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Use of Cubic B-Spline in Approximating Solutions of Boundary Value Problems

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Abstract

Here we investigate the use of cubic B-spline functions in solving boundary value problems. First, we derive the linear, quadratic, and cubic B-spline functions. Then we use the cubic B-spline functions to solve second order linear boundary value problems. We consider constant coefficient and variable coefficient cases with non-homogeneous boundary conditions for ordinary differential equations. We also use this numerical method for the space variable to obtain solutions for second order linear partial differential equations. Numerical results for various cases are presented and compared with exact solutions.

Keywords: Cubic spline, B-spline, Runge-Kutta method, differential equations, boundary value

MSC 2010 No.: 34K28, 65D07, 65D25, 65L06, 65L10

1. Introduction

The use of B-splines has become very popular among many areas of mathematics, engineering, and computer science in recent years. Originally B-splines were used for approximation purposes, but its popularity has extended their applications. The most popular B-Spline is the cubic B-spline. The first mathematician who introduced the concept of splines was Isaac Jacob Schoenberg in 1946. Schoenberg (1946, 1982) is known for his discovery of splines. His work was a motivation to other mathematicians such Carl de Boor who worked directly with Schoenberg. In the early 1970s de Boor (1962, 1972, 1978) introduced a recursive definition for splines. Birkhoff and de Boor (1964) studied the error bound and convergence of spline interpolation. Now splines, especially B-splines, play an important role in the areas of mathematics and engineering. Splines are popular in computer graphing due to their smoothness, flexibility, and accuracy.

Approximated solutions of differential equations have been obtained using different types of methods. Fang, Tsuchiya, and Yamamoto (2002) presented solutions to second order boundary value problems with homogeneous boundary conditions using three methods, the finite difference, the finite element, and finite volume methods, with the help of an inversion formula of a non-singular tridiagonal matrix. Farago and Horvath (1999) obtained numerical solutions of the heat equation using the finite difference method. Bhatti and Bracken (2006) presented approximate solutions to linear and nonlinear ordinary differential equations using Bernstein polynomials. Bhatta and Bhatti (2006) obtained numerical solution of KdV equation using modified Bernstein polynomials via Galerkin method. This last method and the cubic B-spline method share similar properties. However, the advantage of cubic B-splines is that the polynomials are always of degree three, while in the case of Bernstein polynomials, the degree is quite high which depends on the number of subintervals. Munguia et. al. (2014) discussed usage of cubic B-spline functions in interpolation.

In this case, we seek to approximate solutions to second order linear boundary value problems using cubic B-splines. Derivations of the cubic B-spline functions are presented in Section 2. Section 3 deals with the procedure to obtain numerical solution and results for a non-homogeneous linear second order boundary value problem with non-homogeneous boundary conditions. Numerical solution procedures and results for linear second order partial differential equations using cubic B-spline functions are discussed in Section 4.

2. Cubic B-Spline Formulation

Let us consider a partition $\Delta_N : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ on a given interval $[a, b]$ and let $h = \frac{b-a}{N}$ be the mesh size of the partition. Given Δ_N, a piecewise polynomial function s on the interval $[a, b]$ is called a spline of degree k if $s \in C^{k-1}[a, b]$ and s is a polynomial of degree at most k on each subinterval $[x_i, x_{i+1}]$. Let $S_k(\Delta_N)$ denote the set of all polynomials of degree k associated with Δ_N. This set is a linear space with respect to Δ_N of dimension $N + k$.

Now that we have defined spline functions, we introduce a special kind of spline functions called B-splines of degree 3. B-splines are defined by a recursive relation introduced by Carl de Boor (1972, 1978) in the early 1970s. The B-splines of degree zero are defined by

$$B_i^0(x) = \begin{cases} 1 & \text{if } x_i \le x < x_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

and those of degree $k \in \mathbb{Z}^+$ are defined recursively in terms of B-splines of degree $k-1$ by

$$B_i^k(x) = \left(\frac{x - x_i}{x_{i+k} - x_i}\right) B_i^{k-1}(x) + \left(\frac{x_{i+k+1} - x}{x_{i+k+1} - x_{i+1}}\right) B_{i+1}^{k-1}(x), \tag{2}$$

for $i = 0, \pm 1, \pm 2, \pm 3, \ldots$ (Phillips, 2003). The basis functions B_i^k as defined by (2) are called B-splines of degree k.

Using the recurrence relation (2) and assuming the partition Δ_N, the non-uniform B-splines up to degree 3 are given by:

(a) Linear B-spline:

$$B_i^1(x) = \begin{cases} \frac{x - x_i}{x_{i+1} - x_i} & \text{if } x_i \le x < x_{i+1}, \\ \frac{x_{i+2} - x}{x_{i+2} - x_{i+1}} & \text{if } x_{i+1} \le x < x_{i+2}, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

(b) Quadratic B-spline:

$$B_i^2(x) = \begin{cases} \frac{(x - x_i)^2}{(x_{i+2} - x_i)(x_{i+1} - x_i)} & \text{if } x_i \le x < x_{i+1}, \\ \frac{(x - x_i)(x_{i+2} - x)}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})} + \frac{(x_{i+3} - x)(x - x_{i+1})}{(x_{i+3} - x_{i+1})(x_{i+2} - x_{i+1})} & \text{if } x_{i+1} \le x < x_{i+2}, \\ \frac{(x_{i+3} - x)^2}{(x_{i+3} - x_{i+1})(x_{i+3} - x_{i+2})} & \text{if } x_{i+2} \le x < x_{i+3}, \\ 0 & \text{otherwise}. \end{cases} \tag{4}$$

(c) Cubic B-spline:

$$B_i^3(x) = \begin{cases} \frac{(x-x_i)^3}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+1}-x_i)} & \text{if } x_i \le x < x_{i+1}, \\ \frac{(x-x_i)^2(x_{i+2}-x)}{(x_{i+3}-x_i)(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} + \frac{(x-x_i)(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} \\ \quad + \frac{(x_{i+4}-x)(x-x_{i+1})^2}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \text{if } x_{i+1} \le x < x_{i+2}, \\ \frac{(x-x_i)(x_{i+3}-x)^2}{(x_{i+3}-x_i)(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} + \frac{(x_{i+4}-x)(x-x_{i+1})(x_{i+3}-x)}{(x_{i+4}-x_{i+1})(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} \\ \quad + \frac{(x_{i+4}-x)^2(x-x_{i+2})}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+3}-x_{i+2})} & \text{if } x_{i+2} \le x < x_{i+3}, \\ \frac{(x_{i+4}-x)^3}{(x_{i+4}-x_{i+1})(x_{i+4}-x_{i+2})(x_{i+4}-x_{i+3})} & \text{if } x_{i+3} \le x < x_{i+4}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The last equation is a cubic spline with knots $x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}$. Note that the cubic B-spline is zero except on the interval $[x_i, x_{i+4})$. This is true for all B-splines. In fact, $B_i^k(x) = 0$ if $x \notin [x_i, x_{i+k+1})$, otherwise $B_i^k(x) > 0$ if $x \in (x_i, x_{i+k+1})$.

