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Variational Iteration Method For Solving Two-Parameter Singularly Perturbed Two Point Boundary Value Problem

Marwan Taiseer Alquran
Mathematics and Statistics
Faculty of Science
Jordan University of Science and Technology
Irbid, 22110 Jordan
marwan04@just.edu.jo

Nurettin Doğan
Electronics and Computer Education Department
Technical Education Faculty
Gazi University
06500 Teknikokullar
Ankara, Turkey
ndogan@gazi.edu.tr

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Abstract

In this paper, He's Variational iteration method (VIM) is used for the solution of singularly perturbed two-point boundary value problems with two small parameters multiplying the derivatives. Some problems are solved to demonstrate the applicability of the method. This paper suggests a patern for choosing the freely selected initial approximation in the VIM that leads to a very well approximation by only one iteration.

Keywords: Variational iteration method; Singularly perturbed two-point boundary value problems; Two parameter; Approximate solution

MSC (2000) No: 35A 15, 34B 10

1. Introduction

We consider the two-parameter singularly perturbed boundary value problems:

$$\varepsilon y''(x)+\mu a(x)y'(x)+b(x)y(x)=f(x),\quad x\in[0,1], \tag{1.1}$$

with the boundary conditions

$$y(0)=\alpha,\ y(1)=\beta, \tag{1.2}$$

where α,β are real constants and ε,μ are small positive parameters $(0<\varepsilon<<1, 0<\mu<<1)$. We assume that $a(x)$, $b(x)$ and $f(x)$ are sufficiently continuously differentiable functions on $[0,1]$. This problem encloses both the reaction–diffusion problem when μ = 0 and the convection–diffusion problem when μ = 1. Different numerical methods have been proposed by various authors for two parameter problems, such as O'Malley (1967), Gracia, O'Riordan and Pickett (2006), Torsten and Roos (2004), Valanarasu and Ramanujam (2003), Kadalbajoo and Yadaw (2008) and Valarmathi and Ramanujam (2003). In this paper we introduce He's variational iteration method as an alternative to existing methods.

2. Varitaional Iteration Method

The variational iteration method VIM is proposed by the Chinese mathematician Ji-Huan He (1997 - 2004) and it has recently been intensively studied by many scientists to apply different type of problems. By He's method we introduce the following correction functional corresponding to equation (1.1)

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(x,t)\left\{\varepsilon y_n''(t)+\mu a(t)y_n'(t)+b(t)y_n(t)-f(t)\right\}dt, \tag{2.1}$$

where λ is a general Lagrange multiplier He's (1997 – 2004), which can be identified optimally via variational theory. The superscript n denotes the n^{th} iteration. We now consider the convergence proof of the iterative solution of (2.1) for equation (1.1) by Banach's fixed-point theorem. Let (X,d) be a nonempty complete metric space, and let $T:X\to X$ be a contraction mapping on X, i.e., there exists a nonnegative real number $q<1$ such that $Tx^*=x^*$, which can be determined by starting with an arbitrary $x^0\in X$, and the iterative sequence $Tx^k=Tx^{k-1}$, $k=1,2,3,...,$ converges with limit x^*. In our case, for the convergence of equation (2.1), we require that

$$\int_0^x \lambda(x,t)\left\{\varepsilon y_n''(t)+\mu a(t)y_n'(t)+b(t)y_n(t)-f(t)\right\}dt\,,$$

is contractive mapping and its convergence is ensured by Banach's fixed point theorem. By making the correction functional stationary with restricted variation $\delta y_n(0)=0$ and $\delta y_n'(0)=0$, we obtain:

$$\begin{aligned}\delta y_{n+1}(x)&=\delta y_n(x)+\delta\int_0^x \lambda(x,t)\left\{\varepsilon y_n''(t)+\mu a(t)y_n'(t)+b(t)y_n(t)-f(t)\right\}dt \\ &=\delta y_n(x)+\varepsilon\int_0^x \lambda(x,t)\frac{d^2}{dt^2}\delta y_n(t)dt+\mu\int_0^x \lambda(x,t)\frac{d}{dt}\delta a(t)y_n(t)dt+\int_a^x \lambda(x,t)\delta b(t)y_n(t)dt.\end{aligned} \tag{2.2}$$

