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Modelling the Role of Cloud Density on the Removal of Gaseous Pollutants and Particulate Matters from the Atmosphere

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Abstract

In this paper, a six dimensional nonlinear mathematical model is proposed to study the effect of the density of cloud droplets (formed due to the presence of vapors in the atmosphere) on the removal of pollutants, both gaseous and particulate, from the atmosphere. We assume that there exist six nonlinearly interacting phases in the atmosphere i.e. the vapor phase, the phase of cloud droplets, the phase of raindrops, the phase of gaseous pollutants, the phase of particulate matters and the phase of gaseous pollutants absorbed in raindrops. It is further assumed that the dynamics of the system undergo ecological type growth and nonlinear interactions. The model is analyzed qualitatively using the stability theory of ordinary differential equations and computer simulations. By analyzing the model, it is shown that under appropriate conditions, gaseous pollutants and particulate matters would be removed from the atmosphere and their respective equilibrium levels would depend upon the intensity of rain caused by cloud droplets, emission rate of pollutants, the rate of raindrops falling on the ground, etc. It is pointed out that, if due to unfavorable atmospheric conditions cloud droplets are not formed, rain may not occur and pollutants would not be removed.

Keywords: Gaseous pollutants, particulate matters, cloud droplets, raindrops, stability, simulation

AMS-MSC 2010 No.: 34D20, 93A30

1. Introduction

Environmental pollution has been recognized as a most challenging problem for developed and developing countries due to the increasing quantities of gaseous pollutants and particulate matters emitted into the atmosphere from different sources like industrial emissions, household discharges, vehicular exhausts, etc. Precipitation scavenging due to rain is one of the most important mechanisms for removal of pollutants from the atmosphere. Under appropriate atmospheric conditions, cloud droplets are formed due to cooling water vapor. Rain/precipitation occurs when growing cloud droplets are transformed into raindrops which may then remove pollutants (both gaseous and particulate matters) from the atmosphere. It is noted here that the density of the raindrops depends upon the density of cloud droplets; the denser the cloud, the more intense is the rain fall. In the atmosphere, during rain, gaseous pollutants are removed by the process of absorption by raindrops falling on the ground while particulate matters are removed by process of impaction.

Several experimental investigations have been made to study the removal of pollutants from the atmosphere by precipitation [Davies (1976), Sharma et al. (1983), Kleinman et al. (1992), Pandey et al. (1992), Pillai et al. (2001), Goncalves et al. (2002), Ravindra et al. (2003), Moore et al. (2007)]. For example, Davies (1976) studied the removal of sulfur dioxide by precipitation in an industrial area of Sheffield, U.K. and found significant reduction in its concentration after rain. Pandey et al. (1992) measured the concentrations of ozone, nitrogen dioxide, sulfur dioxide and the total suspended particulate matters (TSP) in the urban area of Varanasi city in India during 1989 and found a decrease in their concentrations in the rainy season. Pillai et al. (2001) studied wet deposition and dust fall in the city of Pune, India and emphasized the importance of wet removal. Goncalves et al. (2002) investigated atmospheric scavenging processes considering a numerical simulation through the model Regional Atmospheric Modeling System (RAMS) coupled with a one-dimensional below-cloud scavenging model in order to simulate in-cloud and below-cloud scavenging processes in the Serra Do Mar region in southeastern Brazil. The average concentration of three chemical species, SO_4^{--}, NO_3^- and NH_4^+ found in rain water, scavenged from the atmosphere, have been predicted. The variation in the spatial pattern of criteria air pollutants (SO_2, NO_2, O_3) before and during the initial rain of the monsoon at Shahdara National Ambient Air Quality Monitoring (NAAQM) station in Delhi, India in 1999 is studied and a considerable decrease in the air pollutants concentration after the initial and subsequent rain of the monsoon is obtained, Ravindra et al. (2003).

Several researchers have studied the phenomenon of the removal of pollutants by precipitation scavenging due to rain, snow or fog using mathematical models [Hales et al. (1973), Slinn (1977), Kumar (1985), Arora et al. (1991)]. In particular, Hales et al. (1973) proposed a model for predicting the rain washout of gaseous pollutants from the atmosphere. Some approximations for the wet and dry removal of particles and gases from the atmosphere have also been presented, Slinn (1977). An Eulerian model has been studied to describe the simultaneous process of removal of trace gas from the atmosphere and its absorption in raindrops by considering the precipitation scavenging of the gas present below the cloud, Kumar (1985). Kumar (1986), further, extended the above model by taking into account the process of absorption of multiple species and chemical reactions within the droplets.

Some investigations have been conducted to study the phenomenon of removal of gaseous pollutants and particulate matters using nonlinear mathematical models [Pandis and Seinfeld (1990), Naresh (2003), Naresh et al. (2007), Shukla et al. (2008a, b), Naresh and Sundar (2007, 2010)]. In this regard, Naresh et al. (2007) presented a nonlinear mathematical model to study the removal of gaseous pollutants and particulate matters from the atmosphere of a city by precipitation. They have shown that, under appropriate conditions, gaseous pollutants and particulate matters can be washed out from the atmosphere. Naresh and Sundar (2007) studied an ecological type nonlinear mathematical model for the removal of gaseous pollutants and two distinct particulate matters (smaller and larger particulate matters) by rain to see the effect of precipitation on the equilibrium levels of these pollutants in the atmosphere. Shukla et al. (2008a) presented a mathematical model for the removal of a gaseous pollutant and two particulate matters (one being formed from gaseous pollutants) by rain but did not consider the effect of vapour or cloud droplets phase. Shukla et al. (2008b) further modeled and analyzed a nonlinear mathematical model for the removal of gaseous pollutants and particulate matters from the atmosphere by precipitation considering the effect of cloud density but did not consider the effect of vapor phase.