Since we are only referring to B-splines of degree 3, we write B_i instead of B_i^3. In our case, we restrict our attention to equally-spaced knots. Therefore, after including four additional knots, we assume that $\Delta : x_{-2} < x_{-1} < x_0 < x_1 < \cdots < x_{N-1} < x_N < x_{N+1} < x_{N+2}$ is a uniform grid partition. Using (5) and letting $h = x_{i+1} - x_i$ for any $0 \le i \le N$, we define the uniform cubic B-spline $B_i(x)$ as

$$B_i(x) = \frac{1}{6h^3} \begin{cases} (x - x_{i-2})^3 & \text{if } x_{i-2} \le x < x_{i-1}, \\ -3(x - x_{i-1})^3 + 3h(x - x_{i-1})^2 + 3h^2(x - x_{i-1}) + h^3 & \text{if } x_{i-1} \le x < x_i, \\ -3(x_{i+1} - x)^3 + 3h(x_{i+1} - x)^2 + 3h^2(x_{i+1} - x) + h^3 & \text{if } x_i \le x < x_{i+1}, \\ (x_{i+2} - x)^3 & \text{if } x_{i+1} \le x < x_{i+2}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

If we choose $h = 1$, then in the interval $[-2, 2]$, we have the following

$$B_0(x) = \frac{1}{6} \begin{cases} (x+2)^3 & \text{if } -2 \le x < -1, \\ 4 - 6x^2 - 3x^3 & \text{if } -1 \le x < 0, \\ 4 - 6x^2 + 3x^3 & \text{if } 0 \le x < 1, \\ (2-x)^3 & \text{if } 1 \le x < 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and its graph is shown in Figure 1. We know that B_i lies in the interval $[x_i, x_{i+1})$. This interval has nonzero contributions from B_{i-1}, B_i, B_{i+1} and B_{i+2}. We have a better understanding of this from Figure 2.

Next we derive the cubic B-spline method for approximating solutions to second-order linear

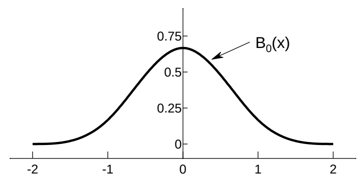

Figure 1: Graph of the cubic B-spline $B_0(x)$ in the interval $[-2, 2]$

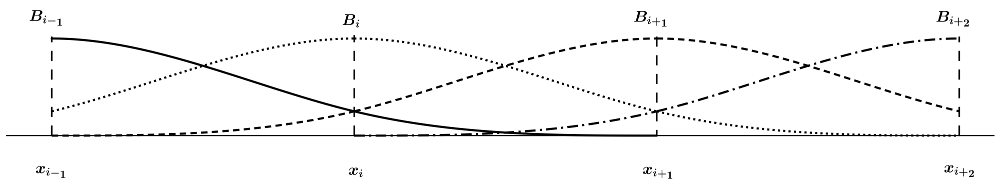

Figure 2: Graphs of the cubic B-splines needed for the interval $[x_i, x_{i+1})$

boundary value problems for ordinary differential equations and present some numerical results.

3. Ordinary Differential Equations

Cubic B-Spline Procedure

In this section, we study the use of cubic B-splines to solve second-order linear boundary value problems (BVP) of the form

$$a_1(x)y'' + a_2(x)y' + a_3(x)y = f(x), \tag{8}$$

with boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{9}$$

where $a_1(x) \neq 0$, $a_2(x)$, $a_3(x)$ and $f(x)$ are continuous real-valued functions on the interval $[a, b]$.

To approximate the solution of this BVP using cubic B-splines, we let $Y(x)$ be a cubic spline with knots Δ. Then $Y(x)$ can be written as linear combinations of $B_i(x)$

$$Y(x) = \sum_{i=-1}^{N+1} c_i B_i(x), \tag{10}$$

where the constants c_i are to be determined and the $B_i(x)$ are defined in (6). It is required that (10) satisfies our BVP (8-9) at $x = x_i$ where x_i is an interior point. That is

$$a_1(x_i)Y''(x_i) + a_2(x_i)Y'(x_i) + a_3(x_i)Y(x_i) = f(x_i), \tag{11}$$

and the boundary conditions are

$$
\begin{aligned}
Y(x_0) &= \alpha \ \text{ for } x_0 = a,\\
Y(x_N) &= \beta \ \text{ for } x_N = b.
\end{aligned}
$$

From (10), we have

$$
\begin{aligned}
Y(x_i) &= c_{i-1}B_{i-1}(x_i) + c_iB_i(x_i) + c_{i+1}B_{i+1}(x_i) + c_{i+2}B_{i+2}(x_i),\\
Y'(x_i) &= c_{i-1}B'_{i-1}(x_i) + c_iB'_i(x_i) + c_{i+1}B'_{i+1}(x_i) + c_{i+2}B'_{i+2}(x_i),\\
Y''(x_i) &= c_{i-1}B''_{i-1}(x_i) + c_iB''_i(x_i) + c_{i+1}B''_{i+1}(x_i) + c_{i+2}B''_{i+2}(x_i),
\end{aligned}
\tag{12}
$$

and these yield

$$
\begin{aligned}
&c_{i-1}[a_1(x_i)B''_{i-1}(x_i) + a_2(x_i)B'_{i-1}(x_i) + a_3(x_i)B_{i-1}(x_i)]\\
&\quad + c_i[a_1(x_i)B''_i(x_i) + a_2(x_i)B'_i(x_i) + a_3(x_i)B_i(x_i)]\\
&\quad + c_{i+1}[a_1(x_i)B''_{i+1}(x_i) + a_2(x_i)B'_{i+1}(x_i) + a_3(x_i)B_{i+1}(x_i)]\\
&\quad + c_{i+2}[a_1(x_i)B''_{i+2}(x_i) + a_2(x_i)B'_{i+2}(x_i) + a_3(x_i)B_{i+2}(x_i)] = f(x_i),
\end{aligned}
\tag{13}
$$

also by the properties of cubic B-spline functions, we obtain the following

$$
\begin{aligned}
&B''_{i-1}(x_i) = \frac{1}{h^2}, && B'_{i-1}(x_i) = -\frac{1}{2h}, && B_{i-1}(x_i) = \frac{1}{6},\\
&B''_i(x_i) = -\frac{2}{h^2}, && B'_i(x_i) = 0, && B_i(x_i) = \frac{2}{3},\\
&B''_{i+1}(x_i) = \frac{1}{h^2}, && B'_{i+1}(x_i) = \frac{1}{2h}, && B_{i+1}(x_i) = \frac{1}{6},\\
&B''_{i+2}(x_i) = 0, && B'_{i+2}(x_i) = 0, && B_{i+2}(x_i) = 0.
\end{aligned}
\tag{14}
$$

If we combine (13) and (14), we obtain

$$
\begin{aligned}
&c_{i-1}[6a_1(x_i) - 3a_2(x_i)h + a_3(x_i)h^2] + c_i[-12a_1(x_i) + 4a_3(x_i)h^2]\\
&+c_{i+1}[6a_1(x_i) + 3a_2(x_i)h + a_3(x_i)h^2] = 6h^2f(x_i).
\end{aligned}
\tag{15}
$$

