it has geometric distribution then the renewal converts to Poisson process.

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