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Uniform Stabilization of *n*-Dimensional Vibrating Equation Modeling 'Standard Linear Model' of Viscoelasticity

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Abstract

In this paper, we deal with the elastic vibrations of flexible structures modeled by the 'standard linear model' of viscoelasticity in n-dimensional space. We study the uniform exponential stabilization of such kind of vibrations after incorporating separately very small amount of passive viscous damping and internal material damping of Kelvin-Viogt type in the model. Explicit forms of exponential energy decay rates are obtained by a direct method, for the solution of such boundary value problems without having to introduce any boundary feedback.

Keywords: Uniform stabilization; Kelvin-Voigt damping; Standard linear model of viscoelasticity; Exponential energy decay estimate

MSC 2000 No.: 35L35, 37L15, 74H55, 93D20

1. Introduction and Mathematical Formulation

The mathematical theory of stabilization of distributed parameter system is currently a subject of interest in several practical fields. In fact, recent studies on stabilization of mechanical systems have gained in importance due to application of vibration control in various structural elements. The dynamics of linear vibrations of elastic structure are mathematically governed by the wave equation

$$y''(x,t) = c^2 \Delta y(x,t) \tag{1}$$

in some suitable domain, where Δ denotes the Laplacian taken in space variable x, prime ($'$) the differentiation with respect to time coordinate t and $c > 0$ is the constant wave velocity. The dynamical equation (1) is formulated on the basis of Hook's law, in which stress σ is simply proportional to strain e, that means $\sigma = Ee$, E being the Young's modulus of the elastic structure. But the dynamics of elastic vibrations of flexible structures are actually nonlinear in practice. It is rather cumbersome for analytical treatment of a non-linear problem and then the result so obtained will not generally in precise form. The linearized mathematical models which describe a true physical phenomenon almost accurately to some extent are much sought after purely for simplicity and for concise results.

In this paper, we are looking into a more realistic linear model of nonlinear vibrations of elastic structure commonly known as the 'standard linear model' of viscoelasticity (cf. Fung (1968)), in which stress σ is not simply proportional to strain e. In this model, a linear spring is connected in series with a combination of another linear spring and a dashpot in parallel and the corresponding stress-strain formula of the elastic structure is described by the constitutive relation (cf. Fung (1968) and Rabotnov (1980))

$$\sigma + \lambda\sigma' = E(e + \mu e'). \tag{2}$$

Here λ, μ are small constants satisfying $0 < \lambda < \mu$. As a result, the dynamics of vibrations of the elastic structures are governed more accurately by the third order differential equation

$$y''(x,t) + \lambda y'''(x,t) = c^2\left(\Delta y(x,t) + \mu\Delta y'(x,t)\right) \tag{3}$$

than the simple wave equation. Our aim is to study stabilization of the mathematical problem (3) in a domain $\Omega \times \mathbb{R}^+$, under undamped mixed boundary conditions

$$y = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad \frac{\partial y}{\partial v} = 0 \text{ on } \Gamma_1 \times \mathbb{R}^+ \tag{4}$$

together with the initial conditions

$$y(x,0) = y_0(x), \;\; y'(x,0) = y_1(x) \;\text{ and }\; y''(x,0) = y_2(x) \;\text{ in }\; \Omega, \tag{5}$$

where Ω is a bounded connected set in $\mathbb{R}^n$ ($n \geq 1$) having a smooth boundary $\Gamma = \partial\Omega$, consisting of two parts Γ_0 and Γ_1 such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. Here, v denotes the unit normal of Γ pointing towards exterior of Ω and $\mathbb{R}^+ := (0, \infty)$.

Theoretical studies on uniform energy decay at exponential rate of vibrating structures modeled by 'standard linear model' of viscoelasticity have become extremely important for the purpose of design of various material structures. Physically, the partial differential equation (3) occurs in the study of vibrations of a elastic structure modeling 'standard linear model' of viscoelasticity in a bounded domain in $\mathbb{R}^n$. The constitutive relation (2) provides the dynamics of vibrations of a real elastic material better than ordinary Hooke's Law (cf. Fung (1968) and Rabotnov (1980)). The boundary conditions considered here are of mixed Dirichlet and Neumann type, and the boundary is absolutely free from any action on it. Such kind of boundary conditions was used by Gorain (1997), (2006), (2007) to establish uniform stability results of an internally damped wave and a nonlinear Kirchhoff wave in a bounded domain in $\mathbb{R}^n$. The mathematical model like (3) was studied earlier by Bose and Gorain (1998) incorporating a viscous feedback damping on the boundary Γ_1.