In the above mentioned models, the vapor phase forming cloud droplets has not been taken into account to model the phenomenon of removal of pollutants from the atmosphere by precipitation [Sundar and Naresh (2012)]. They (2012) studied the removal of primary gaseous pollutants forming secondary species, from the atmosphere by precipitation due to rain with the assumption that the growth of raindrops is directly proportional to the density of cloud droplets, which are formed due to presence of vapor phase in the atmosphere. It may be noted that clouds form when atmospheric water vapor condenses into small liquid droplets. The phenomenon of interaction of gaseous pollutants with raindrops depends upon the temperature of gaseous pollutants. Due to high temperature of gaseous pollutants the depletion of raindrops may also take place by evaporation, thus enhancing the growth of vapors.

Therefore, in this paper, we propose and analyze a nonlinear mathematical model for the removal of gaseous pollutants and particulate matters from the atmosphere by precipitation incorporating the vapor phase with the above considerations. The proposed model study is limited to the situations with certain underlying assumptions. However, various generalizations can be made in further studies. For example, the interaction of one phase with the other can be taken in a more general form then a simple law of mass action as in the present study with recycling phenomena. The rate of emission of gaseous pollutants and particulate matters are taken to be constant and homogeneously distributed as the atmosphere is assumed to be calm but the effects of convection, wind speed and diffusion can also be incorporated in the modeling study. The chemical characteristics of gaseous pollutants and particulate matters and their chemical affinity with water whether in the form of rain or cloud droplets can also be taken into account in further studies.

2. Mathematical Model

In this study, our main aim is to emphasize the role of the density of cloud droplets (caused by water vapors) on the removal of gaseous pollutants and particulate matters from the atmosphere by rain. It is noted here that, when rainfall occurs (due to condensation of cloud droplets), raindrops interact with gaseous pollutants and particulate matters and remove them from the atmosphere. To model the phenomenon, the following assumptions are made:

1. In the atmosphere, water vapor is formed naturally.
2. The growth rate of cloud droplets is in direct proportion to the density of water vapor.
3. The growth rate of raindrops is in direct proportion to the density of cloud droplets.
4. The depletion of raindrops takes place due to chemical interaction with gaseous pollutants and by other natural processes.
5. The rate of emission of gaseous pollutants and particulate matters is taken to be constant, though it may be a function of time.
6. The atmosphere, under consideration, is assumed to be calm and therefore the effects of convection and diffusion in the atmosphere have not been taken into account.
7. If the pollutant species (gaseous) are hot, the raindrops upon interaction with these gaseous pollutants get vaporized and a fraction of it may re-enter the atmosphere enhancing the growth of the vapor phase.

Let $C_v(t)$, $C_d(t)$ and $C_r(t)$ be the densities of water vapor, cloud droplets and raindrops in the atmosphere respectively, $C(t)$ and $C_p(t)$ be the cumulative concentrations of gaseous pollutants and particulate matters in the atmosphere and $C_a(t)$ be the concentration of gaseous pollutants in absorbed phase. Let Q and Q_p be the cumulative emission rates of gaseous pollutants and particulate matters with their natural depletion rates δC and $\delta_p C_p$ respectively. It is assumed that the absorption of gaseous pollutants by raindrops is proportional to the concentration of gaseous pollutants and the density of raindrops (i.e. $\alpha C C_r$) with its natural depletion rate coefficient k. Further, the removal of gaseous pollutants in absorbed phase due to falling raindrops on the ground is assumed to be proportional to the density of raindrops as well as to the concentration of gaseous pollutants in absorbed phase (i.e., $\nu C_a C_r$).

Thus, the dynamics of the system is governed by the following nonlinear differential equations,

$$\frac{dC_v}{dt} = q - \mu_0 C_v + \mu r_1 C_r C, \tag{2.1}$$

$$\frac{dC_d}{dt} = \lambda C_v - \lambda_0 C_d, \tag{2.2}$$

$$\frac{dC_r}{dt} = r C_d - r_0 C_r - r_1 C_r C, \tag{2.3}$$

$$\frac{dC}{dt} = Q - \delta C - \alpha C C_r, \tag{2.4}$$

$$\frac{dC_p}{dt} = Q_p - \delta_p C_p - \alpha_p C_p C_r, \tag{2.5}$$

$$\frac{dC_a}{dt} = \alpha C C_r - k C_a - \nu C_a C_r, \tag{2.6}$$

with

$$C_v(0) \geq 0, C_d(0) \geq 0, C_r(0) \geq 0, C(0) \geq 0, C_p(0) \geq 0, C_a(0) \geq 0 .$$

In the model, let q be the rate of formation of vapors and $\mu_0 C_v$ the depletion of vapor phase caused by natural factors as well as by formation of cloud droplets. Let λ $(\lambda \leq \mu_0)$ be the growth rate of cloud droplets (formed due to the presence of vapor phase) and $\lambda_0 C_d$ the depletion of cloud droplets caused by natural factors as well as by formation of raindrops. Let r $(r \leq \lambda_0)$ be the growth rate of raindrops (due to cloud droplets) and r_0 its natural depletion rate coefficient.

The depletion of raindrops is assumed to be in direct proportion to the number density of raindrops as well as the concentration of gaseous pollutants (i.e., $r_1 C_r C$) and a part of it (i.e., $\mu r_1 C_r C$, $0 \leq \mu \leq 1$) may re-enter the atmosphere enhancing the growth of vapors. The constants δ, δ_p and k are the natural removal rate coefficients of C, C_p and C_a respectively and the constants α, α_p and ν are the removal rate coefficients of C, C_p and C_a respectively due to interactions with C_r. All the constants considered here are taken to be non-negative.

It is remarked here that if r_1 is very large for a given concentration C, due to unfavorable atmospheric conditions, $\frac{dC_r}{dt}$ may become negative. In such a case, no raindrops formation would take place and pollutants would not be removed from the atmosphere. It is also remarked here that, if due to unfavorable atmospheric conditions, there is no cloud formation, rain may not occur and the pollutants would not be removed from the atmosphere.