Now we apply the boundary conditions:

$$
\begin{aligned}
Y(x_0) &= c_{-1}B_{-1}(x_0) + c_0B_0(x_0) + c_1B_1(x_0) + c_2B_2(x_0) = \alpha,\\
Y(x_N) &= c_{N-1}B_{N-1}(x_N) + c_NB_N(x_N) + c_{N+1}B_{N+1}(x_N) + c_{N+2}B_{N+2}(x_N) = \beta,
\end{aligned}
\tag{16}
$$

where the value of $B_i(x)$ at $x = x_0$ and $x = x_N$ are given below

$$
\begin{aligned}
B_{-1}(x_0) &= \frac{1}{6} = B_{N-1}(x_N),\\
B_0(x_0) &= \frac{4}{6} = B_N(x_N),\\
B_1(x_0) &= \frac{1}{6} = B_{N+1}(x_N),\\
B_2(x_0) &= 0 = B_{N+2}(x_N).
\end{aligned} \tag{17}
$$

Therefore,

$$c_{-1} + 4c_0 + c_1 = 6\alpha, \tag{18}$$

$$c_{N-1} + 4c_N + c_{N+1} = 6\beta. \tag{19}$$

Now that we have found all the constant coefficients in (15), (18), and (19), we can write a system of $N+1$ linear equations in $N+1$ unknowns. This system is represented in (20) where the coefficient matrix is an $(N+1) \times (N+1)$ matrix.

$$
\begin{pmatrix}
o_1 & o_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
p_1 & q_1 & r_1 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & p_2 & q_2 & r_2 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & p_{N-2} & q_{N-2} & r_{N-2} & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & p_{N-1} & q_{N-1} & r_{N-1}\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & o_3 & o_4
\end{pmatrix}
\begin{pmatrix}
c_0\\ c_1\\ c_2\\ \vdots\\ c_{N-2}\\ c_{N-1}\\ c_N
\end{pmatrix}
= 6
\begin{pmatrix}
z_0\\ h^2 f(x_1)\\ h^2 f(x_2)\\ \vdots\\ h^2 f(x_{N-2})\\ h^2 f(x_{N-1})\\ z_N
\end{pmatrix}, \tag{20}
$$

where p_i, q_i, and r_i are defined below

$$
\begin{aligned}
p_i &= 6a_1(x_i) - 3a_2(x_i)h + a_3(x_i)h^2,\\
q_i &= -12a_1(x_i) + 4a_3(x_i)h^2,\\
r_i &= 6a_1(x_i) + 3a_2(x_i)h + a_3(x_i)h^2,\\
o_1 &= q_0 - 4p_0,\\
o_2 &= r_0 - p_0,\\
o_3 &= p_N - r_N,\\
o_4 &= q_N - 4r_N,\\
z_0 &= h^2 f(x_0) - \alpha p_0,\\
z_N &= h^2 f(x_N) - \beta r_N.
\end{aligned} \tag{21}
$$

The cubic B-spline approximation for the BVP (8-9) is obtained using (10), where the constant coefficients c_i satisfy the system defined in (20).

A. *Results and Discussions*

Here we demonstrate some numerical results with some specific examples of the boundary value problem presented in the previous section.

Example 1.

We consider a linear boundary value problem with constant coefficients

$$y'' + y' - 6y = x \quad \text{for} \quad 0 < x < 1, \tag{22}$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 1. \tag{23}$$

The exact solution to boundary value problem is

$$y(x) = \frac{(43 - e^2)e^{-3x} - (43 - e^{-3})e^{2x}}{36(e^{-3} - e^2)} - \frac{1}{6}x - \frac{1}{36}. \tag{24}$$

We approximate the solution of (22) with the boundary conditions (23) using the cubic B-spline method with $N = 20$.

In order to use (10), we first need to find the constant coefficients c_i for $i = -1, 0, 1, \ldots, 21$ using the system of linear equations (20) where the coefficient matrix is an 21×21 matrix and using (18) and (19) to find c_{-1} and c_{21} respectively. These coefficients are given below

$c_{-1} = -0.028608719$	$c_0 = 0.000234498$	$c_1 = 0.027670728$	$c_2 = 0.054293511$	$c_3 = 0.080655427$
$c_4 = 0.107278209$	$c_5 = 0.134661904$	$c_6 = 0.163293271$	$c_7 = 0.193653567$	$c_8 = 0.226225882$
$c_9 = 0.261502130$	$c_{10} = 0.299989841$	$c_{11} = 0.342218834$	$c_{12} = 0.388747909$	$c_{13} = 0.440171619$
$c_{14} = 0.497127249$	$c_{15} = 0.560302075$	$c_{16} = 0.630440999$	$c_{17} = 0.708354660$	$c_{18} = 0.794928112$
$c_{19} = 0.891130170$	$c_{20} = 0.998023523$	$c_{21} = 1.116775737$		

and therefore the cubic polynomials are as follows

$$Y(x) = \begin{cases} 0.79138x^3 - 0.28140x^2 + 0.56279x & \text{for } x \in [0.00, 0.05) \\ 0.73677x^3 - 0.27321x^2 + 0.56238x + 0.00001 & \text{for } x \in [0.05, 0.10) \\ 0.69564x^3 - 0.26087x^2 + 0.56115x + 0.00005 & \text{for } x \in [0.10, 0.15) \\ 0.66673x^3 - 0.24786x^2 + 0.55920x + 0.00015 & \text{for } x \in [0.15, 0.20) \\ 0.64901x^3 - 0.23722x^2 + 0.55707x + 0.00029 & \text{for } x \in [0.20, 0.25) \\ 0.64168x^3 - 0.23172x^2 + 0.55570x + 0.00040 & \text{for } x \in [0.25, 0.30) \\ 0.64412x^3 - 0.23392x^2 + 0.55636x + 0.00034 & \text{for } x \in [0.30, 0.35) \\ 0.65589x^3 - 0.24628x^2 + 0.56068x - 0.00017 & \text{for } x \in [0.35, 0.40) \\ 0.67670x^3 - 0.27126x^2 + 0.57067x - 0.00150 & \text{for } x \in [0.40, 0.45) \\ 0.70643x^3 - 0.31139x^2 + 0.58873x - 0.00421 & \text{for } x \in [0.45, 0.50) \\ 0.74506x^3 - 0.36934x^2 + 0.61771x - 0.00904 & \text{for } x \in [0.50, 0.55) \\ 0.79274x^3 - 0.44800x^2 + 0.66097x - 0.01697 & \text{for } x \in [0.55, 0.60) \\ 0.84971x^3 - 0.55056x^2 + 0.72251x - 0.02928 & \text{for } x \in [0.60, 0.65) \\ 0.91637x^3 - 0.68053x^2 + 0.80699x - 0.04758 & \text{for } x \in [0.65, 0.70) \\ 0.99320x^3 - 0.84189x^2 + 0.91994x - 0.07394 & \text{for } x \in [0.70, 0.75) \\ 1.08085x^3 - 1.03910x^2 + 1.06785x - 0.11091 & \text{for } x \in [0.75, 0.80) \\ 1.18007x^3 - 1.27723x^2 + 1.25835x - 0.16171 & \text{for } x \in [0.80, 0.85) \\ 1.29175x^3 - 1.56201x^2 + 1.50041x - 0.23030 & \text{for } x \in [0.85, 0.90) \\ 1.41692x^3 - 1.89996x^2 + 1.80457x - 0.32155 & \text{for } x \in [0.90, 0.95) \\ 1.55675x^3 - 2.29849x^2 + 2.18317x - 0.44144 & \text{for } x \in [0.95, 1.00). \end{cases} \tag{25}$$

Table I: Cubic B-spline results and absolute errors for Example 1.

