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Application of Homotopy analysis method to fourth-order parabolic partial differential equations

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Abstract

In this paper, by means of the homotopy analysis method (HAM), the solutions of some fourth-order parabolic partial differential equations are exactly obtained in the form of convergent Taylor series. The HAM contains the auxiliary parameter h that provides a convenient way of controlling the convergent region of series solutions. This analytical method is employed to solve linear examples to obtain the exact solutions. The results reveal that the proposed method is very effective and simple.

Keywords: Fourth-order parabolic partial differential equations, Homotopy analysis method, Variational iteration method, Homotopy perturbation method

MSC 2000 No.: 65L05, 65M99

1. Introduction

In 1992, Liao see Liao (2003) employed the basic ideas of the homotopy in topology to propose a general analytic method for linear and nonlinear problems, namely the homotopy analysis method (HAM), see Liao (2003), (2004a), and (2005a). This method has been successfully applied to solve many types of nonlinear problems, see Hayat (2004b), (2004c), (2005b), and (2005c). The HAM offers certain advantages over routine numerical methods. HAM method is better since it does not involve discretization of the variables. Hence, is free from rounding off errors and does not require large computer memory or time.

In this paper, we consider the fourth-order parabolic partial differential equations

$$\frac{\partial^2 u}{\partial t^2}+\mu(x,y,z)\frac{\partial^4 u}{\partial x^4}+\frac{1}{y}\lambda(x,y,z)\frac{\partial^4 u}{\partial y^4}+\frac{1}{z}\eta(x,y,z)\frac{\partial^4 u}{\partial z^4}=g(x,y,z),\ a<x,y,z<b,\ \ t>0, \tag{1}$$

where $\mu(x,y,z)$ and $\lambda(x,y,z)$ are positive with the initial conditions

$$u(x,y,z,0)=g_0(x,y,z) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,z,0)=f_0(x,y,z), \tag{2}$$

and boundary conditions

$$\begin{aligned}
&u(a,y,z,t)=g_0(y,z,t), && u(b,y,z,t)=g_1(y,z,t),\\
&u(x,a,z,t)=k_0(x,z,t), && u(x,b,z,t)=k_1(x,z,t),\\
&u(x,y,a,t)=h_0(x,y,t), && u(x,y,b,t)=h_1(x,y,t),\\
&\frac{\partial^2 u}{\partial x^2}(a,y,z,t)=\bar{g}_0(y,z,t), && \frac{\partial^2 u}{\partial x^2}(b,y,z,t)=\bar{g}_1(y,z,t),\\
&\frac{\partial^2 u}{\partial y^2}(x,a,z,t)=\bar{k}_0(x,z,t), && \frac{\partial^2 u}{\partial y^2}(x,b,z,t)=\bar{k}_1(x,z,t),\\
&\frac{\partial^2 u}{\partial z^2}(x,y,a,t)=\bar{h}_0(x,y,t), && \frac{\partial^2 u}{\partial z^2}(x,y,b,t)=\bar{h}_1(x,y,t),
\end{aligned}$$

where the functions $f_i, g_i, k_i, h_i, \bar{g}_i, \bar{k}_i$, and $\bar{h}_i$, $i=0,1,$ are continuous, $\mu(x,y,z)>0$ is the ratio of flexural rigidity [see Khaliq (1987)] of the beam to its mass per unit length, see Wazwaz (2001), Khaliq (1987), Andrade (1977), Gorman (1975) and the references therein. The functions $f_0(x),\ f_1(x), g_0(x), g_1(x), h_0(x)$ and $h_1(x)$ are continuous functions. We apply the homotopy analysis method for solving the singular fourth-order parabolic partial differential equation (1) with variable coefficient.

2. Basic Idea of HAM

Consider the following differential equation

$$N[u(\tau)]=0, \tag{3}$$

where N is a nonlinear operator, τ is independent variable, and $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao (2003) constructs the so called zero-order deformation equation

$$(1-p)L[\phi(\tau;p)-u_0(\tau)] = p\hbar H(\tau)N[\phi(\tau;p)], \tag{4}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(\tau) \neq 0$ is a nonzero auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, and $\phi(\tau;p)$ is a unknown function. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\varphi(\tau;0) = u_0(\tau), \text{ and } \varphi(\tau;1) = u(\tau). \tag{5}$$

Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ by Taylor series with respect to p, we have

$$\varphi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)\, p^m, \tag{6}$$

where

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\Big|_{p=0} . \tag{7}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (6) converges at $p=1$, then we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{+\infty} u_m(r,t), \tag{8}$$

which must be one of solutions of the original nonlinear equation, as proved by Liao (2003). As $\hbar = -1$ and $H(\tau) = 1$, equation (4) becomes

$$(1-p)L[\phi(\tau;p)-u_0(\tau)] + pN[\phi(\tau;p)] = 0, \tag{9}$$

which is used mostly in the homotopy perturbation method [see He (2000)], where as the solution obtained directly, without using Taylor series, see He (2006a) and (2006b). According to the definition (7), the governing equation can be deduced from the zero-order deformation equation (4). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating equation (4) m times with respect to the embedding parameter p, then setting $p = 0$ and finally dividing them by $m!$, we obtain the m^{th} order deformation equation

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{u}_{m-1}), \tag{10}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}}, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}. \tag{12}$$

It should be emphasized that $u_m(\tau)$ for $m \ge 1$ is governed by the linear equation (10) under the linear boundary conditions that come from original problem, which can be easily symbolically solved by Matlab computer software. For the convergence of the above method we refer the reader to Liao (2003). If equation (3) admits unique solution, then this method will produce the unique solution. If equation (3) does not possess a unique solution, the HAM will give a solution among many other (possible) solutions.

3. Applications

In order to assess the advantages and the accuracy of homotopy analysis method for solving linear equations, we will consider the following three examples.

Example 1.

Consider the following singular fourth-order parabolic equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4} = 0, \qquad \frac{1}{2} < x < 1,\ t > 0, \tag{13}$$

with the initial conditions

$$u(x,0) = 0,\ \frac{\partial u}{\partial x}(x,0) = 1 + \frac{x^4}{120},$$

and the boundary conditions

$$u\left(\frac{1}{2},t\right)=\left(1+\frac{(1/2)^{5}}{120}\right)\sin t,\qquad u(1,t)=\left(\frac{121}{120}\right)\sin t,\quad t>0,$$

and

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right)=\frac{1}{6}\left(\frac{1}{2}\right)^3\sin t,\qquad \frac{\partial^2 u}{\partial x^2}(1,t)=\left(\frac{1}{6}\right)\sin t,\quad t>0.$$

To solve the Equation (13) by means of homotopy analysis method, we choose the linear operator

$$L[\phi(x,t;p)]=\frac{\partial^2\phi(x,t;p)}{\partial t^2},\tag{14}$$

with the property

$$L[c_1+tc_2]=0,$$

where c_1 and c_2 are integral constants. The inverse operator L^{-1} is given by

$$L^{-1}(\cdot)=\int_0^t\int_0^t(\cdot)\,dtdt.\tag{15}$$

We now define a nonlinear operator as

$$N[\phi(x,t;p)]=\frac{\partial^2\phi(x,t;p)}{\partial t^2}+\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4\phi(x,t;p)}{\partial x^4}.\tag{16}$$

Using the above definition, we construct the zeroth-order deformation equation

$$(1-p)L[\phi(x,t;p)-u_0(x,t)]=p\hbar H(x,t)N[\phi(x,t;p)].$$

For $p=0$ and $p=1$, we can write

$$\phi(x,t;0)=u_0(x,t),\qquad \phi(x,t;1)=u(x,t).\tag{17}$$

Thus, we obtain the m^{th} order deformation equations

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=\hbar H(x,t)R_m(\vec{u}_{m-1}),\ (m\geq 1),\ u_m(x,0)=0,\ (u_m)_t(x,0)=0,\tag{18}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u_{m-1}}{\partial x^4}.$$

Now, the solution of the m^{th} order deformation equations (14) are

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar H(x,t) L^{-1}[R_m(\vec{u}_{m-1})], \quad (m \geq 1). \tag{19}$$

We start with an initial approximation s$u_0(x,t) = t\left(1 + \frac{x^4}{120}\right)$, by means of the iteration formula (19), if $\hbar = -1$ and $H(x,t) = 1$, we can obtain directly the other components as

$$u_1(x,t) = -\frac{t^3}{3!}\left(1 + \frac{x^5}{120}\right),$$
$$u_2(x,t) = \frac{t^5}{5!}\left(1 + \frac{x^5}{120}\right),$$
$$u_3(x,t) = -\frac{t^7}{7!}\left(1 + \frac{x^5}{120}\right),$$
$$\vdots$$

Thus, the components which constitute $u(x,t)$ are written like this

$$u(x,t) = \sum_{m=1}^{+\infty} u_m(x,t) = \left(1 + \frac{x^5}{120}\right)\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right\}.$$

Continuing the expansion to the last term gives the solution of (13) as

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t, \tag{20}$$

which is exactly the same as obtained by variational homotopy perturbation method [see Noor (2009)] and homotopy perturbation method, see Fazeli (2008).