Now, for the sake of simplicity, we set

$$u := y + \lambda y'. \tag{6}$$

The equivalent form of the governing differential equation (3) is then

$$u'' = c^2 \Delta u + c^2 (\mu - \lambda) \Delta y' \quad \text{ in } \Omega \times \mathbb{R}^+. \tag{7}$$

The corresponding boundary conditions in (4) become

$$u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+ \tag{8}$$

and the initial conditions in (5) reduce to

$$u(x,0) = u_0(x) \quad \text{ and } \quad u'(x,0) = u_1(x) \quad \text{ in } \Omega, \tag{9}$$

where

$$u_0 = y_0 + \lambda y_1 \quad \text{ and } \quad u_1 = y_1 + \lambda y_2 \text{ in } \Omega, \tag{10}$$

by virtue of (6). The corresponding system (7)–(9) together with the relation (6) is thus equivalent to the original system (3)–(5).

There has been extensive work in the last two decades on the problems of boundary stabilization for the solution of wave equation in a bounded domain by means of uniform energy decay estimate (cf. Chen (1981), Chen and Zhou (1990), Komornik (1991), Komornik and Zuazua (1990), Lagnese (1988), Lions (1988) and the references therein). All their investigations have shown the stability of wave equation, clamped at one end and a feedback viscous damping at the other end. On the contrary, the problem of viscoelastic structure of Kelvin-Voigt model with a movable mass and a viscous damper at the held end is treated by Gorain and Bose (1998), (2002) for torsional and flexural modes of vibrations.

Recent advances in material sciences have provided new means for the suppression of vibrations of elastic structures. The three most common classes of vibration control mechanism are of passive, active and hybrid type. In fact, the passive vibration control mechanism plays an important role for the suppression of vibrations, which uses resistive device that absorbs vibration energy. Viewed in the context of recent developments, we are concerned about uniform stabilization of two mathematical problems governed by the following partial differential equations and the boundary-initial conditions:

$$u'' + 2\delta u' = c^2 \Delta u + c^2 (\mu - \lambda)\Delta y' \quad \text{in } \Omega \times \mathbb{R}^+. \tag{11}$$

$$u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+ \tag{12}$$

$$u(x,0) = u_0(x), \qquad u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{13}$$

and

$$u'' = c^2 \Delta u + c^2 (\mu - \lambda)\Delta y' + 2\beta \Delta u' \quad \text{in } \Omega \times \mathbb{R}^+. \tag{14}$$

$$u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+ \tag{15}$$

$$u(x,0) = u_0(x), \qquad u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{16}$$

where the parameters $\delta > 0$, $\beta > 0$ are very small constants and u is related by y according to (6). The term $2\delta u'$ in (11) is the distributed damping of passive viscous type where as the term $2\beta\Delta u'$ in (14) is the internal damping of Kelvin-Voigt type of the structure. In fact, the internal material damping mechanism usually non-linear in nature is always present, however small it may be, in real materials (cf. Christensen (1971)) as long as the system vibrates. The most acceptable two linearized forms of that are incorporated here, for just to avoid the non-linearity.

In this paper, we proceed with the idea as in Gorain (2006) to study separately, the uniform stabilization of above two mathematical systems (11)–(13) and (14)–(16). In other words, we wish to show exponential decay of the energy functional E: $[0, \infty) \to [0, \infty)$ defined by (cf. Bose and Gorain (1998))

$$E(t) = \frac{1}{2}\int_{\Omega} [u'^2 + c^2 |\nabla u|^2 + c^2 \lambda(\mu - \lambda) |\nabla y'|^2] dx, \tag{17}$$

for every solution of the systems (11)–(13) and (14)–(16) separately. To establish such a result, our approach is to be a direct method by constructing suitable Lyapunov functional related to the energy functional E, without going through the literature of semigroup theory (cf. Pazy (1983)).

