Variational Iteration Method For Solving Two-Parameter Singularly Perturbed Two Point Boundary Value Problem

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Received: September 17, 2009; Accepted: March 11, 2010

Abstract

In this paper, He’s Variational iteration method (VIM) is used for the solution of singularly perturbed two-point boundary value problems with two small parameters multiplying the derivatives. Some problems are solved to demonstrate the applicability of the method. This paper suggests a pattern for choosing the freely selected initial approximation in the VIM that leads to a very well approximation by only one iteration.

Keywords: Variational iteration method; Singularly perturbed two-point boundary value problems; Two parameter; Approximate solution

1. Introduction

We consider the two-parameter singularly perturbed boundary value problems:

\[ \varepsilon y''(x) + \mu a(x)y'(x) + b(x)y(x) = f(x), \quad x \in [0,1], \]

with the boundary conditions

\[ y(0) = \alpha, \quad y(1) = \beta, \]

where \( \alpha, \beta \) are real constants and \( \varepsilon, \mu \) are small positive parameters \((0 < \varepsilon \ll 1, 0 < \mu \ll 1)\). We assume that \( a(x), b(x) \) and \( f(x) \) are sufficiently continuously differentiable functions on \([0,1]\). This problem encloses both the reaction–diffusion problem when \( \mu = 0 \) and the convection–diffusion problem when \( \mu = 1 \). Different numerical methods have been proposed by various authors for two parameter problems, such as O’Malley (1967), Gracia, O’Riordan and Pickett (2006), Torsten and Roos (2004), Valanarasu and Ramanujam (2003), Kadalbajoo and Yadaw (2008) and Valarmathi and Ramanujam (2003). In this paper we introduce He’s variational iteration method as an alternative to existing methods.

2. Variational Iteration Method

The variational iteration method VIM is proposed by the Chinese mathematician Ji-Huan He (1997–2004) and it has recently been intensively studied by many scientists to apply different type of problems. By He’s method we introduce the following correction functional corresponding to equation (1.1)

\[ y_{n+1}(x) = y_n(x) + \int_0^x \lambda(x,t) \left[ \varepsilon y_n''(t) + \mu a(t)y_n'(t) + b(t)y_n(t) - f(t) \right] dt, \]

where \( \lambda \) is a general Lagrange multiplier He's (1997–2004), which can be identified optimally via variational theory. The superscript \( n \) denotes the \( n^{th} \) iteration. We now consider the convergence proof of the iterative solution of (2.1) for equation (1.1) by Banach’s fixed-point theorem. Let \((X,d)\) be a nonempty complete metric space, and let \( T : X \rightarrow X \) be a contraction mapping on \( X \), i.e., there exists a nonnegative real number \( q < 1 \) such that \( Tx^* = x^* \), which can be determined by starting with an arbitrary \( x^0 \in X \), and the iterative sequence \( Tx^k = Tx^{k-1}, k = 1, 2, 3, \ldots \), converges with limit \( x^* \). In our case, for the convergence of equation (2.1), we require that
\[
\int_0^x \lambda(x,t) \left\{ \varepsilon y''(t) + \mu a(t)y'(t) + b(t)y(t) - f(t) \right\} dt,
\]

is contractive mapping and its convergence is ensured by Banach’s fixed point theorem. By making the correction functional stationary with restricted variation \(\delta y_n(0) = 0\) and \(\delta y_n'(0) = 0\), we obtain:

\[
\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(x,t) \left\{ \varepsilon y''(t) + \mu a(t)y'(t) + b(t)y(t) - f(t) \right\} dt
\]

\[
= \delta y_n(x) + \varepsilon \int_0^x \lambda(x,t) \frac{d^2}{dt^2} \delta y_n(t) dt + \mu \int_0^x \lambda(x,t) \frac{d}{dt} \delta a(t) y_n(t) dt + \int_0^x \lambda(x,t) \delta b(t) y_n(t) dt.
\]