In the following, we analyze the nonlinear model (2.1) – (2.6) by using the stability theory of differential equations.

3. Boundedness of Solutions

To analyze the model (2.1) – (2.6), we need the bounds of the dependent variables involved in the dynamical system. For this, we state the region of attraction in the form of following lemma,

Lemma 3.1

Let the initial conditions be $C_v(0) \geq 0, C_d(0) \geq 0, C_r(0) \geq 0, C(0) \geq 0, C_p(0) \geq 0, C_a(0) \geq 0$ for all $t \geq 0$, then the set

$$\Omega = \left\{ (C_v, C_d, C_r, C, C_p, C_a) : 0 \leq C_v + C_d + C_r \leq \frac{q}{\lambda_m},\ 0 \leq C + C_a \leq \frac{Q}{\delta_m}, 0 \leq C_p \leq \frac{Q_p}{\delta_p} \right\}$$

attracts all solutions initiating in the interior of the positive octant, where $\lambda_m = \min\{\mu_0 - \lambda, \lambda_0 - r, r_0\}$ and $\delta_m = \min\{\delta, k\}$.

Proof:

From equations (2.1) – (2.3), we have

$$\frac{d}{dt}(C_v + C_d + C_r) = q - (\mu_0 - \lambda)C_v - (\lambda_0 - r)C_d - r_0 C_r - (1-\mu) r_1 C_r C$$

$$\leq q - (\mu_0 - \lambda)C_v - (\lambda_0 - r)C_d - r_0 C_r$$

$$\leq q - \lambda_m (C_v + C_d + C_r),$$

where

$$\lambda_m = \min\{\mu_0 - \lambda, \lambda_0 - r, r_0\}.$$

Thus, we have

$$\operatorname*{Lim}_{t\to\infty} \sup(C_v + C_d + C_r) \leq \frac{q}{\lambda_m}.$$

Again, from equations (2.4) and (2.6), we have

$$\frac{d}{dt}(C + C_a) = Q - \delta C - k C_a - \nu C_a C_r$$

$$\leq Q - \delta C - k C_a$$

$$\leq Q - \delta_m (C + C_a),$$

where

$$\delta_m = \min\{\delta, k\}.$$

Thus, we have, $\operatorname*{Lim}_{t\to\infty} \sup(C + C_a) \leq \dfrac{Q}{\delta_m}$. Similarly, from equation (2.5), we get, $\operatorname*{Lim}_{t\to\infty} \sup(C_p) \leq \dfrac{Q_p}{\delta_p}$. Hence, the lemma.

4. Equilibrium and Stability Analysis

The model has only one equilibrium namely $E^*(C_v^*, C_d^*, C_r^*, C^*, C_p^*, C_a^*)$, where $C_v^*, C_d^*, C_r^*, C^*, C_p^*$ and C_a^* are the positive solutions of the following algebraic equations:

$$C_v = \frac{q + \mu r_1 C_r C}{\mu_0} = \frac{q + \mu r_1 C_r f(C_r)}{\mu_0}, \tag{4.1}$$

$$C_d = \frac{\lambda}{\lambda_0} C_v = \frac{\lambda}{\lambda_0 \mu_0} \{q + \mu r_1 C_r f(C_r)\}, \tag{4.2}$$

$$r C_d - r_0 C_r - r_1 C_r C = 0, \tag{4.3}$$

$$C = \frac{Q}{\delta + \alpha C_r} = f(C_r), \tag{4.4}$$

$$C_p = \frac{Q_p}{\delta_p + \alpha_p C_r}, \tag{4.5}$$

$$C_a = \frac{\alpha C C_r}{k + \nu C_r} = \frac{\alpha C_r f(C_r)}{k + \nu C_r}. \tag{4.6}$$

To show the existence and uniqueness of E^*, we write equation (4.3) as follows:

$$F(C_r) = \frac{r\lambda q}{\lambda_0 \mu_0} - r_1 C_r f(C_r)\left\{1 - \frac{r\lambda\mu}{\lambda_0 \mu_0}\right\} - r_0 C_r. \tag{4.7}$$

It is known that $F(C_r) = 0$ has a unique root in Ω, if $F(0) > 0$, $F\left(\frac{q}{\lambda_m}\right) < 0$ and $F'(C_r) < 0$ in this region.

It can be easily checked from equation (4.7) that

$$F(0) > 0 \text{ and } F\left(\frac{q}{\lambda_m}\right) < 0.$$

Also from equation (4.7) we note that

$$F'(C_r) = -\left[r_0 + \frac{r_1 \delta Q C_r}{(\delta + \alpha C_r)^2}\left\{1 - \frac{r\lambda\mu}{\lambda_0 \mu_0}\right\}\right] < 0, \text{ since } r \le \lambda_0, \lambda \le \mu_0 \text{ and } 0 \le \mu \le 1.$$

Hence, there exists a unique root (say C_r^*) in $0 \le C_r \le \frac{q}{\lambda_m}$ without any condition. Using C_r^* we can evaluate C_v^*, C_d^*, C^*, C_p^* and C_a^* from equations (4.1), (4.2), (4.4), (4.5) and (4.6), respectively.

In the following, we check the characteristics of various phases with respect to relevant parameters.

From equations (4.1) – (4.4), we have

$$r_0 \alpha C_r^{\ 2} - \left(\frac{r\lambda}{\lambda_0 \mu_0} (q\alpha + \mu r_1 Q) - r_1 Q - r_0 \delta \right) C_r - \frac{r\lambda q \delta}{\lambda_0 \mu_0} = 0 . \tag{4.8}$$

4.1. Variation of C with q

Differentiating equation (4.8) with respect to ‘ q ’ we get:

$$\frac{dC_r}{dq} = \frac{\frac{r\lambda}{\lambda_0 \mu_0} (\delta + \alpha C_r) C_r}{r_0 \alpha C_r^{\ 2} + \frac{r\lambda q \delta}{\lambda_0 \mu_0}} > 0 .$$

This implies that, the density of raindrops (C_r) increases as the rate of formation of water vapor (q) increases in the atmosphere. Also, from equation (4.4), we note that $\frac{dC}{dC_r} < 0$.