In the sequel, we use following two inequalities: For any real number $\alpha > 0$, we have by the Cauchy-Schwartz's inequality

$$|f \cdot g| \le \frac{1}{2\alpha}(|f|^2 + \alpha^2 |g|^2). \tag{18}$$

Let k be the smallest positive constant independent of t (depends only on Ω) satisfying the Poincaré inequality (cf. Aubin (1979))

$$\int_{\Omega} u^2 dx \le k \int_{\Omega} |\nabla u|^2 \, dx, \tag{19}$$

for every $u \in H^1_{\Gamma_0}(\Omega)$, where $H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_0\}$, the subspace of the classical Sobolev space $H^1(\Omega)$ of real valued functions of order one.

For example, let $u(x, t)$ be a function defined on $[0, L] \times \mathbb{R}^+$ with $u(0, t) = u(L, t) = 0$ for all $t \in \mathbb{R}^+$, then Poincaré type Scheffer's inequality (cf. Mitrinović et al. (1991)) yields

$$\int_0^L u^2 dx \le \frac{L^2}{\pi^2} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx, \tag{20}$$

which estimates a value $\frac{L^2}{\pi^2}$ for the constant k depending on the interval $[0, L]$.

2. Uniform Stability on Account of Passive Viscous Damping

Taking time derivative of (17) and using the governing equation (11), we obtain

$$E'(t) = c^2 \int_{\Omega} \left[u'\left(\Delta u + (\mu - \lambda)\Delta y'\right) + (\nabla u \cdot \nabla u') + \lambda(\mu - \lambda)(\nabla y' \cdot \nabla y'') \right] dx - 2\delta \int_{\Omega} (u')^2 \, dx.$$

Applying Green's formula, we get

$$E'(t) = c^2 \int_{\Gamma} u' \left[\frac{\partial u}{\partial \nu} + (\mu - \lambda) \frac{\partial y'}{\partial \nu} \right] d\Gamma$$
$$+ c^2 (\mu - \lambda) \int_{\Omega} \left[\lambda(\nabla y' \cdot \nabla y'') - (\nabla u' \cdot \nabla y') \right] dx - 2\delta \int_{\Omega} (u')^2 \, dx.$$

Using the relation (6) and the boundary conditions (12), we finally obtain

$$E'(t) = -c^2 (\mu - \lambda) \int_{\Omega} |\nabla y'|^2 dx - 2\delta \int_{\Omega} (u')^2 \, dx \qquad \forall \; t \in \mathbb{R}^+. \tag{21}$$

Thus, it follows from (21) that $E'(t) < 0$ for every $t \in \mathbb{R}^+$, which implies that energy E of the system (11)–(13) is a decreasing function of time and hence as a whole, the system (11)–(13) is energy dissipating. Hence, we have

$$E(t) < E(0) \qquad \forall \;\; t \in \mathbb{R}^+, \tag{22}$$

where

$$E(0)=\frac{1}{2}\int_{\Omega}\left[(u_1)^2+c^2\,|\nabla u_0|^2+c^2\lambda(\mu-\lambda)\,|\nabla y_1|^2\right]dx. \tag{23}$$

Remark 2.1: According to the definition of u_0 and u_1 in (10), it is clear that E(0) < ∞ for every initial data $y_0\in H^1_{\Gamma_0}(\Omega)$, $y_1\in H^1_{\Gamma_0}(\Omega)$ and $y_2\in L^2(\Omega)$. Because, in this case,

$$\int_{\Omega}|\nabla y_0|^2\,dx<\infty,\ \int_{\Omega}|\nabla y_1|^2\,dx<\infty,\ \text{and}\ \int_{\Omega}(y_1)^2\,dx<\infty$$

and, hence,

$$u_0\in H^1_{\Gamma_0}(\Omega)\quad\text{and}\quad \mathrm{u}_1\in L^2(\Omega).$$

As the system evolves from its initial state $\{u_0, u_1\}$, the energy $E(t)$ at any time $t\in\mathbb{R}^+$ diminishes from its initial value $E(0)$ driven by the work done by the small passive viscous damping. The uniform stability of the system (11)–(13) by means of exponential energy decay estimate can be obtained by the following theorem.