Integrating (2.2) by parts yields

\[
\delta y_{n+1}(x) = \left( 1 - \varepsilon \frac{\partial \lambda(x,t)}{\partial t} + \mu a(t) \lambda(x,t) \right) \delta y_n(x) \bigg|_{t=x} + \varepsilon \lambda(x,t) \frac{d}{dt} \delta y_n(t) \bigg|_{t=x}
\]

\[
+ \varepsilon \int_0^x \left[ \lambda(x,t) \frac{\partial^2 \lambda(x,t)}{\partial t^2} - \mu a(t) \frac{\partial \lambda(x,t)}{\partial t} + b(t) \lambda(x,t) \right] \delta y_n(t) dt.
\]

Therefore, the general Lagrange multiplier satisfies the following system of equations:

\[
\left( 1 - \varepsilon \frac{\partial \lambda(x,t)}{\partial t} + \mu a(t) \lambda(x,t) \right) \bigg|_{t=x} = 0,
\]

\[
\lambda(x,t) \bigg|_{t=x} = 0,
\]

\[
\varepsilon \frac{\partial^2 \lambda(x,t)}{\partial t^2} - \mu a(t) \frac{\partial \lambda(x,t)}{\partial t} + b(t) \lambda(x,t) = 0.
\]

3. Applications and Results

To incorporate our discussion above, four special cases of the singularly perturbed two-point boundary problem (1.1) – (1.2) will be studied. In this paper we suggest a pattern for choosing the freely selected initial approximation in the VIM \(y_0\) by giving it the general form of the obtained Lagrange multiplier function described in system (2.4). We do comparison between
one- iterative variational iteration solutions and the exact solutions for each application. All the results are calculated by using the symbolic calculus software Mathematica.

**Example 1.**

We consider the following example Gracia, O’Riordan and Pickett (2006):

\[ \varepsilon y^*(x) + \mu y' - y(x) = 1, \]
\[ y(0) = 0, \quad y(1) = 0. \]  

(3.1)

Solving System (2.4) using the coefficients: \( a(x) = 1, b(x) = -1 \), then \( \lambda \) can be easily identified as:

\[ \lambda(x,t) = \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{\mu^2+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} \right]. \]

Therefore, we have the following iteration formula

\[ y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{\mu^2+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} \right] \left\{ \varepsilon y_n''(t) + \mu y_n'(t) - y_n(t) - 1 \right\} dt. \]

Now, we begin with an arbitrary initial approximation:

\[ y_0(x) = C_1 e^{\frac{1}{2\varepsilon}(\mu+\sqrt{\mu^2+4\varepsilon})x} + C_2 e^{\frac{1}{2\varepsilon}(\mu-\sqrt{\mu^2+4\varepsilon})x}, \]

where \( C_1 \) and \( C_2 \) are constants to be determined. By the variational iteration formula, we have

\[ y_1(x) = y_0(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ e^{\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} - e^{\frac{\mu^2+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}(t-x)} \right] \left\{ \varepsilon y_0''(t) + \mu y_0'(t) - y_0(t) - 1 \right\} dt. \]

By imposing the boundary conditions at \( x = 0 \) and \( x = 1 \), Mathematica software identifies the function \( y_1(x) \) to be considered as the approximate solution to BVP (3.1). Figures 1, 2, 3 show the errors compared with the exact solution for different values of \( \varepsilon \) and \( \mu \).
Figure 1. Example 1. The errors using the approximate solution $y_1(x), 0 \leq x \leq 1$, for the case $\varepsilon = 0.005$ $0 < \mu < 0.005$.

Figure 2. Example 1. The errors using the approximate solution $y_1(x), 0 \leq x \leq 1$, for the case $\mu = 0.005$ and $0.002 < \varepsilon < 0.005$. 
Example 1.

The errors using the approximate solution $y_1(x), x = 0.5$, for the case $0.005 < \mu < 0.009$ and $0.005 < \varepsilon < 0.009$.

Example 2.