Integrating (2.2) by parts yields

$$\begin{aligned}\delta y_{n+1}(x)&=\left(1-\varepsilon\frac{\partial\lambda(x,t)}{\partial t}+\mu a(t)\lambda(x,t)\right)\delta y_n(t)\Big|_{t=x}+\varepsilon\lambda(x,t)\frac{d}{dt}\delta y_n(t)\Big|_{t=x} \\ &\quad+\int_0^x\left[\varepsilon\frac{\partial^2\lambda(x,t)}{\partial t^2}-\mu a(t)\frac{\partial\lambda(x,t)}{\partial t}+b(t)\lambda(x,t)\right]\delta y_n(t)dt.\end{aligned} \tag{2.3}$$

Therefore, the general Lagrange multiplier satisfies the following system of equations:

$$\begin{aligned}&\left(1-\varepsilon\frac{\partial\lambda(x,t)}{\partial t}+\mu a(t)\lambda(x,t)\right)_{t=x}=0, \\ &\lambda(x,t)\big|_{t=x}=0, \\ &\varepsilon\frac{\partial^2\lambda(x,t)}{\partial t^2}-\mu a(t)\frac{\partial\lambda(x,t)}{\partial t}+b(t)\lambda(x,t)=0.\end{aligned} \tag{2.4}$$

3. Applications and Results

To incorporate our discussion above, four special cases of the singularly perturbed two-point boundary problem (1.1) – (1.2) will be studied. In this paper we suggests a pattern for choosing the freely selected initial approximation in the VIM y_0 by giving it the general form of the obtained Lagrange multiplier function described in system (2.4). We do comparison between one-iterative variational iteration solutions and the exact solutions for each application. All the results are calculated by using the symbolic calculus software *Mathematica*.

Example 1.

We consider the following example Gracia, O'Riordan and Pickett (2006):

$$\begin{aligned} &\varepsilon y''(x)+\mu y'-y(x)=1, \\ &y(0)=0,\ y(1)=0. \end{aligned} \tag{3.1}$$

Solving System (2.4) using the coefficients: $a(x)=1$, $b(x)=-1$, then λ can be easily identified as:

$$\lambda(x,t)=\frac{1}{\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}-e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}\right].$$

Therefore, we have the following iteration formula

$$y_{n+1}(x)=y_n(x)+\int_0^x\frac{1}{\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}-e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}\right]\left\{\varepsilon y_n''(t)+\mu y_n'(t)-y_n(t)-1\right\}dt.$$

Now, we begin with an arbitrary initial approximation:

$$y_0(x)=C_1\,e^{-\frac{1}{2\varepsilon}\left(\mu+\sqrt{\mu^2+\varepsilon}\right)x}+C_2\,e^{-\frac{1}{2\varepsilon}\left(\mu-\sqrt{\mu^2+\varepsilon}\right)x},$$

where C_1 and C_2 are constants to be determined. By the variational iteration formula, we have

$$y_1(x)=y_0(x)+\int_0^x\frac{1}{\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}-e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}\right]\left\{\varepsilon y_0''(t)+\mu y_0'(t)-y_0(t)-1\right\}dt.$$

By imposing the boundary conditions at $x=0$ and $x=1$, *Mathematica* software identifies the function $y_1(x)$ to be considered as the approximate solution to BVP (3.1). Figures 1, 2, 3 show the errors compared with the exact solution for different values of ε and μ.