Theorem 2.1.: If $y = y(x, t)$ be a solution of the system (11) − (13) with initial values $y_0\in H^1_{\Gamma_0}(\Omega)$, $y_1\in H^1_{\Gamma_0}(\Omega)$ and $y_2\in L^2(\Omega)$, then the energy $E(t)$ of the system decays uniformly exponentially with time, that means

$$E(t)<M_\delta\exp(-\xi_\delta t)E(0)\qquad\forall\ \mathrm{t}\in\mathbb{R}^+, \tag{24}$$

for some finite reals $M_\delta>1$ and $\xi_\delta>0$ defined later, both dependent on the damping parameter δ.

Firstly, we establish the following lemma:

Lemma 2.1.: For every solution $y = y(x, t)$ of the system (11)−(13), the time derivative of the functional G_δ (cf. Bose and Gorain (1998), Gorain (2006), Komornik and Zuazua (1990)) defined by

$$G_\delta(t)=\int_{\Omega}\left[uu'+\delta u^2+\frac{c^2}{2}(\mu-\lambda)\,|\nabla y|^2\right]dx \tag{25}$$

satisfies

$$G'_\delta(t)=2\int_{\Omega}(u')^2\,dx-2E(t)\qquad\forall t\in\mathbb{R}^+. \tag{26}$$

Proof:

If we differentiate (25) with respect to t and replace u'' by the relation (11), then

$$G'_\delta(t) = c^2 \int_\Omega \left[u\left(\Delta u + (\mu-\lambda)\Delta y'\right) + (\mu-\lambda)(\nabla y \cdot \nabla y')\right] dx + \int_\Omega (u')^2\, dx.$$

On application of Green's formula, the above gives

$$\begin{aligned} G'_\delta(t) &= c^2 \int_\Gamma u\left[\frac{\partial u}{\partial \nu} + (\mu-\lambda)\frac{\partial y'}{\partial \nu}\right] d\Gamma - c^2 \int_\Omega \left[\left(\nabla u + (\mu-\lambda)\nabla y'\right)\cdot \nabla u\right] dx \\ &\quad + c^2(\mu-\lambda)\int_\Omega \lambda(\nabla y \cdot \nabla y')dx + \int_\Omega (u')^2\, dx. \\ &= \int_\Omega \left[(u')^2 - c^2 \,|\nabla u|^2\right] dx - c^2\lambda(\mu-\lambda)\int_\Omega |\nabla y'|^2\, dx, \end{aligned} \tag{27}$$

by the use of the boundary conditions (12) and the relation (6). Invoking $E(t)$ as defined in (17) into (27), the lemma follows immediately.

***Proof of Theorem* 2.1:**

Proceeding as in Gorain (2006) (see also in Bose and Gorain (1998), Gorain (2006), Komornik (1991)), we define an energy like Lyapunov functional V_δ: $[0, \infty) \to [0, \infty)$ by

$$V_\delta(t) = E(t) + \delta G_\delta(t). \tag{28}$$

With the help of (17) and (25), the above can be written as

$$\begin{aligned} V_\delta(t) &= \frac{1}{2}\int_\Omega (u' + \delta u)^2\, dx + \frac{1}{2}\delta^2 \int_\Omega u^2 dx \\ &\quad + \frac{c^2}{2}\int_\Omega \left[|\nabla u|^2 + \lambda(\mu-\lambda)\,|\nabla y'|^2 + \delta(\mu-\lambda)\,|\nabla y|^2\right] dx, \geq 0 \qquad \forall t \geq 0. \end{aligned} \tag{29}$$

Now, using the inequalities (18) and (19), we can write

$$\left|\int_\Omega uu'dx\right| \leq \frac{\sqrt{k}}{2c}\int_\Omega \left[(u')^2 + \frac{c^2}{k}u^2\right] dx \leq \frac{\sqrt{k}}{c} E(t) \tag{30}$$

and

$$0 \leq \delta \int_\Omega u^2 dx \leq \frac{2\delta k}{c^2} E(t) \qquad \forall t \geq 0. \tag{31}$$