Now, $\frac{dC}{dq} = \frac{dC}{dC_r} \frac{dC_r}{dq} < 0$, since $\frac{dC_r}{dq} > 0$.

Therefore, the concentration (C) of gaseous pollutants decreases as the rate of formation of vapors (i.e. q) increases.

4.2. Variation of C with λ

Differentiating equation (4.8), with respect to λ we get:

$$\frac{dC_r}{d\lambda} = \frac{\dfrac{r}{\lambda_0 \mu_0}\{q(\delta + \alpha C_r) + \mu r_1 Q C_r\} C_r}{r_0 \alpha C_r^{\ 2} + \dfrac{r \lambda q \delta}{\lambda_0 \mu_0}} > 0 .$$

This implies that, the density of raindrops (C_r) increases as the growth rate of cloud droplets (λ) increases in the atmosphere. Again, from equation (4.4), $\dfrac{dC}{dC_r} < 0$.

Now, $\dfrac{dC}{d\lambda} = \dfrac{dC}{dC_r}\dfrac{dC_r}{d\lambda} < 0$, since $\dfrac{dC_r}{d\lambda} > 0$.

Therefore, the concentration (C) of gaseous pollutants decreases as the growth rate of cloud droplets (λ) increases.

4.3. Variation of C with r

Differentiating equation (4.8), with respect to r we get:

$$\frac{dC_r}{dr} = \frac{\dfrac{\lambda}{\lambda_0 \mu_0}\{q(\delta + \alpha C_r) + \mu r_1 Q C_r\} C_r}{r_0 \alpha C_r^{\ 2} + \dfrac{r \lambda q \delta}{\lambda_0 \mu_0}} > 0 .$$

This implies that the density of raindrops (C_r) increases as the growth rate of raindrops increases in the atmosphere. Again, from equation (4.4), $\dfrac{dC}{dC_r} < 0$.

Now $\dfrac{dC}{dr} = \dfrac{dC}{dC_r}\dfrac{dC_r}{dr} < 0$, since $\dfrac{dC_r}{dr} > 0$.

Therefore, the concentration of gaseous pollutants decreases as the growth rate of raindrops (r) increases.

4.4 Variation of C_p with λ

From equation (4.5), we note that $\dfrac{dC_p}{dC_r} < 0$.

Now $\dfrac{dC_p}{d\lambda} = \dfrac{dC_p}{dC_r}\dfrac{dC_r}{d\lambda} < 0$, since $\dfrac{dC_r}{d\lambda} > 0$

Therefore, the concentration (C_p) of particulate matters decreases as the growth rate of cloud droplets (λ) increases.

Similarly we can also show that $\frac{dC_p}{dq} < 0, \frac{dC_a}{d\lambda} < 0, \frac{dC_p}{dr} < 0, \frac{dC_a}{dr} < 0$, etc.

Thus, from the above analysis, it is noted that the density of raindrops increases but the cumulative concentration of gaseous pollutants and particulate matters decreases as the growth rate of cloud droplets increases. This decrease in the concentration of pollutants is due to increased level of density of raindrops.

We also note that,

1. If the coefficient α and α_p are very large, then $\frac{dC}{dt}$ and $\frac{dC_p}{dt}$ respectively may become negative and the pollutants (gaseous and particulate) would be removed from the atmosphere.

2. If the coefficient k and ν are very large, then $\frac{dC_a}{dt} < 0$ and the formation of absorbed phase is very transient and it may not exist.

To see the stability behavior of E^*, we state the following theorems.

Theorem 4.1.

Let the following inequalities

$$(r_1 C_r^* + \alpha C^*)^2 < \frac{4}{15}(r_0 + r_1 C^*)(\delta + \alpha C_r^*) \tag{4.9}$$

$$\frac{15}{4}\frac{\lambda^2 r^2}{\mu_0 \lambda_0^2 (r_0 + r_1 C^*)} < k_1 < \frac{4}{3}\frac{\mu_0}{(\mu r_1)^2}\min\left\{\frac{(r_0 + r_1 C^*)}{5C^{*2}}, \frac{(\delta + \alpha C_r^*)}{3C_r^{*2}}\right\} \tag{4.10}$$

hold, then E^* is locally stable (See Appendix A for proof).

Theorem 4.2.

If the following inequalities are satisfied inside the region of attraction Ω,

$$(r_1 C_r^* + \alpha C^*)^2 < \frac{4}{15} r_0 \delta, \tag{4.11}$$

$$\frac{15}{4}\frac{\lambda^2 r^2}{\mu_0 \lambda_0{}^2 r_0} < m_1 < \frac{4}{3}\frac{\mu_0}{(\mu r_1)^2}\min\left\{\frac{r_0}{5C^{*2}}, \frac{\delta}{3(q/\lambda_m)^2}\right\}, \tag{4.12}$$

then E^* is globally asymptotically stable with respect to all solutions initiating in the interior of the positive octant (See Appendix B for proof).

The above theorems imply that under certain conditions, the gaseous pollutants and particulate matters would be removed from the atmosphere and the removal rate increases as the densities of vapors and cloud droplets increase.

Remark:

If $\lambda = 0$, then cloud droplets may not be formed and hence due to non-occurrence of rain, α and r_1 will be assumed to be zero. In such a case, the inequalities (4.9) – (4.12) are satisfied automatically. It shows that, in absence of these parameters, the pollutants would be removed from the atmosphere due to gravitational effect (natural removal).