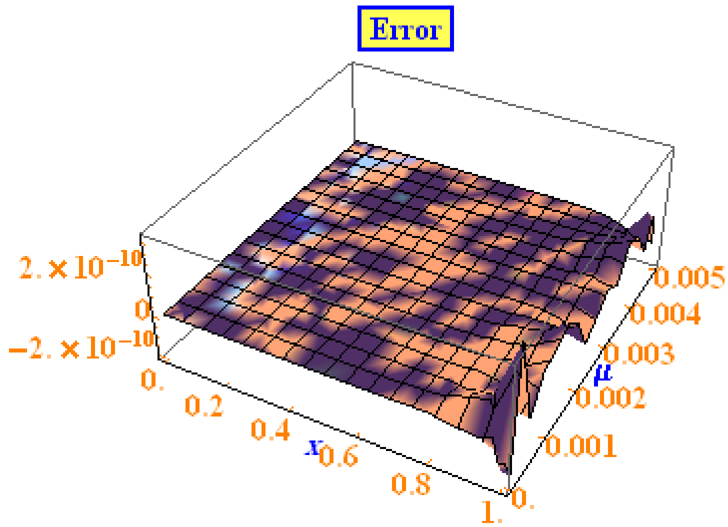

Figure 1. *Example 1.* The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\varepsilon = 0.005$ $0 < \mu < 0.005$.

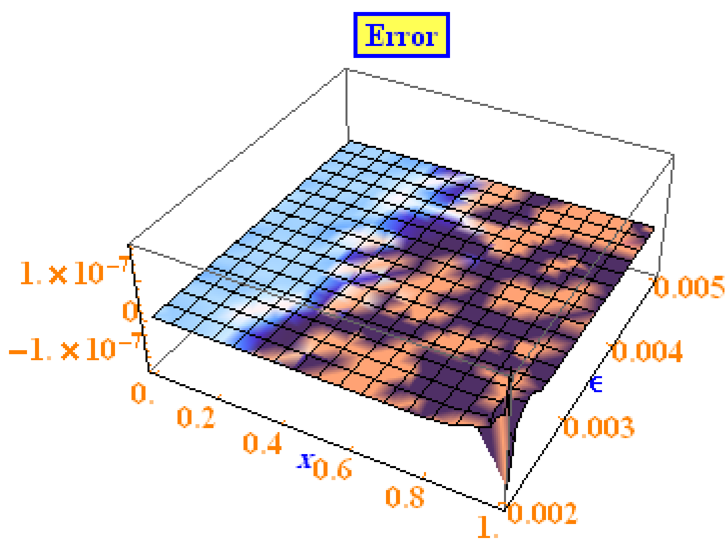

Figure 2. *Example 1.* The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\mu = 0.005$ and $0.002 < \varepsilon < 0.005$.

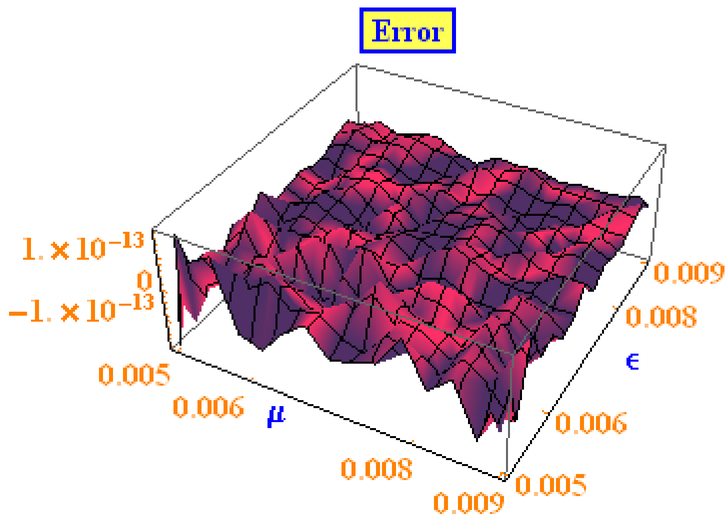

Figure 3. *Example 1.* The errors using the approximate solution $\boldsymbol{y}_1(\boldsymbol{x})$, $\boldsymbol{x} = 0.5$, for the case $0.005 < \mu < 0.009$ and $0.005 < \varepsilon < 0.009$.

Example 2.