Again by the relation (6), we can write

$$
\begin{aligned}
|\nabla y|^2 &= (\nabla u-\lambda\nabla y')^2 = |\nabla u|^2+\lambda^2|\nabla y'|^2-2\lambda(\nabla u\cdot\nabla y') \\
&\le |\nabla u|^2+\lambda^2|\nabla y'|^2+\frac{\lambda}{\mu-\lambda}\left[|\nabla u|^2+(\mu-\lambda)^2|\nabla y'|^2\right] \\
&= \frac{\mu}{\mu-\lambda}\left[|\nabla u|^2+\lambda(\mu-\lambda)|\nabla y'|^2\right]
\end{aligned}
$$

by the help of the inequality (18). Hence,

$$
0 \le \frac{c^2}{2}(\mu-\lambda)\int_\Omega |\nabla y|^2\,dx \le \mu E(t) \qquad \forall t \ge 0. \tag{32}
$$

On account of the inequalities (30), (31) and (32), G_δ (as defined by (25)) can be estimated as

$$
-\frac{\sqrt{k}}{c}E(t) \le G_\delta(t) \le \left(\frac{\sqrt{k}}{c}+\frac{2\delta k}{c^2}+\mu\right)E(t) \qquad \forall t \ge 0. \tag{33}
$$

Hence, it follows from (28) that

$$
\left(1-\frac{\sqrt{k}}{c}\delta\right)E(t) \le V_\delta(t) \le \left[1+\left(\frac{\sqrt{k}}{c}+\mu\right)\delta+\frac{2k}{c^2}\delta^2\right]E(t), \qquad \forall t \ge 0. \tag{34}
$$

For smallness of δ, we assume that

$$
\delta < \frac{c}{\sqrt{k}} \tag{35}
$$

so that (34) yields $V_\delta(t) > 0$ for every $t \ge 0$.

Next, differentiating (28) with respect to t and using the expression (21) and the Lemma 2.1, we obtain

$$
V_\delta'(t) = -2\delta E(t) - c^2(\mu-\lambda)\int_\Omega |\nabla y'|^2 dx, < -2\delta E(t), \qquad \forall t \in \mathbb{R}^+. \tag{36}
$$

Invoking the right inequality of (34), we obtain the differential inequality

$$
V_\delta'(t)+\xi_\delta V_\delta(t) < 0, \quad \forall t \in < -2\delta E(t), \quad \forall t \in \mathbb{R}^+, \tag{37}
$$

where

$$\xi_\delta = \frac{2\delta}{1+(\frac{\sqrt{k}}{c}+\mu)\delta+\frac{2k}{c^2}\delta^2} > 0, \tag{38}$$

the denominator of ξ_δ being a quadratic function of the passive viscous damping parameter δ. Multiplying (37) by $\exp(\xi_\delta t)$ and integrating from 0 to t, we obtain the estimate

$$V_\delta(t) < \exp(-\xi_\delta t)V_\delta(0) \qquad \forall t \in \mathbb{R}^+. \tag{39}$$

Again applying the inequality (34) in (39), we finally obtain for every $\forall t \in \mathbb{R}^+$,

$$E(t) < M_\delta \exp(-\xi_\delta t)E(0), \tag{40}$$

where

$$M_\delta = \frac{1+(\frac{\sqrt{k}}{c}+\mu)\delta+\frac{2k}{c^2}\delta^2}{1-\frac{\sqrt{k}}{c}\delta} > 1. \tag{41}$$

Hence, the theorem.