5. Numerical Simulation and Discussion

In this section we present the results of computer simulations of system (2.1) – (2.6) for different values of parameters to study the behavior of the model system. For that the system (2.1) – (2.6) is integrated numerically with the help of MAPLE 7 by considering the following set of parameter values:

$$q = 5,\ \mu_0 = 0.8,\ \mu = 0.0002, r_1 = 0.08,\ \lambda = 0.7,$$

$$\lambda_0 = 0.6, r = 0.5, r_0 = 0.07, Q = 20, Q_p = 10,\ \delta = 0.15,$$

$$\alpha = 0.65,\ \delta_p = 0.25, \alpha_p = 0.50,\ k = 0.30,\ \nu = 0.55.$$

The equilibrium E^* is calculated as,

$$C_v{}^* = 6.250607,\ C_d{}^* = 7.292375, C_r{}^* = 17.384243,$$

$$C^* = 1.746761,\ C_p{}^* = 1.118302,\ C_a{}^* = 2.001553.$$

Eigenvalues corresponding to E^* are obtained as:

$$-9.377595,\ -7.352455,\ -6.872420,\ -1.712506, -0.676462, -0.740252.$$

Since all the eigenvalues corresponding to E^* are negative, therefore E^* is locally asymptotically stable.

The global stability behavior of E^* in $C_d - C_r$ plane is shown in Figure 5.1. In Figures 5.2 – 5.4, the variation of density of cloud droplets C_d, concentration of gaseous pollutants C and particulate matters C_p with time 't' is shown for different values of rate of formation of vapors (i.e., at $q = 0,3,5$) respectively. From these figures, it is visualized that if the rate of formation of vapors is zero i.e., $q = 0$, the density of cloud droplets will be zero (Figure 5.2) and the concentration of the gaseous pollutants and particulate matters would increase continuously attaining their respective equilibria (Figures 5.3– 5.4). Further, the density of cloud droplets increases but the concentrations of gaseous pollutants and particulate matters decrease as q increases. In Figures 5.5 – 5.8, the variation of density of raindrops C_r, the concentrations of gaseous pollutants C and particulate matters C_p, and the concentration of gaseous pollutants in absorbed phase C_a with time 't' is shown for different values of growth rate of cloud droplets λ (i.e., at $\lambda = 0, 0.6, 0.7$) respectively. From Figure 5.5 at $\lambda = 0$, it is seen that the formation of raindrops phase is very transient and may not exist but the density of raindrops increases as the growth rate of cloud droplets increases. In Figures 5.6 and 5.7, it is shown that if the growth rate of cloud droplets is zero i.e. $\lambda = 0$, the concentrations of gaseous pollutants C and particulate matters C_p increase continuously attaining their respective equilibria and pollutants would not be removed from the atmosphere. Further, as the density of cloud droplets increases, the concentrations of these pollutants decrease.