x_i	Cubic B-spline	Exact	Absolute Error
0.00	0.0000000000	0.0000000000	0.0000000000
0.05	0.0275351538	0.0275370031	0.0000018493
0.10	0.0542500333	0.0542570003	0.0000069670
0.15	0.0806989046	0.0807133503	0.0000144457
0.20	0.1074050277	0.1074285617	0.0000235340
0.25	0.1348698493	0.1349034523	0.0000336029
0.30	0.1635814257	0.1636255435	0.0000441179
0.35	0.1940222368	0.1940768511	0.0000546144
0.40	0.2266765374	0.2267412146	0.0000646773
0.45	0.2620373739	0.2621112965	0.0000739226
0.50	0.3006133878	0.3006953693	0.0000819815
0.55	0.3429355142	0.3430239998	0.0000884855
0.60	0.3895636812	0.3896567348	0.0000930536
0.65	0.4410936054	0.4411888843	0.0000952789
0.70	0.4981637819	0.4982584988	0.0000947169
0.75	0.5614627582	0.5615536314	0.0000908731
0.80	0.6317367884	0.6318199790	0.0000831905
0.85	0.7097979583	0.7098689951	0.0000710368
0.90	0.7965328796	0.7965865702	0.0000536906
0.95	0.8929120525	0.8929423791	0.0000303266
1.00	1.0000000000	1.0000000000	0.0000000000

Now we obtain the results shown in Table I.

Example 2.

We consider a linear boundary value problem with constant coefficients

$$y'' + 2y' + 5y = 6\cos(2x) - 7\sin(2x) \quad \text{for} \quad 0 < x < \frac{\pi}{4}, \tag{26}$$

with boundary conditions

$$y(0) = 4, \quad y(\frac{\pi}{4}) = 1. \tag{27}$$

The exact solution to boundary value problem is

$$y(x) = 2(1 + e^{-x})\cos(2x) + \sin(2x). \tag{28}$$

We approximate the solution of this BVP using $N = 20$. We first need to find the constant coefficients c_i for $i = -1, 0, 1, \ldots, 21$ using (18), (19) and the system of linear equations (20) where the coefficient matrix is an 21×21 matrix. These coefficients are given below

$c_{-1} = 3.992827220$	$c_0 = 4.003597990$	$c_1 = 3.992780821$	$c_2 = 3.961243944$	$c_3 = 3.909893955$
$c_4 = 3.839671695$	$c_5 = 3.751548462$	$c_6 = 3.646522513$	$c_7 = 3.525615838$	$c_8 = 3.389871153$
$c_9 = 3.240349099$	$c_{10} = 3.078125595$	$c_{11} = 2.904289322$	$c_{12} = 2.719939298$	$c_{13} = 2.526182533$
$c_{14} = 2.324131701$	$c_{15} = 2.114902848$	$c_{16} = 1.899613069$	$c_{17} = 1.679378165$	$c_{18} = 1.455310254$
$c_{19} = 1.228515306$	$c_{20} = 1.000090621$	$c_{21} = 0.771122212$		

and therefore the cubic B-spline polynomials are as follows

$$Y(x)=\begin{cases} 2.38948x^3-6.99941x^2-0.00059x+4.00000 & \text{for } x\in[0,\frac{\pi}{80}) \\ 2.49507x^3-7.01185x^2-0.00010x+3.99999 & \text{for } x\in[\frac{\pi}{80},\frac{\pi}{40}) \\ 2.58932x^3-7.03406x^2+0.00164x+3.99995 & \text{for } x\in[\frac{\pi}{40},\frac{3\pi}{80}) \\ 2.67314x^3-7.06368x^2+0.00513x+3.99981 & \text{for } x\in[\frac{3\pi}{80},\frac{\pi}{20}) \\ 2.74734x^3-7.09864x^2+0.01062x+3.99952 & \text{for } x\in[\frac{\pi}{20},\frac{5\pi}{80}) \\ 2.81265x^3-7.13712x^2+0.01818x+3.99903 & \text{for } x\in[\frac{5\pi}{80},\frac{3\pi}{40}) \\ 2.86970x^3-7.17744x^2+0.02768x+3.99828 & \text{for } x\in[\frac{3\pi}{40},\frac{7\pi}{80}) \\ 2.91902x^3-7.21812x^2+0.03886x+3.99726 & \text{for } x\in[\frac{7\pi}{80},\frac{\pi}{10}) \\ 2.96107x^3-7.25775x^2+0.05131x+3.99595 & \text{for } x\in[\frac{\pi}{10},\frac{9\pi}{80}) \\ 2.99619x^3-7.29499x^2+0.06447x+3.99440 & \text{for } x\in[\frac{9\pi}{80},\frac{\pi}{8}) \\ 3.02465x^3-7.32851x^2+0.07764x+3.99268 & \text{for } x\in[\frac{\pi}{8},\frac{11\pi}{80}) \\ 3.04663x^3-7.35700x^2+0.08994x+3.99091 & \text{for } x\in[\frac{11\pi}{80},\frac{3\pi}{20}) \\ 3.06223x^3-7.37906x^2+0.10034x+3.98928 & \text{for } x\in[\frac{3\pi}{20},\frac{13\pi}{80}) \\ 3.07150x^3-7.39325x^2+0.10758x+3.98804 & \text{for } x\in[\frac{13\pi}{80},\frac{7\pi}{40}) \\ 3.07439x^3-7.39802x^2+0.11021x+3.98756 & \text{for } x\in[\frac{7\pi}{40},\frac{15\pi}{80}) \\ 3.07083x^3-7.39173x^2+0.10650x+3.98829 & \text{for } x\in[\frac{15\pi}{80},\frac{\pi}{5}) \\ 3.06069x^3-7.37261x^2+0.09448x+3.99081 & \text{for } x\in[\frac{\pi}{5},\frac{17\pi}{80}) \\ 3.04378x^3-7.33874x^2+0.07188x+3.99584 & \text{for } x\in[\frac{17\pi}{80},\frac{9\pi}{40}) \\ 3.01991x^3-7.28813x^2+0.03610x+4.00427 & \text{for } x\in[\frac{9\pi}{40},\frac{19\pi}{80}) \\ 2.98885x^3-7.21861x^2-0.01577x+4.01717 & \text{for } x\in[\frac{19\pi}{80},\frac{\pi}{4}). \end{cases} \tag{29}$$

After finding the set of cubic polynomials, we obtain the results shown in Table II.

Example 3.