Remark 2.2.: This result shows that the solution of the system decays uniformly exponentially with time, for every initial data $y_0 \in H^1_{\Gamma_0}(\Omega)$, $y_1 \in H^1_{\Gamma_0}(\Omega)$ and $y_2 \in L^2(\Omega)$. As the system is uniformly stable, it is controllable in particular, from an arbitrary initial state to a desired final state in the energy space. Again, the expression for ξ_δ in (38) as a function of passive damping parameter δ shows that decay rate will be maximum for $\delta = \frac{c}{\sqrt{2k}}$, satisfying the restriction given by (35). Because

$$\frac{d\xi_\delta}{d\delta} = \frac{2(1-\frac{2\delta^2 k}{c^2})}{\left[1+(\frac{\sqrt{k}}{c}+\mu)\delta+\frac{2k}{c^2}\delta^2\right]^2}, \tag{42}$$

that implies $\frac{d\xi_\delta}{d\delta} = 0$ and $\frac{d^2\xi_\delta}{d\delta^2} < 0$ for $\delta = \frac{c}{\sqrt{2k}}$. The corresponding maximum value of ξ_δ is given by

$$\xi_\delta(\max) = \frac{2c}{(2\sqrt{2}+1)\sqrt{k}+\mu c} \tag{43}$$

and the value of M_δ is, then,

$$M_\delta = \frac{(2\sqrt{2}+1)\sqrt{k}+\mu c}{(\sqrt{2}-1)\sqrt{k}}. \tag{44}$$

In practical cases, the value of the passive damping parameter δ is much smaller than its upper bound $\dfrac{c}{\sqrt{k}}$ given in (35).

3. Uniform Stability on Account of Internal Damping of Kelvin-Voigt Type

If we differentiate (17) with respect to time t and use the governing equation (14), then

$$E'(t) = c^2\int_\Omega \left[u'\left(\Delta u + (\mu-\lambda)\Delta y'\right) + (\nabla u\cdot\nabla u') + \lambda(\mu-\lambda)(\nabla y'\cdot\nabla y'')\right]dx + 2\beta\int_\Omega u'\Delta u' dx.$$

Applying Green's formula, we have from above

$$E'(t) = c^2\int_\Gamma u'\left[\frac{\partial u}{\partial \nu} + (\mu-\lambda)\frac{\partial y'}{\partial \nu}\right]d\Gamma + 2\beta\int_\Gamma u'\frac{\partial u'}{\partial \nu}d\Gamma$$
$$+ c^2(\mu-\lambda)\int_\Omega \left[\lambda(\nabla y'\cdot\nabla y'') - (\nabla u'\cdot\nabla y')\right]dx - 2\beta\int_\Omega |\nabla u'|^2\, dx.$$

Using the boundary conditions in (15) and the relation (6), we finally get

$$E'(t) = -c^2(\mu-\lambda)\int_\Omega |\nabla y'|^2 dx - 2\beta\int_\Omega |\nabla u'|^2\, dx < 0, \qquad \forall\ t\in\mathbb{R}^+. \tag{45}$$

The negativity of right hand side of (45) shows that this system also is an energy dissipating and thus the system (14)–(16) is a non-conserving. So naturally, the question arises as to whether, the solution of this system also decays with time uniformly or not. An affirmative answer can be found by the following theorem.

Theorem 3.1.: If $y = y(x, t)$ be a solution of the system (14) − (16) with initial values $(y_0, y_1, y_2) \in H^1_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$, then the solution of the system decays uniformly exponentially with time, that means, $\forall\ t\in\mathbb{R}^+$, the energy $E(t)$ satisfies the result

$$E(t) < M_\beta \exp(-\xi_\beta t)E(0), \tag{46}$$

for some reals $M_\beta > 1$ and $\xi_\beta > 0$ defined later, dependent on the damping parameter β.

Before establishing the theorem, we now prove the following lemma.

Lemma 3.1.: For every solution $y = y(x, t)$ of the system (14)−(16), the time derivative of the functional G_β (cf. Bose and Gorain (1998), Gorain (2006), Komornik and Zuazua (1990)) defined by

$$G_\beta(t) = \int_\Omega \left[uu' + \beta |\nabla u|^2 + \frac{c^2}{2}(\mu - \lambda)|\nabla y|^2 \right] dx \tag{47}$$

satisfies

$$G'_\delta(t) \le 2k \int_\Omega |\nabla u'|^2 \, dx - 2E(t), \qquad \forall t \in \mathbb{R}^+, \tag{48}$$

where k is a constant defined by (19).