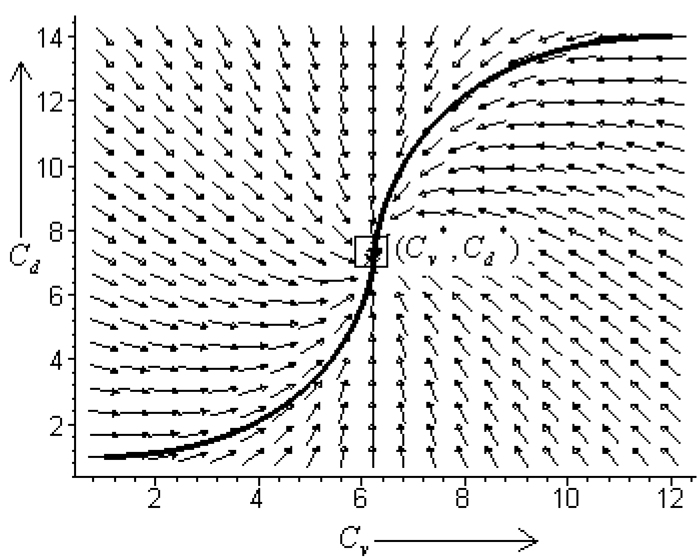

Figure 5.1. Global stability in $C_v - C_d$ plane

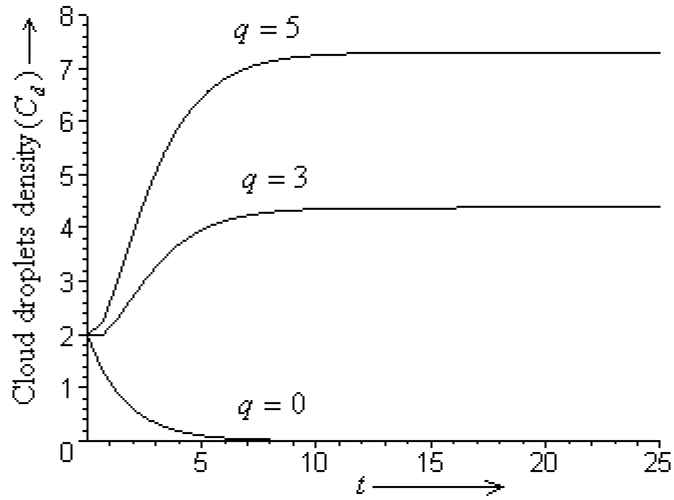

Figure 5.2. Variation of C_d with time 't' for different values of q

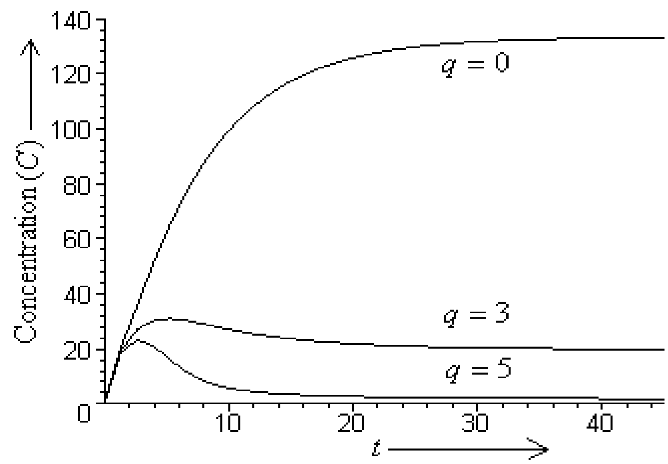

Figure 5.3. Variation of C with time 't' for different values of q

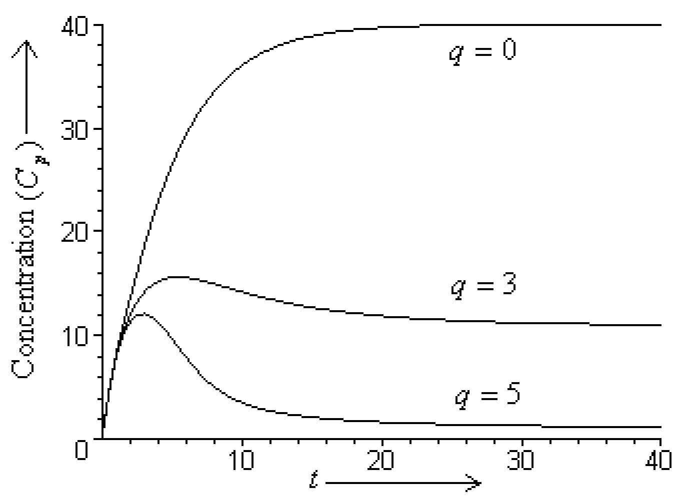

Figure 5.4. Variation of C_p with time 't' for different values of q

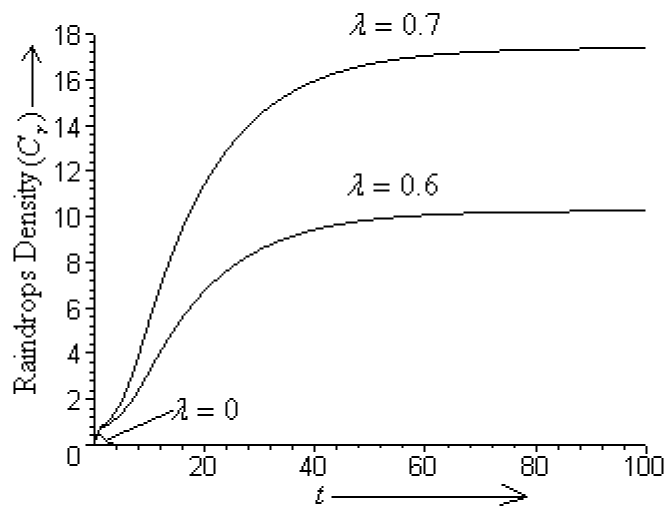

Figure 5.5. Variation of C_r with time $'t'$ for different values of λ

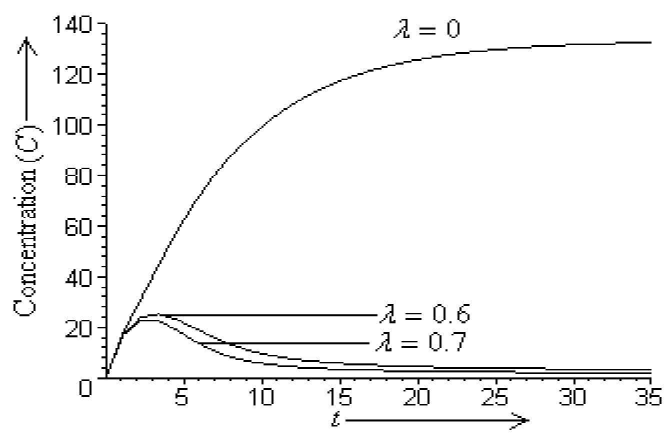

Figure 5.6. Variation of C with time $'t'$ for different values of λ

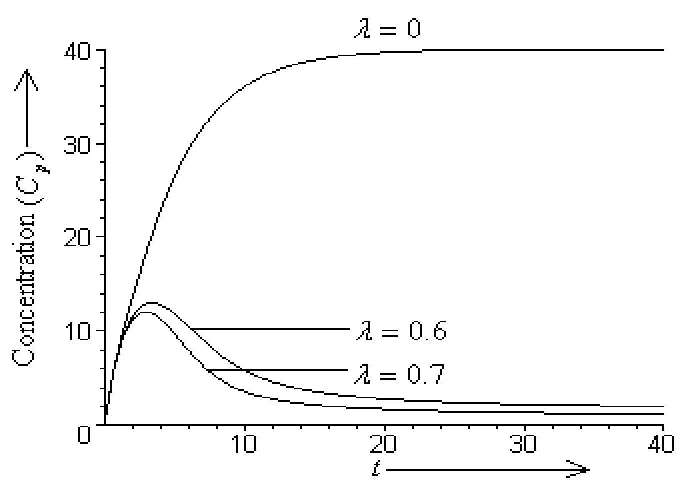

Figure 5.7. Variation of C_p with time $'t'$ for different values of λ

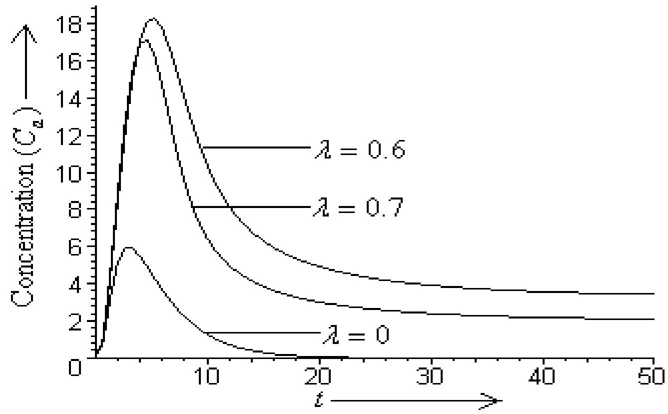

Figure 5.8. Variation of C_a with time 't' for different values of λ

In Figure 5.8, it is shown that at $\lambda = 0$ the formation of the absorbed phase is very transient. It is also depicted that the concentration of gaseous pollutants in the absorbed phase (i.e., C_a) decreases as the growth rate of cloud droplets increases. Further, if the cloud droplets density is very large, the removal of gaseous pollutants as well as particulate matters is quite significant due to enhanced rainfall.