We consider the following linear boundary value problem with variable coefficients

$$x^2y''+3xy'+3y=0 \text{ for } 1<x<2, \tag{30}$$

with boundary conditions

$$y(1)=5, \quad y(2)=0. \tag{31}$$

The exact solution to this boundary value problem is

$$y(x)=\frac{5}{x}[\cos(\sqrt{2}\ln x)-\cot(\sqrt{2}\ln 2)\sin(\sqrt{2}\ln x)]. \tag{32}$$

Table II: Cubic B-spline results and absolute errors for Example 2.

x_i	Cubic B-spline	Exact	Absolute Error
0	4.0000000000	4.0000000000	0.0000000000
$\frac{\pi}{80}$	3.9893275364	3.9893481701	0.0000206336
$\frac{\pi}{40}$	3.9579417589	3.9579782444	0.0000364855
$\frac{3\pi}{80}$	3.9067485765	3.9067967056	0.0000481291
$\frac{\pi}{20}$	3.8366881995	3.8367442991	0.0000560995
$\frac{5\pi}{80}$	3.7487313426	3.7487922376	0.0000608949
$\frac{3\pi}{40}$	3.6438757255	3.6439387036	0.0000629781
$\frac{7\pi}{80}$	3.5231428361	3.5232056151	0.0000627790
$\frac{\pi}{10}$	3.3875749246	3.3876356206	0.0000606960
$\frac{9\pi}{80}$	3.2382321907	3.2382892895	0.0000570988
$\frac{\pi}{8}$	3.0761901337	3.0762424637	0.0000523300
$\frac{11\pi}{80}$	2.9025370300	2.9025837374	0.0000467074
$\frac{3\pi}{20}$	2.7183715079	2.7184120337	0.0000405258
$\frac{13\pi}{80}$	2.5248001883	2.5248342470	0.0000340587
$\frac{7\pi}{40}$	2.3229353642	2.3229629245	0.0000275603
$\frac{15\pi}{80}$	2.1138926936	2.1139139602	0.0000212666
$\frac{\pi}{5}$	1.8987888813	1.8988042779	0.0000153966
$\frac{17\pi}{80}$	1.6787393307	1.6787494845	0.0000101538
$\frac{9\pi}{40}$	1.4548557478	1.4548614743	0.0000057265
$\frac{19\pi}{80}$	1.2282436831	1.2282459716	0.0000022885
$\frac{\pi}{4}$	1.0000000000	1.0000000000	0.0000000000

We want to approximate the solution of this BVP using $N = 20$. In this case, the constant coefficients are computed as

$$
\begin{array}{lllll}
c_{-1} = 5.498922492 & c_0 = 4.994073274 & c_1 = 4.524784412 & c_2 = 4.089302834 & c_3 = 3.685654832 \\
c_4 = 3.311784751 & c_5 = 2.965642493 & c_6 = 2.645237403 & c_7 = 2.348670027 & c_8 = 2.074149336 \\
c_9 = 1.820000442 & c_{10} = 1.584666183 & c_{11} = 1.366704832 & c_{12} = 1.164785437 & c_{13} = 0.977681805 \\
c_{14} = 0.804265806 & c_{15} = 0.643500436 & c_{16} = 0.494432924 & c_{17} = 0.356188065 & c_{18} = 0.227961883 \\
c_{19} = 0.109015687 & c_{20} = -0.001329460 & c_{21} = -0.103697849 & &
\end{array}
$$

The following cubic polynomials are derived:

$$
Y(x) = \begin{cases}
-2.33743x^3 + 14.12436x^2 - 30.97781x + 24.19088 & \text{for } x \in [1.00, 1.05) \\
-2.63161x^3 + 15.05103x^2 - 31.95081x + 24.53143 & \text{for } x \in [1.05, 1.10) \\
-2.74087x^3 + 15.41159x^2 - 32.34744x + 24.67686 & \text{for } x \in [1.10, 1.15) \\
-2.73346x^3 + 15.38604x^2 - 32.31805x + 24.66559 & \text{for } x \in [1.15, 1.20) \\
-2.65421x^3 + 15.10071x^2 - 31.97566x + 24.52864 & \text{for } x \in [1.20, 1.25) \\
-2.53260x^3 + 14.64470x^2 - 31.40564x + 24.29113 & \text{for } x \in [1.25, 1.30) \\
-2.38804x^3 + 14.08090x^2 - 30.67270x + 23.97352 & \text{for } x \in [1.30, 1.35) \\
-2.23318x^3 + 13.45373x^2 - 29.82604x + 23.59252 & \text{for } x \in [1.35, 1.40) \\
-2.07621x^3 + 12.79446x^2 - 28.90305x + 23.16180 & \text{for } x \in [1.40, 1.45) \\
-1.92230x^3 + 12.12495x^2 - 27.93225x + 22.69258 & \text{for } x \in [1.45, 1.50) \\
-1.77460x^3 + 11.46030x^2 - 26.93528x + 22.19409 & \text{for } x \in [1.50, 1.55) \\
-1.63492x^3 + 10.81079x^2 - 25.92853x + 21.67394 & \text{for } x \in [1.55, 1.60) \\
-1.50417x^3 + 10.18318x^2 - 24.92437x + 21.13839 & \text{for } x \in [1.60, 1.65) \\
-1.38267x^3 + 9.58175x^2 - 23.93200x + 20.59259 & \text{for } x \in [1.65, 1.70) \\
-1.27036x^3 + 9.00897x^2 - 22.95827x + 20.04081 & \text{for } x \in [1.70, 1.75) \\
-1.16694x^3 + 8.46601x^2 - 22.00809x + 19.48653 & \text{for } x \in [1.75, 1.80) \\
-1.07197x^3 + 7.95315x^2 - 21.08496x + 18.93265 & \text{for } x \in [1.80, 1.85) \\
-0.98492x^3 + 7.47005x^2 - 20.19121x + 18.38151 & \text{for } x \in [1.85, 1.90) \\
-0.90525x^3 + 7.01592x^2 - 19.32836x + 17.83504 & \text{for } x \in [1.90, 1.95) \\
-0.83239x^3 + 6.58969x^2 - 18.49721x + 17.29479 & \text{for } x \in [1.95, 2.00).
\end{cases} \tag{33}
$$

Table III compares the cubic B-spline results with the exact solution.

Table III: Cubic B-spline results and absolute errors for Example 3.

x_i	Cubic B-spline	Exact	Absolute Error
1.00	5.0000000000	5.0000000000	0.0000000000
1.05	4.5304189596	4.5304962322	0.0000772726
1.10	4.0946084301	4.0947693502	0.0001609202
1.15	3.6906178188	3.6908571967	0.0002393779
1.20	3.3164060550	3.3167126115	0.0003065565
1.25	2.9699320212	2.9702917258	0.0003597046
1.30	2.6492103552	2.6496084276	0.0003980724
1.35	2.3523444743	2.3527665477	0.0004220734
1.40	2.0775446353	2.0779773959	0.0004327606
1.45	1.8231362142	1.8235677149	0.0004315007
1.50	1.5875616676	1.5879814418	0.0004197742
1.55	1.3693784917	1.3697775482	0.0003990565
1.60	1.1672547313	1.1676254805	0.0003707492
1.65	0.9799630775	0.9802992219	0.0003361444
1.70	0.8063742444	0.8066706529	0.0002964084
1.75	0.6454500792	0.6457026575	0.0002525783
1.80	0.4962366996	0.4964422651	0.0002055654
1.85	0.3578578444	0.3580140078	0.0001561633
1.90	0.2295085473	0.2296136048	0.0001050575
1.95	0.1104491952	0.1105020313	0.0000528360
2.00	0.0000000000	0.0000000000	0.0000000000

Example 4.