Proof:

A differentiation with respect to t of (47) gives

$$G'_\delta(t) = c^2 \int_\Omega \left[u\left(\Delta u + (\mu - \lambda)\Delta y'\right) + (\mu - \lambda)(\nabla y \cdot \nabla y') \right] dx$$
$$+ \int_\Omega \left[(u')^2 + 2\beta(u\Delta u' + \nabla u \cdot \nabla u') \right] dx$$

by the use of (14). Applying Green's formula, we have

$$G'_\delta(t) = c^2 \int_\Gamma u \left[\frac{\partial u}{\partial \nu} + (\mu - \lambda)\frac{\partial y'}{\partial \nu} \right] d\Gamma + 2\beta \int_\Gamma u \frac{\partial u}{\partial \nu} d\Gamma - c^2 \int_\Omega \left[\left(\nabla u + (\mu - \lambda)\nabla y'\right) \cdot \nabla u \right] dx$$
$$+ c^2(\mu - \lambda)\int_\Omega (\nabla y \cdot \nabla y') dx + \int_\Omega (u')^2 dx$$
$$= \int_\Omega \left[(u')^2 - c^2 |\nabla u|^2 \right] dx - c^2\lambda(\mu - \lambda)\int_\Omega |\nabla y'|^2 \, dx, \tag{49}$$

where we have used the boundary conditions (15) and the relation (6). After introducing $E(t)$ in (49), the lemma follows immediately.

***Proof of Theorem* 3.1:**

Proceeding as in the last section, we consider an energy like Lyapunov functional $V_\beta : [0, \infty) \to [0, \infty)$ by

$$V_\beta(t) = E(t) + \frac{\beta}{k} G_\beta(t) \tag{50}$$

$$\begin{aligned} &\geq \frac{1}{2}\int_\Omega \left(u' + \frac{\beta}{k}u\right)^2 dx + \frac{\beta^2}{2k^2}\int_\Omega u^2 dx \\ &\qquad + \frac{c^2}{2}\int_\Omega \left[|\nabla u|^2 + \lambda(\mu-\lambda)|\nabla y'|^2 + \frac{\beta}{k}(\mu-\lambda)|\nabla y|^2 \right] dx \ \geq 0, \quad \forall t \geq 0 \end{aligned} \tag{51}$$

by the use of (17), (47) and the inequality (19).

Now, according to the previous section, we can estimate three terms of G_β in (47) as

$$\left|\int_\Omega uu'dx\right| \leq \frac{\sqrt{k}}{c} E(t), \tag{52}$$

$$0 \leq \beta\int_\Omega |\nabla u|^2 \, dx \leq \frac{2\beta}{c^2} E(t) \tag{53}$$

and

$$0 \leq \frac{c^2}{2}(\mu-\lambda)\int_\Omega |\nabla y|^2 \, dx \leq \mu E(t). \tag{54}$$

Hence, (47) yields the estimate

$$-\frac{\sqrt{k}}{c} E(t) \leq G_\beta(t) \leq \left(\frac{\sqrt{k}}{c} + \frac{2\beta}{c^2} + \mu\right) E(t), \qquad \forall t \geq 0, \tag{55}$$

and so V_β defined by (50) can be estimated as

$$\left(1 - \frac{\beta}{c\sqrt{k}}\right) E(t) \leq V_\beta(t) \leq \left[1 + \left(\frac{1}{c\sqrt{k}} + \frac{\mu}{k}\right)\beta + \frac{2\beta^2}{c^2 k}\right] E(t), \qquad \forall t \geq 0. \tag{56}$$