In tables 1 and 2, the variation of equilibrium values is shown for different values of rate of the formation of vapors q and the growth rate of cloud droplets λ respectively. From table 1, it is clear that the densities of cloud droplets and raindrops increase as the rate of formation of vapors increases but the concentrations of gaseous pollutants and particulate matters decrease. From table 2, it is seen that the density of raindrops increases but the concentration of pollutants decreases, with increase in the growth rate of cloud droplets.

Table 1. Variation of C_d, C_r, C and C_p with rate of formation of water vapor q

q	3	4	5	6
C_d	4.3756	5.8340	7.2923	8.7507
C_r	1.3205	7.5497	17.3842	27.6315
C	19.8342	3.9546	1.7467	1.1043
C_p	10.9857	2.4845	1.1183	0.7109

Table 2. Variation of C_r, C and C_p with growth rate of cloud droplets λ

λ	0.5	0.6	0.7	0.8
C_r	3.9719	10.2561	17.3842	24.6904
C	7.3213	2.9340	1.7467	1.2346
C_p	4.4723	1.8594	1.1183	0.7939

6. Conclusion

In this paper, an attempt has been made to study the role of cloud droplets, caused by water vapors, on the removal of gaseous pollutants and particulate matters from the atmosphere using a nonlinear mathematical model. The model is analyzed using the stability theory of differential equations and numerical simulations. It is shown, analytically and numerically, that the density of raindrops increases while the cumulative concentrations of gaseous pollutants and particulate matters decrease as the growth rate of cloud droplets increases. It has also been shown numerically that the densities of cloud droplets and raindrops increase while the concentrations of gaseous pollutants and particulate matters decrease as the rate of formation of water vapor increases. Further, the magnitude of pollutants removed by rainfall depends upon the intensity of rain caused by cloud droplets formation, but the remaining equilibrium amount would depend upon the rate of emission of pollutants, the rate of formation of water vapors, the growth rate of cloud droplets and raindrops, the rate of falling raindrops on the ground and other interaction parameters. It has also been shown that if there is no cloud formation, raindrops may not be formed and pollutants would not be removed from the atmosphere. The results, so obtained, are qualitatively in line with the experimental observations, Davies (1976), Sharma (1983), and Pandey et al. (1992).

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Appendix A

Proof of Theorem 4.1.

Using the following positive definite function in the linearized system of (2.1) – (2.6),

$$V = \frac{1}{2}(k_1 C_{v1}{}^2 + k_2 C_{d1}{}^2 + k_3 C_{r1}{}^2 + k_4 C_1{}^2 + k_5 C_{p1}{}^2 + k_6 C_{a1}{}^2), \tag{A.1}$$

where $C_{v1}, C_{d1}, C_{r1}, C_1, C_{p1}, C_{a1}$ are small perturbations from E^*, as follows

$$C_v = C_v{}^* + C_{v1}, C_d = C_d{}^* + C_{d1}, C_r = C_r{}^* + C_{r1}, C = C^* + C_1,\ C_p = C_p{}^* + C_{p1}, C_a = C_a{}^* + C_{a1}.$$

Differentiating (A.1) with respect to $'t'$ we get, in the linearized system corresponding to E^*

$$\dot{V} = -k_1\mu_0 C_{v1}{}^2 - k_2\lambda_0 C_{d1}{}^2 - k_3(r_0 + r_1 C^*)C_{r1}{}^2 - k_4(\delta + \alpha C_r{}^*)C_1{}^2$$

$$- k_5(\delta_p + \alpha_p C_r{}^*)C_{p1}{}^2 - k_6(k + \nu C_r{}^*)C_{a1}{}^2$$

$$+ k_2\lambda C_{v1}C_{d1} + k_1\mu r_1 C^* C_{v1}C_{r1} + k_1\mu r_1 C_r{}^* C_{v1}C_1$$

$$+ k_3 r C_{d1}C_{r1} - (k_3 r_1 C_r{}^* + k_4\alpha C^*)\ C_{r1}C_1$$

$$- k_5\ \alpha_p\ C_p{}^* C_{r1}C_{p1} + k_6(\alpha C^* - \nu C_a{}^*)C_{r1}C_{a1} + k_6\alpha C_r{}^* C_1 C_{a1}.$$

Now $\dot{V}$ will be negative definite under the following conditions:

$$k_2\lambda^2 < \frac{2}{3}k_1\mu_0\lambda_0, \tag{A.2}$$

$$k_1(\mu r_1 C^*)^2 < \frac{4}{15}k_3\mu_0(r_0 + r_1 C^*), \tag{A.3}$$

$$k_1(\mu r_1 C_r{}^*)^2 < \frac{4}{9}k_4\mu_0(\delta + \alpha C_r{}^*), \tag{A.4}$$

$$k_3 r^2 < \frac{2}{5}k_2\lambda_0(r_0 + r_1 C^*), \tag{A.5}$$

$$(k_3 r_1 C_r{}^* + k_4\alpha C^*)^2 < \frac{4}{15}k_3 k_4(r_0 + r_1 C^*)(\delta + \alpha C_r{}^*), \tag{A.6}$$

$$k_5(\alpha_p C_p{}^*)^2 < \frac{4}{5}k_3(r_0 + r_1 C^*)(\delta_p + \alpha_p C_r{}^*), \tag{A.7}$$

$$k_6(\alpha C^* - \nu C_a{}^*)^2 < \frac{2}{5}k_3(r_0 + r_1 C^*)(k + \nu C_r{}^*), \tag{A.8}$$

$$k_6(\alpha C_r{}^*)^2 < \frac{2}{3}k_4(\delta + \alpha C_r{}^*)(k + \nu C_r{}^*). \tag{A.9}$$