Now we consider the following example [Valarmathi and Ramanujam (2003)]:

\[ \varepsilon y''(x) + \mu y'(x) - y(x) = -x; \quad y(0) = 1, y(1) = 0. \]  

Solving System (2.4) using the coefficients: $a(x) = 1, b(x) = -1$, then $\lambda$ can be easily identified as:

\[ \lambda(x, t) = \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon} (t-x) - \frac{\mu - \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon} (t-x) \right]. \]

Therefore, we have the following iteration formula:

\[ y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ e^{\frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon} (t-x)} - e^{\frac{\mu - \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon} (t-x)} \right] \left\{ \varepsilon y_n''(t) + \mu y_n'(t) - y_n(t) + t \right\} dt. \]
Now, we begin with an arbitrary initial approximation:

\[ y_0(x) = C_1 e^{\frac{1}{2x} (\mu + \sqrt{\mu^2 + \varepsilon}) x} + C_2 e^{\frac{1}{2x} (\mu - \sqrt{\mu^2 + \varepsilon}) x}, \]

where \( C_1 \) and \( C_2 \) are constants to be determined. By the variational iteration formula, we have

\[ y_1(x) = y_0(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ e^{\frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{2x}(t-x)} - e^{\frac{\mu - \sqrt{\mu^2 + 4\varepsilon}}{2x}(t-x)} \right] \left\{ \varepsilon y_0''(t) + \mu y_0'(t) - y_0(t) + t \right\} dt. \]

By imposing the boundary conditions at \( x = 0 \) and \( x = 1 \), Mathematica software identifies the function \( y_1(x) \) to be considered as the approximate solution to BVP (3.2). Figures 4, 5, 6 show the errors compared with the exact solution for different values of \( \varepsilon \) and \( \mu \).

**Figure 4. Example 2.** The errors using the approximate solution \( y_1(x) \), \( 0 \leq x \leq 1 \), for the case \( \varepsilon = 0.005 \) and \( 0 < \mu < 0.005 \).
Figure 5. Example 2. The errors using the approximate solution \( y_1(x), 0 \leq x \leq 1 \), for the case \( \mu = 0.005 \) and \( 0.002 < \epsilon < 0.005 \).

Figure 6. Example 2. The errors using the approximate solution \( y_1(x), x = 0.5 \), for the case \( 0.005 < \mu < 0.009 \) and \( 0.005 < \epsilon < 0.009 \).
Example 3.

Now we consider the following example Valarmathi and Ramanujam (2003):

\[ \varepsilon y^*(x) + 2\mu y'(x) - 4y(x) = -x; \quad y(0) = 0, y(1) = 1, \]  

(3.3)

Solving system (2.4), using the coefficients: \( a(x) = 2, b(x) = -4 \), then \( \lambda \) can be easily be identified as:

\[ \lambda(x,t) = \frac{1}{2\sqrt{\mu^2 + 4\varepsilon}} \left[ \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{\varepsilon} (t-x) - e^{\frac{-\mu - \sqrt{\mu^2 + 4\varepsilon}}{\varepsilon} (t-x)} \right]. \]

Therefore, we have the following iteration formula:

\[ y_{n+1}(x) = y_n(x) + \int_0^x \frac{1}{2\sqrt{\mu^2 + 4\varepsilon}} \left[ \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{\varepsilon} (t-x) - e^{\frac{-\mu - \sqrt{\mu^2 + 4\varepsilon}}{\varepsilon} (t-x)} \right] \left\{ \varepsilon y_n''(t) + 2\mu y_n'(t) - 4y_n(t) + t \right\} dt. \]

Now, we begin with an arbitrary initial approximation:

\[ y_0(x) = C_1 e^{\frac{-1}{\varepsilon}(\mu + \sqrt{\mu^2 + 4\varepsilon})x} + C_2 e^{\frac{-1}{\varepsilon}(\mu - \sqrt{\mu^2 + 4\varepsilon})x}, \]

where \( C_1 \) and \( C_2 \) are constants to be determined. By the variational iteration formula, we have

\[ y_1(x) = y_0(x) + \int_0^x \frac{1}{2\sqrt{\mu^2 + 4\varepsilon}} \left[ \frac{\mu + \sqrt{\mu^2 + 4\varepsilon}}{\varepsilon} (t-x) - e^{\frac{-\mu - \sqrt{\mu^2 + 4\varepsilon}}{\varepsilon} (t-x)} \right] \left\{ \varepsilon y_0''(t) + 2\mu y_0'(t) - 4y_0(t) + t \right\} dt. \]

By imposing the boundary conditions at \( x = 0 \) and \( x = 1 \), Mathematica software identifies the function \( y_1(x) \) to be considered as the approximate solution to BVP (3.3). Figures 7, 8, 9 show the errors compared with the exact solution for different values of \( \varepsilon \) and \( \mu \).
Figure 7. Example 3. The errors using the approximate solution $y_1(x), 0 \leq x \leq 1$, for the case $\varepsilon = 0.005$ and $0 < \mu < 0.005$.