Now we consider the following linear boundary value problem with variable coefficients

$$xy'' + y' = x \quad \text{for} \quad 1 < x < 2, \tag{34}$$

with boundary conditions

$$y(1) = 1, \quad y(2) = 1. \tag{35}$$

The exact solution to this boundary value problem is

$$y(x) = \frac{x^2}{4} - \frac{3\ln x}{4\ln 2} + \frac{3}{4}. \tag{36}$$

We want to approximate the solution of this BVP using $N = 20$. In this case, the constant coefficients are computed as

$$
\begin{array}{lllll}
c_{-1} = 1.030435187 & c_0 = 0.999340695 & c_1 = 0.972202032 & c_2 = 0.948767493 & c_3 = 0.928818933 \\
c_4 = 0.912166057 & c_5 = 0.898641870 & c_6 = 0.888099023 & c_7 = 0.880406850 & c_8 = 0.875448938 \\
c_9 = 0.873121127 & c_{10} = 0.873329853 & c_{11} = 0.875990752 & c_{12} = 0.881027495 & c_{13} = 0.888370800 \\
c_{14} = 0.897957588 & c_{15} = 0.909730267 & c_{16} = 0.923636109 & c_{17} = 0.939626723 & c_{18} = 0.957657587 \\
c_{19} = 0.977687650 & c_{20} = 0.999678977 & c_{21} = 1.023596444 & &
\end{array}
$$

The following cubic polynomials are derived as

$$
Y(x) = \begin{cases}
-0.33561x^3 + 1.79799x^2 - 3.17148x + 2.70910 & \text{for } x \in [1.00, 1.05) \\
-0.29086x^3 + 1.65703x^2 - 3.02348x + 2.65730 & \text{for } x \in [1.05, 1.10) \\
-0.25373x^3 + 1.53450x^2 - 2.88870x + 2.60788 & \text{for } x \in [1.10, 1.15) \\
-0.22266x^3 + 1.42731x^2 - 2.76543x + 2.56063 & \text{for } x \in [1.15, 1.20) \\
-0.19646x^3 + 1.33301x^2 - 2.65227x + 2.51536 & \text{for } x \in [1.20, 1.25) \\
-0.17422x^3 + 1.24960x^2 - 2.54801x + 2.47192 & \text{for } x \in [1.25, 1.30) \\
-0.15522x^3 + 1.17548x^2 - 2.45165x + 2.43017 & \text{for } x \in [1.30, 1.35) \\
-0.13888x^3 + 1.10931x^2 - 2.36232x + 2.38997 & \text{for } x \in [1.35, 1.40) \\
-0.12476x^3 + 1.04999x^2 - 2.27927x + 2.35122 & \text{for } x \in [1.40, 1.45) \\
-0.11248x^3 + 0.99661x^2 - 2.20187x + 2.31381 & \text{for } x \in [1.45, 1.50) \\
-0.10177x^3 + 0.94841x^2 - 2.12956x + 2.27765 & \text{for } x \in [1.50, 1.55) \\
-0.09238x^3 + 0.90472x^2 - 2.06185x + 2.24267 & \text{for } x \in [1.55, 1.60) \\
-0.08410x^3 + 0.86501x^2 - 1.99832x + 2.20878 & \text{for } x \in [1.60, 1.65) \\
-0.07679x^3 + 0.82881x^2 - 1.93859x + 2.17593 & \text{for } x \in [1.65, 1.70) \\
-0.07030x^3 + 0.79572x^2 - 1.88233x + 2.14405 & \text{for } x \in [1.70, 1.75) \\
-0.06452x^3 + 0.76538x^2 - 1.82924x + 2.11308 & \text{for } x \in [1.75, 1.80) \\
-0.05936x^3 + 0.73751x^2 - 1.77906x + 2.08298 & \text{for } x \in [1.80, 1.85) \\
-0.05474x^3 + 0.71183x^2 - 1.73157x + 2.05369 & \text{for } x \in [1.85, 1.90) \\
-0.05058x^3 + 0.68814x^2 - 1.68655x + 2.02517 & \text{for } x \in [1.90, 1.95) \\
-0.04683x^3 + 0.66622x^2 - 1.64381x + 1.99739 & \text{for } x \in [1.95, 2.00).
\end{cases} \tag{37}
$$

Table IV compares the cubic B-spline results with the exact solution.

Next we derive the cubic B-spline method for the spatial variable to approximate solutions of second-order linear initial-boundary value problems for partial differential equations and present some numerical results. With respect to the time variable, we use the fourth-order Runge-Kutta method.

Table IV: Cubic B-spline results and absolute errors for Example 4.

x_i	Cubic B-spline	Exact	Absolute Error
1.00	1.0000000000	1.0000000000	0.0000000000
1.05	0.9728193862	0.9728330041	0.0000136179
1.10	0.9493484897	0.9493723572	0.0000238675
1.15	0.9293682141	0.9293996041	0.0000313901
1.20	0.9126875054	0.9127241956	0.0000366902
1.25	0.8991387600	0.8991789288	0.0000401688
1.30	0.8885741354	0.8886162826	0.0000421471
1.35	0.8808625601	0.8809054445	0.0000428845
1.40	0.8758872879	0.8759298796	0.0000425917
1.45	0.8735438833	0.8735853248	0.0000414415
1.50	0.8737385485	0.8737781245	0.0000395759
1.55	0.8763867258	0.8764238384	0.0000371126
1.60	0.8814119218	0.8814460712	0.0000341494
1.65	0.8887447136	0.8887754816	0.0000307680
1.70	0.8983219031	0.8983489402	0.0000270371
1.75	0.9100857939	0.9101088085	0.0000230145
1.80	0.9239835710	0.9240023201	0.0000187491
1.85	0.9399667648	0.9399810469	0.0000142822
1.90	0.9579907870	0.9580004361	0.0000096491
1.95	0.9780145272	0.9780194070	0.0000048798
2.00	1.0000000000	1.0000000000	0.0000000000

4. Partial Differential Equations

(a) We consider the following second order linear partial differential equation

$$u_t = k u_{xx}, \tag{38}$$

where the domain of $u = u(x,t)$ is $(0,\, L)$ and k is a constant. The boundary conditions and initial condition are

$$\begin{cases} u(0,t) = 0, \\ u(L,t) = 0, \quad t > 0 \end{cases} \tag{39}$$

$$u(x,0) = \phi(x). \tag{40}$$

We assume an approximate solution of the form

$$Y(x,t) = \sum_{i=-1}^{N+1} c_i(t) B_i(x), \tag{41}$$

where B_i is defined in (6) and $a \leq x \leq b$. Let $Y = Y(x,t)$, then

$$Y_t = \sum_{i=-1}^{N+1} \dot{c_i}(t) B_i(x) \quad \text{and} \quad Y_{xx} = \sum_{i=-1}^{N+1} c_i(t) B_i''(x). \tag{42}$$

Substituting (42) into (38) and letting $x = x_i$, we obtain the following

$$\dot{c}_{i-1}(t)B_{i-1}(x_i) + \dot{c}_i(t)B_i(x_i) + \dot{c}_{i+1}(t)B_{i+1}(x_i) + \dot{c}_{i+2}(t)B_{i+2}(x_i) = k[c_{i-1}(t)B''_{i-1}(x_i) + c_i(t)B''_i(x_i) + c_{i+1}(t)B''_{i+1}(x_i) + c_{i+2}(t)B''_{i+2}(x_i)].$$

Using (14), we obtain

$$\dot{c}_{i-1}(t) + 4\dot{c}_i(t) + \dot{c}_{i+1}(t) = \frac{6k}{h^2}[c_{i-1}(t) - 2c_i(t) + c_{i+1}(t)],$$

with the following boundary conditions:

$$c_{-1}(t)B_{-1}(x_0) + c_0(t)B_0(x_0) + c_1(t)B_1(x_0) + c_2(t)B_2(x_0) = 0,$$
$$c_{N-1}(t)B_{N-1}(x_N) + c_N(t)B_N(x_N) + c_{N+1}(t)B_{N+1}(x_N) + c_{N+2}(t)B_{N+2}(x_N) = 0,$$

where (17) yields

$$c_{-1}(t) + 4c_0(t) + c_1(t) = 0$$
$$c_{N-1}(t) + 4c_N(t) + c_{N+1}(t) = 0.$$

By eliminating c_{-1} and c_{N+1}, we obtain a system for the unknown coefficients $c_0(t),\ c_1(t), \ldots, c_N(t)$ at time t. The left-hand side of the system is given by

$$\begin{pmatrix} 6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \dot{c}_0 \\ \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \\ \dot{c}_{N-3} \\ \dot{c}_{N-2} \\ \dot{c}_{N-1} \\ \dot{c}_N \end{pmatrix},$$

and the right-hand side of which is given by

$$\frac{6k}{h^2} \begin{pmatrix} -6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{N-3} \\ c_{N-2} \\ c_{N-1} \\ c_N \end{pmatrix}.$$

We solve the system using the fourth order Runge-Kutta method.