Now, if we impose a restriction on β by

$$\beta < c\sqrt{k}\,, \tag{57}$$

Then, it follows from (56) that $V_\beta(t) > 0$ for every $t \geq 0$. Next, differentiating (50) with respect to t and using (45) and the Lemma 3.1, we obtain

$$V'_\beta(t) < -\frac{2\beta}{k}E(t). \qquad \forall t \in \mathbb{R}^+.$$

With the help (56), the above yields the differential inequality

$$V'_\beta(t) + \xi_\beta V_\beta(t) < 0, \qquad \forall t \in \mathbb{R}^+, \tag{58}$$

where

$$\xi_\beta = \frac{\frac{2\beta}{k}}{1 + (\frac{1}{c\sqrt{k}} + \frac{\mu}{k})\beta + \frac{2\beta^2}{c^2 k}} > 0, \tag{59}$$

the denominator of ξ_β being also a quadratic function of the Kelvin-Voigt damping parameter β. Multiplying (58) by $\exp(\xi_\beta t)$ and integrating over $(0, t)$, we obtain

$$V_\beta(t) < \exp(-\xi_\beta t)V_\beta(0), \qquad \forall t \in \mathbb{R}^+. \tag{60}$$

Finally, invoking the inequality (56) in (60), we get the result (46), where

$$M_\beta = \frac{1 + (\frac{1}{c\sqrt{k}} + \frac{\mu}{k})\beta + \frac{2\beta^2}{c^2 k}}{1 - \frac{\beta}{c\sqrt{k}}} > 1. \tag{61}$$

Hence, the theorem.

Remark 3.1.: The result of the theorem 3.1 implies that the solution of the system (14)–(16) also converges uniformly to zero as time $t \to +\infty$, at exponentially rate ξ_β for every initial data $y_0 \in H^1_{\Gamma_0}(\Omega)$, $y_1 \in H^1_{\Gamma_0}(\Omega)$ and $y_2 \in L^2(\Omega)$. Hence, it is controllable in particular, from an arbitrary initial state to a desired final state in the energy space. Again, from the expression for ξ_β in (59) as a function of the damping parameter β, we have

$$\frac{d\xi_\beta}{d\beta} = \frac{\frac{2}{k}(1 - \frac{2\beta^2}{c^2 k})}{\left[1 + (\frac{1}{c\sqrt{k}} + \frac{\mu}{k})\beta + \frac{2\beta^2}{c^2 k}\right]^2}. \tag{62}$$

Thus, exponential decay rate $\xi\beta$ will be greatest when $\beta = \frac{c\sqrt{k}}{\sqrt{2}}$ (the second order derivative of ξ_β being negative for this β), satisfying the restriction (57), although the actual value of β in practical cases is much smaller than its upper bound $c\sqrt{k}$. The corresponding maximum value of ξ_β is then given by

$$\xi_{\beta}(\max) = \frac{2c}{(2\sqrt{2}+1)\sqrt{k}+\mu c} \tag{63}$$

and for this $\beta = \frac{c\sqrt{k}}{\sqrt{2}}$, the value of M_β becomes

$$M_{\beta} = \frac{(2\sqrt{2}+1)\sqrt{k}+\mu c}{(\sqrt{2}-1)\sqrt{k}}. \tag{64}$$

Thus, we see that the values found in (63) and (64) are identical corresponding to those values (43) and (44) obtained in the previous section.

4. Conclusions

This study explicitly deals with uniform exponential stabilization of a class of vibration problems modeling 'standard linear model' of vicoelasticity in bounded domain in $\mathbb{R}^n$. To achieve the explicit forms of the results, we incorporate separately very small amount of two damping mechanism – one is of passive viscous damping and the other is of internal damping of Kelvin-Voigt type, without having to introduce any boundary damping. The results are valid even if one of the parts Γ_0 or Γ_1 and is empty. The procedure adopted here is direct method by constructing energy like suitable Lyapunov functional *V*. However, one may imagine such type results by the application of semigroup theory (cf. Pazy (1983)). The significant outcomes in this study are that the possible maximum exponential decay rates in two systems i.e., $\xi_\delta(\max)$ and $\xi_\beta(\max)$ are the same, although the values of ξ_δ and ξ_β according to (38) and (59) are different. Similar result is found for the corresponding M_δ and M_β according to (44) and (64). Since the formulation (3) is more general than the simple wave equation (1), the mathematical theory developed here can be realized for a class of elastic vibrations of flexible structures in *n*-dimensional space satisfying the model equation (11) or (14), such as the vibrations of elastic strings, beams, plates etc. The investigation is motivated by such considerations.

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