Now we consider the following example [Valarmathi and Ramanujam (2003)] :

$$\varepsilon y''(x) + \mu y'(x) - y(x) = -x;\ y(0) = 1, y(1) = 0. \tag{3.2}$$

Solving System (2.4) using the coefficients: $a(x) = 1$, $b(x) = -1$, then λ can be easily identified as:

$$\lambda(x,t) = \frac{1}{\sqrt{\mu^2 + 4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}\right].$$

Therefore, we have the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}\right]\left\{\varepsilon y_n''(t) + \mu y_n'(t) - y_n(t) + t\right\} dt.$$

Now, we begin with an arbitrary initial approximation:

$$y_0(x) = C_1 e^{-\frac{1}{2\varepsilon}\left(\mu+\sqrt{\mu^2+\varepsilon}\right)x} + C_2 e^{-\frac{1}{2\varepsilon}\left(\mu-\sqrt{\mu^2+\varepsilon}\right)x},$$

where C_1 and C_2 are constants to be determined. By the variational iteration formula, we have

$$y_1(x) = y_0(x) + \int_0^x \frac{1}{\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)}\right]\left\{\varepsilon y_0''(t) + \mu y_0'(t) - y_0(t) + t\right\} dt.$$

By imposing the boundary conditions at $x = 0$ and $x = 1$, *Mathematica* software identifies the function $y_1(x)$ to be considered as the approximate solution to BVP (3.2). Figures 4, 5, 6 show the errors compared with the exact solution for different values of ε and μ.

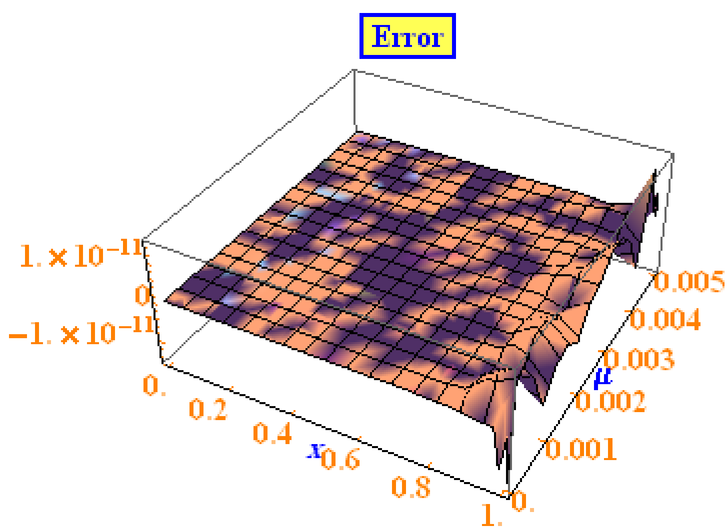

Figure 4. *Example 2.* The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\varepsilon = 0.005$ $0 < \mu < 0.005$.

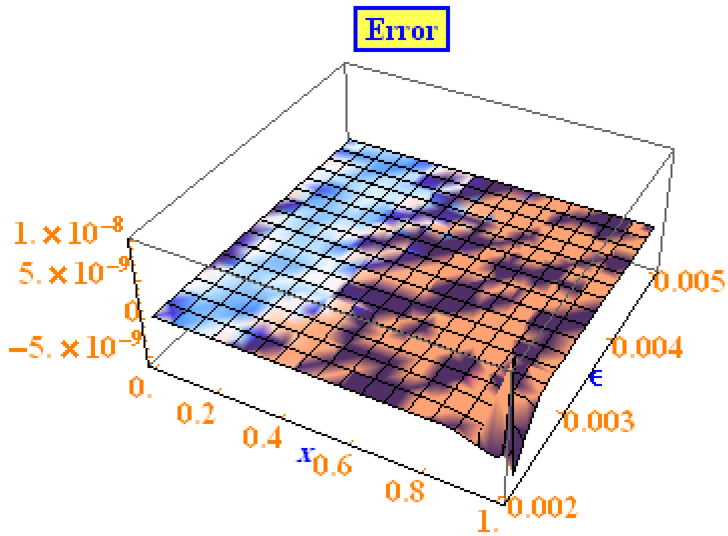

Figure 5. *Example 2.* The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\mu = 0.005$ and $0.002 < \varepsilon < 0.005$.