Figure 8. Example 3. The errors using the approximate solution $y_1(x), 0 \leq x \leq 1$, for the case $\mu = 0.005$ and $0.002 < \varepsilon < 0.005$. 
Example 3. The errors using the approximate solution

\[ y(x) = 0.5, \quad x_0 = 0.005 < \mu < 0.009 \]

and \( 0.005 < \epsilon < 0.009 \).

Example 4.

Now we consider the following example Kadalbajoo and Yadaw (2008):

\[ \varepsilon y^*(x) - \mu y'(x) - y(x) = -\cos \pi x; \quad y(0) = 0, y(1) = 0, \quad (3.4) \]

Solving system (2.4) using the coefficients: \( a(x) = -1, b(x) = -1 \), then \( \lambda \) can be easily be identified as:

\[
\lambda(x,t) = \frac{1}{\sqrt{\mu^2 + 4\epsilon}} \begin{bmatrix}
\frac{-\mu + \sqrt{\mu^2 + 4\epsilon}}{2\epsilon} & \frac{-\mu - \sqrt{\mu^2 + 4\epsilon}}{2\epsilon}
\end{bmatrix}
\]

Therefore, we have the following iteration formula:

\[
y_{n+1}(x) = y_n(x) + \int_0^1 \frac{1}{\sqrt{\mu^2 + 4\epsilon}} \begin{bmatrix}
\frac{-\mu + \sqrt{\mu^2 + 4\epsilon}}{2\epsilon} & \frac{-\mu - \sqrt{\mu^2 + 4\epsilon}}{2\epsilon}
\end{bmatrix}
\begin{bmatrix}
\varepsilon y_n''(t) - \mu y_n'(t) - y_n(t) + \cos \pi t
\end{bmatrix} dt.
\]

Now, we begin with an arbitrary initial approximation:
where $C_1$ and $C_2$ are constants to be determined. By the variational iteration formula, we have

$$y_1(x) = y_0(x) + \int_0^x \frac{1}{\sqrt{\mu^2 + 4\varepsilon}} \left[ e^{\frac{-\mu^2 + 4\varepsilon}{2\varepsilon}(x-t)} - e^{\frac{-\mu^2 + 4\varepsilon}{2\varepsilon}(t-x)} \right] \left\{ \varepsilon y''_0(t) - \mu y'_0(t) - y_0(t) + \cos \pi t \right\} dt.$$ 

By imposing the boundary conditions at $x = 0$ and $x = 1$, Mathematica software identifies the function $y_1(x)$ to be considered as the approximate solution to BVP (3.4). Figures 10, 11, 12 show the errors compared with the exact solution for different values of $\varepsilon$ and $\mu$.

**Figure 10.** Example 4. The errors using the approximate solution $y_1(x), 0 \leq x \leq 1$, for the case $\varepsilon = 0.005$ and $0 < \mu < 0.005$. 

$$y_0(x) = C_1 e^{\frac{-1}{2\varepsilon}(\mu + \sqrt{\mu^2 + 4\varepsilon})x} + C_2 e^{\frac{-1}{2\varepsilon}(\mu - \sqrt{\mu^2 + 4\varepsilon})x},$$
4. Conclusion
He’s Variational iteration method was employed successfully for solving linear singularly perturbed two-point boundary value problems. The results show that a good choice of the freely selected initial approximation in the VIM leads to a very good approximation by considering only one iteration.

Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear two-parameter singularly perturbed two-point boundary value problems.

Acknowledgement

I would like to thank the Editor and the anonymous referees for their in-depth reading, criticism of, and insightful comments on an earlier version of this paper.

REFERENCES