Example 5.

As the next example, we consider the initial boundary value problem (IBVP) stated in 4(a) with $\phi(x) = \sin(\pi x)$ and the domain is $(0, 1)$.

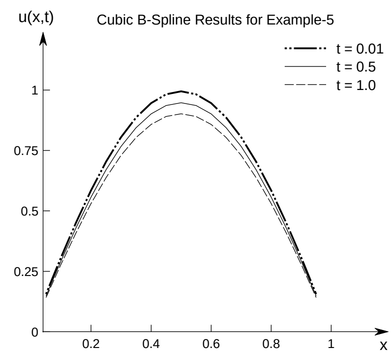

Figure 3: $u(x, t)$ using cubic B-spline at three different times for Example 5.

The exact solution to this IBVP is

$$u(x, t) = e^{-k\pi^2 t} \sin(\pi x).$$

In this example, we approximate the solution to the IBVP using cubic B-splines where the number of intervals is $N = 20$, $k = 0.01$, and the time step for the fourth order Runge-Kutta method is $t = 0.01$.

Table V: Cubic B-spline results and absolute errors at $t = 0.01$ for Example 5.

x_i	Cubic B-spline	Exact	Absolute Error
0.05	0.1556384745	0.1562801466	0.0006416721
0.10	0.3074446132	0.3087121573	0.0012675441
0.15	0.4516804452	0.4535426501	0.0018622049
0.20	0.5847944056	0.5872054177	0.0024110121
0.25	0.7035087869	0.7064092390	0.0029004522
0.30	0.8049004470	0.8082189205	0.0033184735
0.35	0.8864727869	0.8901275698	0.0036547829
0.40	0.9462172247	0.9501183242	0.0039010995
0.45	0.9826626541	0.9867140122	0.0040513581
0.50	0.9949116677	0.9990135264	0.0041018588
0.55	0.9826626541	0.9867140122	0.0040513581
0.60	0.9462172247	0.9501183242	0.0039010995
0.65	0.8864727869	0.8901275698	0.0036547829
0.70	0.8049004470	0.8082189205	0.0033184735
0.75	0.7035087869	0.7064092390	0.0029004522
0.80	0.5847944056	0.5872054177	0.0024110121
0.85	0.4516804452	0.4535426501	0.0018622049
0.90	0.3074446132	0.3087121573	0.0012675441
0.95	0.1556384745	0.1562801466	0.0006416721

Figure 3 presents the numerical solutions of Example 5 using cubic B-spline functions for three different times $t = 0.01,\ 0.5$ and 1.0.

Table VI: Cubic B-spline results and absolute errors at $t = 0.05$ for Example 5.

x_i	Cubic B-spline	Exact	Absolute Error
0.05	0.1482759797	0.1489021154	0.0006261357
0.10	0.2929009127	0.2941377666	0.0012368539
0.15	0.4303136531	0.4321307697	0.0018171166
0.20	0.5571306432	0.5594832792	0.0023526359
0.25	0.6702292279	0.6730594535	0.0028302255
0.30	0.7668245447	0.7700626703	0.0032381256
0.35	0.8445380962	0.8481043884	0.0035662922
0.40	0.9014563169	0.9052629618	0.0038066449
0.45	0.9361776914	0.9401309567	0.0039532654
0.50	0.9478472641	0.9518498074	0.0040025433
0.55	0.9361776914	0.9401309567	0.0039532654
0.60	0.9014563169	0.9052629618	0.0038066449
0.65	0.8445380962	0.8481043884	0.0035662922
0.70	0.7668245447	0.7700626703	0.0032381256
0.75	0.6702292279	0.6730594535	0.0028302255
0.80	0.5571306432	0.5594832792	0.0023526359
0.85	0.4303136531	0.4321307697	0.0018171166
0.90	0.2929009127	0.2941377666	0.0012368539
0.95	0.1482759797	0.1489021154	0.0006261357

(b) Now consider the following equation

$$u_t = k u_{xx}, \tag{43}$$

where the domain of $u = u(x,t)$ is $(0,\, L)$ and k is a constant. The boundary conditions and initial condition are

$$\begin{cases} u_x(0,t) = 0, \\ u_x(L,t) = 0 \qquad t > 0, \end{cases} \tag{44}$$

$$u(x,0) = \phi(x). \tag{45}$$

Table VII: Cubic B-spline results and absolute errors at $t = 1.00$ for Example 5.

x_i	Cubic B-spline	Exact	Absolute Error
0.05	0.1411221309	0.1417324499	0.0006103190
0.10	0.2787693665	0.2799749764	0.0012056099
0.15	0.4095523752	0.4113235899	0.0017712147
0.20	0.5302508452	0.5325440515	0.0022932063
0.25	0.6378927795	0.6406515111	0.0027587316
0.30	0.7298276766	0.7329840043	0.0031563277
0.35	0.8037917941	0.8072679987	0.0034762046
0.40	0.8579638901	0.8616743758	0.0037104858
0.45	0.8910100676	0.8948634701	0.0038534025
0.50	0.9021166202	0.9060180558	0.0039014356
0.55	0.8910100676	0.8948634701	0.0038534025
0.60	0.8579638901	0.8616743758	0.0037104858
0.65	0.8037917941	0.8072679987	0.0034762046
0.70	0.7298276766	0.7329840043	0.0031563277
0.75	0.6378927795	0.6406515111	0.0027587316
0.80	0.5302508452	0.5325440515	0.0022932063
0.85	0.4095523752	0.4113235899	0.0017712147
0.90	0.2787693665	0.2799749764	0.0012056099
0.95	0.1411221309	0.1417324499	0.0006103190

Using (41), the boundary conditions are as follow

$$\begin{aligned} c_{-1}(t)B'_{-1}(x_0) + c_0(t)B'_0(x_0) + c_1(t)B'_1(x_0) + c_2(t)B'_2(x_0) = 0, \\ c_{N-1}(t)B'_{N-1}(x_N) + c_N(t)B'_N(x_N) + c_{N+1}(t)B'_{N+1}(x_N) + c_{N+2}(t)B'_{N+2}(x_N) = 0, \end{aligned}$$

which gives

$$\begin{aligned} &- c_{-1}(t) + c_1(t) = 0, \\ &- c_{N-1}(t) + c_{N+1}(t) = 0. \end{aligned}$$

Now we obtain a system whose left-hand side is

$$\begin{pmatrix} 4 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} \dot{c}_0 \\ \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \\ \vdots \\ \dot{c}_{N-3} \\ \dot{c}_{N-2} \\ \dot{c}_{N-1} \\ \dot{c}_N \end{pmatrix},$$

and right-hand side is

$$\frac{6k}{h^2}\begin{pmatrix} -2 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{N-3} \\ c_{N-2} \\ c_{N-1} \\ c_N \end{pmatrix}.$$

The system is solved using the fourth order Runge-Kutta method.