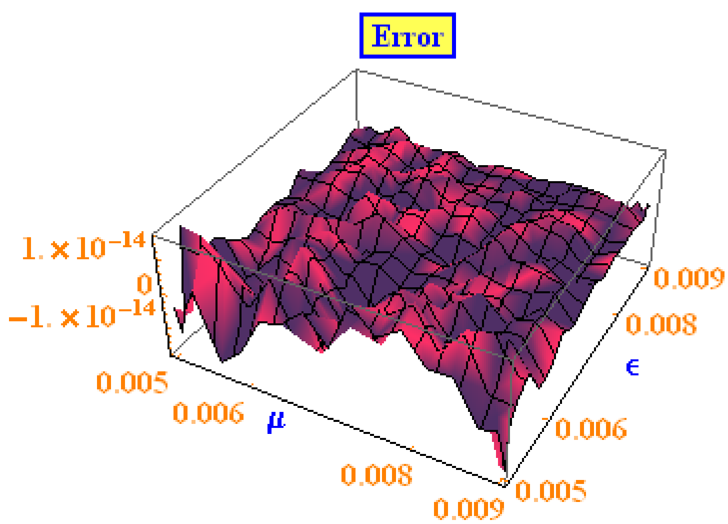

Figure 6. *Example 2.* The errors using the approximate solution $y_1(x)$, $x = 0.5$, for the case $0.005 < \mu < 0.009$ and $0.005 < \varepsilon < 0.009$.

Example 3.

Now we consider the following example Valarmathi and Ramanujam (2003) :

$$\varepsilon y''(x)+2\mu y'(x)-4y(x)=-x\,;\; y(0)=0, y(1)=1, \tag{3.3}$$

Solving system (2.4), using the coefficients: $a(x)=2$, $b(x)=-4$, then λ can be easily be identified as:

$$\lambda(x,t)=\frac{1}{2\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{\varepsilon}(t-x)}-e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{\varepsilon}(t-x)}\right].$$

Therefore, we have the following iteration formula:

$$y_{n+1}(x)=y_n(x)+\int_0^x\frac{1}{2\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{\varepsilon}(t-x)}-e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{\varepsilon}(t-x)}\right]\left\{\varepsilon y_n''(t)+2\mu y_n'(t)-4y_n(t)+t\right\}dt.$$

Now, we begin with an arbitrary initial approximation:

$$y_0(x)=C_1\,e^{-\frac{1}{\varepsilon}\left(\mu+\sqrt{\mu^2+\varepsilon}\right)x}+C_2\,e^{-\frac{1}{\varepsilon}\left(\mu-\sqrt{\mu^2+\varepsilon}\right)x},$$

where C_1 and C_2 are constants to be determined. By the variational iteration formula, we have

$$y_1(x)=y_0(x)+\int_0^x\frac{1}{2\sqrt{\mu^2+4\varepsilon}}\left[e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{\varepsilon}(t-x)}-e^{\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{\varepsilon}(t-x)}\right]\left\{\varepsilon y_0''(t)+2\mu y_0'(t)-4y_0(t)+t\right\}dt.$$

By imposing the boundary conditions at $x=0$ and $x=1$, *Mathematica* software identifies the function $y_1(x)$ to be considered as the approximate solution to BVP (3.3). Figures 7, 8, 9 show the errors compared with the exact solution for different values of ε and μ.

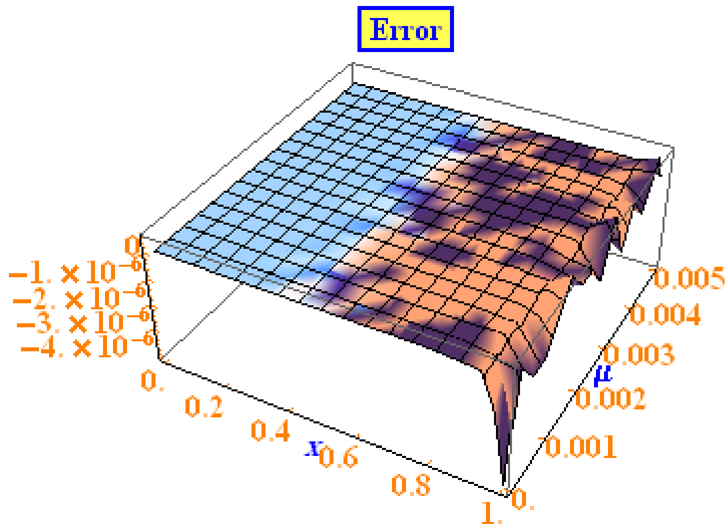

Figure 7. *Example* 3. The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\varepsilon = 0.005$ and $0 < \mu < 0.005$.