Now, choosing $k_3 = k_4 = 1$, $k_5 < \dfrac{4}{5}\dfrac{(r_0 + r_1 C^*)(\delta_p + \alpha_p C_r^*)}{(\alpha_p C_p^*)^2}$

and

$$k_6 < 2(k + \nu C_r^*) \min\left\{\frac{1}{5}\frac{(r_0 + r_1 C^*)}{(\alpha C^* - \nu C_a^*)^2}, \frac{1}{3}\frac{(\delta + \alpha C_r^*)}{(\alpha C_r^*)^2}\right\},$$

the inequalities (A.2) – (A.9) reduce to

$$(r_1 C_r^* + \alpha C^*)^2 < \frac{4}{15}(r_0 + r_1 C^*)(\delta + \alpha C_r^*),$$

$$\frac{15}{4}\frac{\lambda^2 r^2}{\mu_0 \lambda_0^2 (r_0 + r_1 C^*)} < k_1 < \frac{4}{3}\frac{\mu_0}{(\mu r_1)^2} \min\left\{\frac{(r_0 + r_1 C^*)}{5C^{*2}}, \frac{(\delta + \alpha C_r^*)}{3C_r^{*2}}\right\},$$

which are same as stated in the theorem. Thus, $\dot{V}$ will be negative definite provided the conditions (4.9) – (4.10) are satisfied showing that V is a Liapunov function and hence the theorem.

Appendix B

Proof of Theorem 4.2.

Using the following positive definite function,

$$U = \frac{1}{2}[m_1(C_v - C_v^*)^2 + m_2(C_d - C_d^*)^2 + m_3(C_r - C_r^*)^2 + m_4(C - C^*)^2$$

$$+ m_5(C_p - C_p^*)^2 + m_6(C_a - C_a^*)^2. \qquad \text{(B.1)}$$

Differentiating with respect to t, we get:

$$\begin{aligned}
\dot{U} = &-m_1\mu_0(C_v - C_v^*)^2 - m_2\lambda_0(C_d - C_d^*)^2 - m_3(r_0 + r_1 C)(C_r - C_r^*)^2 \\
&- m_4(\delta + \alpha C_r)(C - C^*)^2 - m_5(\delta_p + \alpha_p C_r)(C_p - C_p^*)^2 \\
&- m_6(k + \nu C_r^*)(C_a - C_a^*)^2 \\
&+ m_2\lambda(C_v - C_v^*)(C_d - C_d^*) + m_1\mu r_1 C^*(C_v - C_v^*)(C_r - C_r^*) \\
&+ m_1\mu r_1 C_r(C_v - C_v^*)(C - C^*) + m_3 r(C_d - C_d^*)(C_r - C_r^*) \\
&- (m_3 r_1 C_r^* + m_4\alpha C^*)(C_r - C_r^*)(C - C^*) - m_5\alpha_p C_p^*(C_r - C_r^*)(C_p - C_p^*) \\
&+ m_6(\alpha C - \nu C_a)(C_r - C_r^*)(C_a - C_a^*) + m_6\alpha C_r^*(C - C^*)(C_a - C_a^*).
\end{aligned}$$

Now, $\dot{U}$ will be negative definite under the following conditions:

$$m_2\lambda^2 < \frac{2}{3}m_1\mu_0\lambda_0\,, \tag{B.2}$$

$$m_1(\mu r_1 C^*)^2 < \frac{4}{15}m_3\mu_0(r_0 + r_1C)\,, \tag{B.3}$$

$$m_1(\mu r_1 C_r)^2 < \frac{4}{9}m_4\mu_0(\delta + \alpha C_r)\,, \tag{B.4}$$

$$m_3 r^2 < \frac{2}{5}m_2\lambda_0(r_0 + r_1C)\,, \tag{B.5}$$

$$(m_3 r_1 C_r^{\ *} + m_4\alpha C^*)^2 < \frac{4}{15}m_3m_4(r_0 + r_1 C)(\delta + \alpha C_r)\,, \tag{B.6}$$

$$m_5(\alpha_p C_p^{\ *})^2 < \frac{4}{5}m_3(r_0 + r_1 C)(\delta_p + \alpha_p C_r)\,, \tag{B.7}$$

$$m_6(\alpha C - \nu C_a)^2 < \frac{2}{5}m_3(r_0 + r_1 C)(k + \nu C_r^{\ *})\,, \tag{B.8}$$

$$m_6(\alpha C_r^{\ *})^2 < \frac{2}{3}m_4(\delta + \alpha C_r)(k + \nu C_r^{\ *})\,. \tag{B.9}$$

Maximizing the LHS and minimizing the RHS and choosing the constants such that:

$$m_3 = m_4 = 1,\ m_5 < \frac{4}{5}\frac{r_0\delta_p}{(\alpha_p C_p^{\ *})^2}$$

and

$$m_6 < 2(k + \nu C_r^{\ *})\min\left\{\frac{1}{5}\frac{r_0}{(\alpha+\nu)^2(Q/\delta_m)^2},\ \frac{1}{3}\frac{\delta}{(\alpha C_r^{\ *})^2}\right\},$$

the inequalities (B.2) – (B.9) reduce to

$$(r_1C_r^{\ *} + \alpha C^*)^2 < \frac{4}{15}r_0\delta\,,$$

$$\frac{15}{4}\frac{\lambda^2 r^2}{\mu_0\lambda_0^{\ 2}r_0} < m_1 < \frac{4}{3}\frac{\mu_0}{(\mu r_1)^2}\min\left\{\frac{r_0}{5C^{*2}},\frac{\delta}{3(q/\lambda_m)^2}\right\},$$

which are the same as stated in the theorem. Thus, $\dot{U}$ will be negative definite provided the conditions (4.11) – (4.12) are satisfied inside the region of attraction Ω showing that U is a Liapunov function and hence the theorem.