Example 6.

Another example we consider is the initial boundary value problem (IBVP) stated in 4(b) with $\phi(x) = x(1-x)$ in the domain$(0,\, 1)$.

The exact solution to this IBVP is

$$u(x,t) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2\{(-1)^{n+1}-1\}}{(n\pi)^2} e^{-k(n\pi)^2 t} \cos(n\pi x).$$

To approximate this IBVP using cubic B-splines, we use $N = 20$, $k = 0.01$, and the time step for the fourth order Runge-Kutta method is $t = 0.01$.

Table VIII: Cubic B-spline results and absolute errors at $t = 0.01$ for Example 6.

x_i	Cubic B-spline	Exact	Absolute Error
0.05	0.0466617756	0.0473014352	0.0006396597
0.10	0.0889213691	0.0898000000	0.0008786309
0.15	0.1264771637	0.1273000000	0.0008228363
0.20	0.1589642318	0.1598000000	0.0008357682
0.25	0.1864672344	0.1873000000	0.0008327656
0.30	0.2089665328	0.2098000000	0.0008334672
0.35	0.2264666988	0.2273000000	0.0008333012
0.40	0.2389666587	0.2398000000	0.0008333413
0.45	0.2464666688	0.2473000000	0.0008333312
0.50	0.2489666656	0.2498000000	0.0008333344
0.55	0.2464666688	0.2473000000	0.0008333312
0.60	0.2389666587	0.2398000000	0.0008333413
0.65	0.2264666988	0.2273000000	0.0008333012
0.70	0.2089665328	0.2098000000	0.0008334672
0.75	0.1864672344	0.1873000000	0.0008327656
0.80	0.1589642318	0.1598000000	0.0008357682
0.85	0.1264771637	0.1273000000	0.0008228363
0.90	0.0889213691	0.0898000000	0.0008786309
0.95	0.0466617756	0.0473014352	0.0006396597

Table IX: Cubic B-spline results and absolute errors at $t = 0.50$ for Example 6.

x_i	Cubic B-spline	Exact	Absolute Error
0.05	0.0781968715	0.0770593115	0.0011375600
0.10	0.0968827779	0.0966630941	0.0002196837
0.15	0.1227875746	0.1233613588	0.0005737842
0.20	0.1507670127	0.1516981405	0.0009311278
0.25	0.1769379208	0.1779008274	0.0009629067
0.30	0.1991795065	0.2000764309	0.0008969243
0.35	0.2166608158	0.2175116962	0.0008508803
0.40	0.2291657137	0.2300014291	0.0008357154
0.45	0.2366668173	0.2375001395	0.0008333222
0.50	0.2391667443	0.2400000214	0.0008332771
0.55	0.2366668173	0.2375001395	0.0008333222
0.60	0.2291657137	0.2300014291	0.0008357154
0.65	0.2166608158	0.2175116962	0.0008508803
0.70	0.1991795065	0.2000764309	0.0008969243
0.75	0.1769379208	0.1779008274	0.0009629067
0.80	0.1507670127	0.1516981405	0.0009311278
0.85	0.1227875746	0.1233613588	0.0005737842
0.90	0.0968827779	0.0966630941	0.0002196837
0.95	0.0781968715	0.0770593115	0.0011375600

Figure 4 presents the numerical solutions of Example 6 using cubic B-spline functions for three different times $t = 0.01,\ 0.5$ and 1.0.

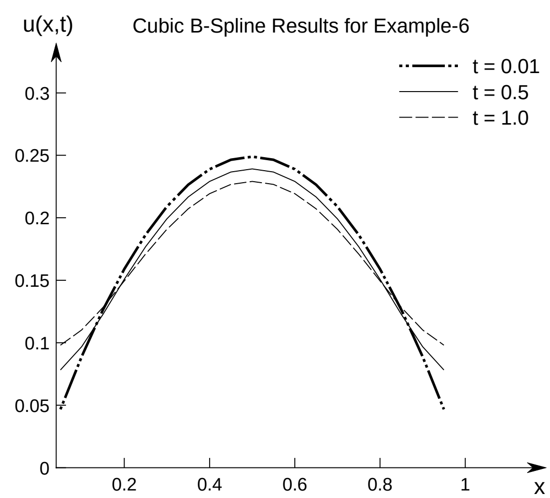

Figure 4: $u(x, t)$ using cubic B-spline at three different times for Example 6.

5. Conclusion

We investigated the application of cubic B-spline functions in solving boundary value problems. After deriving the cubic B-spline functions, we used these functions to solve second order linear boundary value problems. We also used this numerical method with respect to space variable to obtain solution for second order linear partial differential equations. We use the fourth-order Runge-Kutta method for the time variable for the partial differential equation case. Numerical

Table X: Cubic B-spline results and absolute errors at $t = 1.00$ for Example 6.

x_i	Cubic B-spline	Exact	Absolute Error
0.05	0.0980448897	0.0973177324	0.0007271572
0.10	0.1102388626	0.1099282457	0.0003106170
0.15	0.1282806992	0.1284664520	0.0001857528
0.20	0.1494558898	0.1500509087	0.0005950189
0.25	0.1710403591	0.1718771472	0.0008367881
0.30	0.1907993071	0.1917245926	0.0009252855
0.35	0.2071893711	0.2081127066	0.0009233355
0.40	0.2193047658	0.2201962755	0.0008915097
0.45	0.2266946810	0.2275592951	0.0008646141
0.50	0.2291737629	0.2300287048	0.0008549419
0.55	0.2266946810	0.2275592951	0.0008646141
0.60	0.2193047658	0.2201962755	0.0008915097
0.65	0.2071893711	0.2081127066	0.0009233355
0.70	0.1907993071	0.1917245926	0.0009252855
0.75	0.1710403591	0.1718771472	0.0008367881
0.80	0.1494558898	0.1500509087	0.0005950189
0.85	0.1282806992	0.1284664520	0.0001857528
0.90	0.1102388626	0.1099282457	0.0003106170
0.95	0.0980448897	0.0973177324	0.0007271572

results for various cases are presented and compared with exact solutions. The current method is superior to the method in Bhatti et al. (2006) and Bhatta et al. (2006) since the polynomials in the present method always have degree three and it is independent of the number of nodes. In the method of Bhatti et al. (2006) and Bhatta et al. (2006), the degree of the polynomial depends on the number of nodes and the computational complexity increases with increasing number of nodes. In the current method, there is no constraint on the spatial grid size and temporal grid size unlike the finite difference method. A limitation of the current method is that it can be used to solve up to second order differential equations. It cannot be used for solving higher (more than two) order ODEs and PDEs whereas the method in Bhatti et al. (2006) and Bhatta et al. (2006) can be applied to solve higher order differential equations. In future work, we plan to address the error analysis and the optimal value of spatial grid for cubic B-spline method.

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