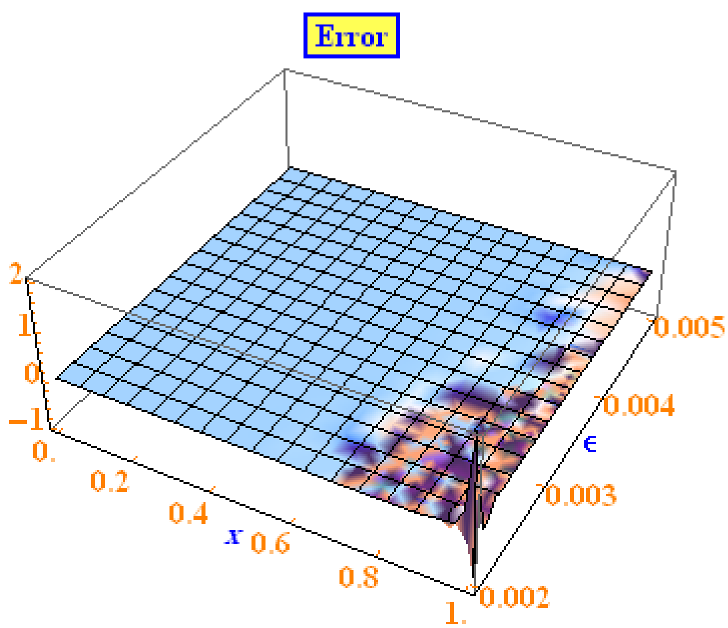

Figure 8. *Example* 3. The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\mu = 0.005$ and $0.002 < \varepsilon < 0.005$.

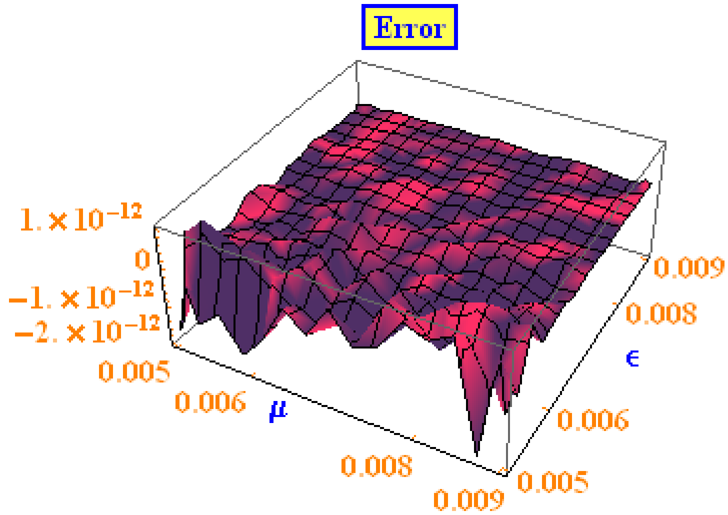

Figure 9. *Example* 3. The errors using the approximate solution $y_1(x)$, $x = 0.5$, for the case $0.005 < \mu < 0.009$ and $0.005 < \varepsilon < 0.009$.

Example 4.