Example 2.

Consider the following singular fourth-order parabolic partial differential equation in two space variables

$$\frac{\partial^4 u}{\partial t}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}=0, \qquad \frac{1}{2}<x,y<1,\ t>0, \tag{21}$$

with the initial conditions

$$u(x,y,0)=0, \qquad \frac{\partial u}{\partial t}(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!},$$

and the boundary conditions

$$u(0.5,y,t)=\left(2+\frac{(0.5)^6}{6!}+\frac{y^6}{6!}\right)\sin t, \qquad u(1,y,t)=\left(2+\frac{1}{6!}+\frac{y^6}{6!}\right)\sin t,$$
$$\frac{\partial^2 u}{\partial x^2}(0.5,y,t)=\frac{(0.5)^4}{24}\sin t, \qquad \frac{\partial^2 u}{\partial x^2}(1,y,t)=\frac{1}{24}\sin t,$$
$$\frac{\partial^2 u}{\partial y^2}(x,0.5,t)=\frac{(0.5)^4}{24}\sin t, \qquad \frac{\partial^2 u}{\partial x^2}(x,1,t)=\frac{1}{24}\sin t.$$

To solve equation (21) by means of homotopy analysis method, we choose the linear operator same as Example 1 and define a nonlinear operator as

$$N[\varphi(x,y,t;p)]=\frac{\partial^2\varphi(x,y,t;p)}{\partial t^2}+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4\varphi(x,t,t;p)}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4\varphi(x,y,t;p)}{\partial y^4}. \tag{22}$$

Using the above definition and explanation in section (2), we have

$$u_m(x,y,t)=\chi_m u_{m-1}(x,y,t)+\hbar H(x,y,t)\,L^{-1}[R_m(\vec{u}_{m-1})], \quad (m\geq 1), \tag{23}$$

where

$$R_m(\vec{u}_{m-1})=\frac{\partial^2 u_{m-1}(x,y,t)}{\partial t^2}+2\left(\frac{1}{x}+\frac{x^4}{6!}\right)\frac{\partial^4 u_{m-1}(x,y,t)}{\partial x^4}+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right).\frac{\partial^4 u_{m-1}(x,y,t)}{\partial y^4}.$$

We start with an initial approximation $u_0(x,y,t)=t\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)$, by means of the iteration formula (23) if $\hbar=-1$ and $H(x,y,t)=1$, we can directly obtain the other components as

$$u_1(x,y,t)=-\frac{t^3}{3!}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right),$$

$$u_2(x,y,t) = \frac{t^5}{5!}\left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right),$$
$$u_3(x,y,t) = -\frac{t^7}{7!}\left(2 + \frac{x^6}{120} + \frac{y^6}{6!}\right),$$
$$\vdots$$

Thus, the components which constitute $u(x,y,t)$ are written as

$$u(x,y,t) = \sum_{m=1}^{+\infty} u_m(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right\}.$$

Continuing the expansion to the last term gives the solution of (21) as

$$u(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!}\right)\sin t, \tag{24}$$

which is exactly the same as obtained by variational homotopy perturbation method see Noor (2009) and homotopy perturbation method see Fazeli (2008).

Example 3.

Consider the following three-dimensional inhomogeneous singular parabolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\frac{1}{4!z}\right)\frac{\partial^4 u}{\partial x^4} + \left(\frac{1}{4!x}\right)\frac{\partial^4 u}{\partial y^4} + \left(\frac{1}{4!y}\right)\frac{\partial^4 u}{\partial z^4} = \left[\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - \frac{1}{x^5} - \frac{1}{y^5} - \frac{1}{z^5}\right]\cos t, \tag{25}$$

with the initial conditions

$$u(x,y,z,0) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \qquad \frac{\partial u}{\partial t}(x,y,z,0) = 0,$$

and boundary conditions

$$u(0.5,y,z,t) = \left(\frac{1}{2y} + \frac{y}{z} + 2z\right)\cos t, \qquad u(1,y,z,t) = \left(\frac{1}{y} + \frac{y}{z} + z\right)\cos t,$$
$$u(x,0.5,z,t) = \left(2x + \frac{1}{2z} + \frac{z}{x}\right)\cos t, \qquad u(x,1,z,t) = \left(x + \frac{1}{z} + \frac{z}{x}\right)\cos t,$$

$$u(x,y,0.5,t)=\left(2y+\frac{x}{y}+\frac{1}{2x}\right)\cos t, \qquad u(x,y,1,t)=\left(y+\frac{x}{y}+\frac{1}{x}\right)\cos t,$$

$$\frac{\partial u}{\partial x}(0.5,y,z,t)=\left(\frac{1}{y}-4z\right)\cos t, \qquad \frac{\partial u}{\partial x}(1,y,z,t)=\left(\frac{1}{y}-z\right)\cos t.$$