Now we consider the following example Kadalbajoo and Yadaw (2008) :

$$\varepsilon y''(x) - \mu y'(x) - y(x) = -\cos \pi x \; ; \; y(0) = 0, y(1) = 0, \tag{3.4}$$

Solving system (2.4) using the coefficients: $a(x) = -1$, $b(x) = -1$, then λ can be easily be identified as:

$$\lambda(x,t) = \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[e^{\frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{-\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} \right].$$

Therefore, we have the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[e^{\frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{-\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} \right] \left\{ \varepsilon y_n''(t) - \mu y_n'(t) - y_n(t) + \cos \pi t \right\} dt.$$

Now, we begin with an arbitrary initial approximation:

$$y_0(x) = C_1 e^{-\frac{1}{2\varepsilon}\left(-\mu+\sqrt{\mu^2+\varepsilon}\right)x} + C_2 e^{-\frac{1}{2\varepsilon}\left(-\mu-\sqrt{\mu^2+\varepsilon}\right)x},$$

where C_1 and C_2 are constants to be determined. By the variational iteration formula, we have

$$y_1(x) = y_0(x) + \int_0^x \frac{1}{\sqrt{\mu^2+4\varepsilon}} \left[e^{\frac{-\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{-\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} \right] \left\{ \varepsilon y_0''(t) - \mu y_0'(t) - y_0(t) + \cos \pi t \right\} dt.$$

By imposing the boundary conditions at $x = 0$ and $x = 1$, *Mathematica* software identifies the function $y_1(x)$ to be considered as the approximate solution to BVP (3.4). Figures 10, 11, 12 show the errors compared with the exact solution for different values of ε and μ.

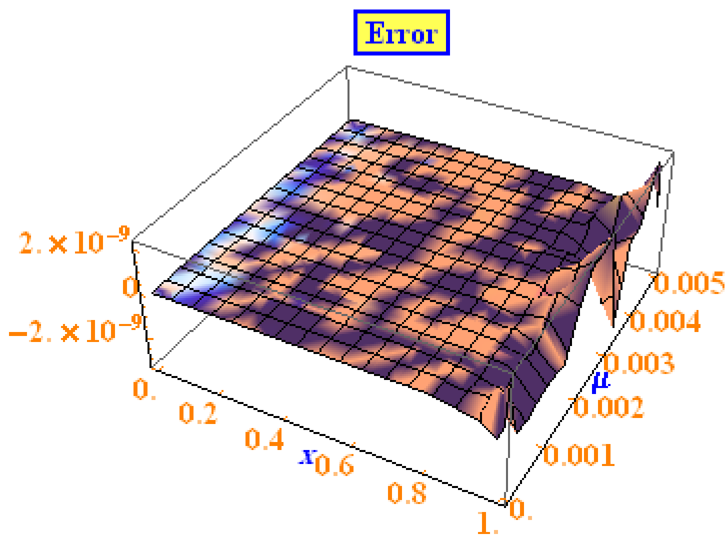

Figure 10. *Example* 4. The errors using the approximate solution $\boldsymbol{y}_1(\boldsymbol{x})$, $0 \le \boldsymbol{x} \le 1$, for the case $\varepsilon = 0.005$ and $0 < \mu < 0.005$.

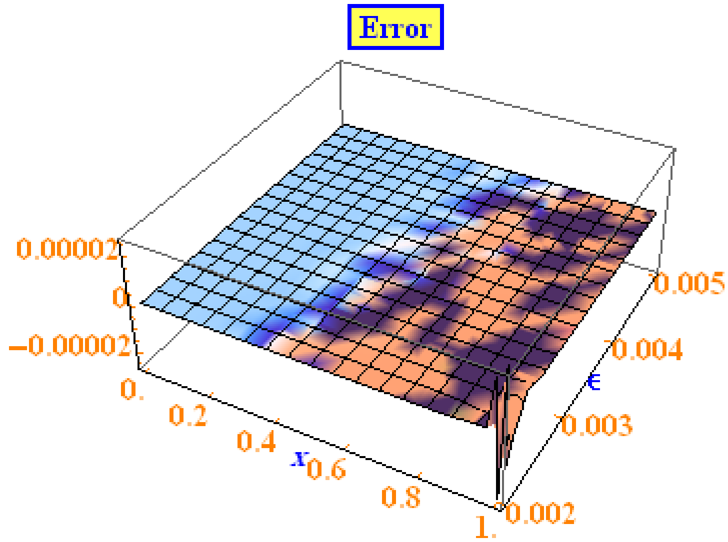

Figure 11. *Example* 4. The errors using the approximate solution $y_1(x)$, $0 \le x \le 1$, for the case $\mu = 0.005$ and $0.002 < \varepsilon < 0.005$.

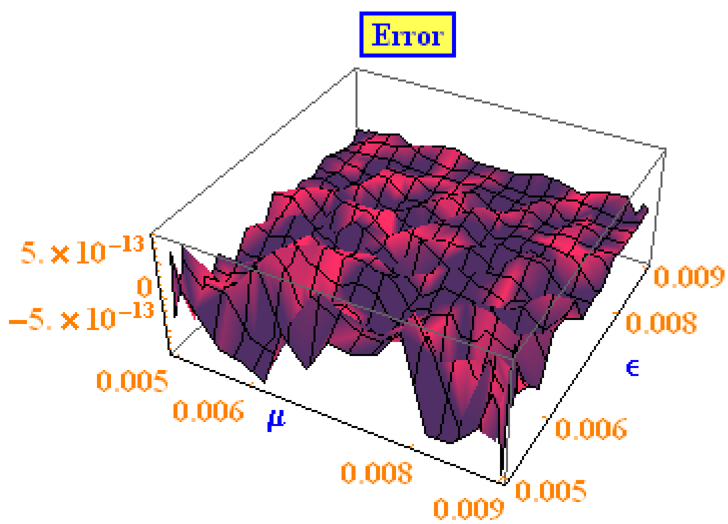

Figure 12. *Example* 4. The errors using the approximate solution $y_1(x)$, $x = 0.5$, for the case $0.005 < \mu < 0.009$ and $0.005 < \varepsilon < 0.009$.

4. Conclusion

He's Variational iteration method was employed successfully for solving linear singularly perturbed two-point boundary value problems. The results show that a good choice of the freely selected initial approximation in the VIM leads to a very good approximation by considering only one iteration.

Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear two-parameter singularly perturbed two-point boundary value problems.

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REFERENCES

Gracia, J.L.; O'Riordan, E., and Pickett, M. L. (2006). A parameter robust second order numerical method for a singularly, perturbed two-parameter problem, Applied Numerical Mathematics, Vol. 56, pp. 962–980.

He, J. H. (1997). Variational iteration method for delay differential equations, Commun. Nonlinear Sci. Numer. Simul, Vol. 2, pp. 235-236.

He, J. H. (1997). A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul. Vol 2, pp. 230-235.

He, J. H. (1998). Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Engrg., Vol. 167, pp. 69-73.

He, J. H. (1998). Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg., Vol. 167, pp. 57-68.

He, J. H. (1999). Variational iteration method_ a kind of non-linear analytical technique: Some examples, Internat. J. Non-Linear Mech., Vol. 34, pp. 699-708.

He, J.H. (2000). Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput., Vol. 114, pp. 115-123.

He, J. H. (2003). Variational approach to the sixth-order boundary value problems, Appl. Math. Comput., Vol. 143, pp. 537-538.

He, J. H. (2004). Variational principle for some nonlinear partial differential equations with variable coefficients, Chaos Solitons Fractals, Vol. 19, pp. 847-851.

Kadalbajoo, Mohan, K. and Yadaw, Arjun Singh (2008). B-Spline collocation method for a two-parameter singularly perturbed convection–diffusion boundary value problems, Applied Mathematics and Computation., Vol. 201, pp. 504–513.

O'Malley, Jr. R. E. (1967). Two-parameter singular perturbation problems for second order equations, J. Math. Mech., Vol.16, pp.1143–1164.

Torsten, Lin B. and Hans-Görg, Roos (2004). Analysis of a finite-difference scheme for a singularly perturbed problem with two small parameters, J. Math. Anal. Appl. Vol. 289, pp. 355–366.

Valanarasu, T. and Ramanujam, N. (2003). Asymptotic initial value methods for two-parameter singularly perturbed boundary value problems for second order ordinary differential equations, Applied Mathematics and Computation., Vol. 137, pp. 549–570.

Valarmathi, S. and Ramanujam, N. (2003). Computational methods for solving two-parameter singularly perturbed boundary value problems for second-order ordinary differential equations, Applied Mathematics and Computation, Vol. 136, pp. 415–441.