To solve the equation (25) by means of homotopy analysis method, we choose the linear operator the same as Example 1 and define a nonlinear operator as

$$N[\varphi(x,y,z,t;p)]=\frac{\partial^2\varphi(x,y,z,t;p)}{\partial t^2}+\left(\frac{1}{4!z}\right)\frac{\partial^4\varphi(x,y,z,t;p)}{\partial x^4}+\left(\frac{1}{4!x}\right)\frac{\partial^4\varphi(x,y,z,t;p}{\partial y^4}$$
$$+\left(\frac{1}{4!y}\right)\frac{\partial^4\varphi(x,y,z,t;p)}{\partial z^4}-\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{1}{x^5}-\frac{1}{y^5}-\frac{1}{z^5}\right]\cos t, \qquad (m\geq 1).$$

Using the above definition and explanation in section (2), we have

$$u_m(x,y,z,t)=\chi_m u_{m-1}(x,y,z,t)+\hbar H(x,y,z,t)\,L^{-1}[R_m(\vec{u}_{m-1})], \quad (m\geq 1), \tag{26}$$

where

$$R_m(\vec{u}_{m-1})=\frac{\partial^2 u_{m-1}}{\partial t^2}+\left(\frac{1}{4!z}\right)\frac{\partial^4 u_{m-1}}{\partial x^4}+\left(\frac{1}{4!x}\right)\frac{\partial^4 u_{m-1}}{\partial y^4}+\left(\frac{1}{4!y}\right)\frac{\partial^4 u_{m-1}}{\partial z^4}$$
$$-(1-\chi_m)\left[\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{1}{x^5}-\frac{1}{y^5}-\frac{1}{z^5}\right]\cos t, \qquad (m\geq 1).$$

We start with an initial approximation $u_0(x,y,z,t)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, by means of the iteration formula (26) if $\hbar=-1$ and $H(x,y,z,t)=1$, we can obtain directly the other components as

$$u_1(x,y,z,t)=\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)(\cos t-1)+\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)(-\cos t+1-\frac{t^2}{2}),$$

$$u_2(x,y,z,t)=\left(\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)(\cos t-1+\frac{t^2}{2})+70\left(\frac{1}{x^9 z}+\frac{1}{xy^9}+\frac{1}{yz^9}\right)(-\cos t+1-\frac{t^2}{2!}+\frac{t^4}{4!}),$$

$$u_3(x,y,z,t) = 70\left(\frac{1}{x^9 z}+\frac{1}{xy^9}+\frac{1}{yz^9}\right)\left(\cos t - 1 + \frac{t^2}{2!} - \frac{t^4}{4!}\right)$$
$$+34650\left(\frac{1}{x^2 y^{13}}+\frac{1}{y^2 z^{13}}+\frac{1}{z^2 x^{13}}\right)\left(-\cos t + 1 - \frac{t^2}{2!} + \frac{t^4}{4!}\right)$$
$$+70\left(\frac{1}{x^5 y^9 z}+\frac{1}{y^5 z^9 x}+\frac{1}{z^5 x^9 y}\right)\left(-\cos t + 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right),$$
$$\vdots$$

Thus, the components which constitute $u(x,y,z,t)$ are written as

$$u(x,y,z,t) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)(\cos t) + 70\left(\frac{1}{x^5 y^9 z}+\frac{1}{y^5 z^9 x}+\frac{1}{z^5 x^9 y}\right)\left(-\cos t + 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\right) + \cdots .$$

The sequences tends to $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t$, as $n \to +\infty$. Therefore, the exact solution is given as

$$u(x,y,z,t) = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\cos t,$$

which is exactly the same as the one obtained by variational homotopy perturbation method see Noor (2009).

4. Conclusion

In this paper, the Homotopy analysis method has been successfully applied to find the exact solution of linear fourth-order parabolic partial differential equation. All the examples show that the results of the present method are in excellent agree with those obtained by the variational homotopy perturbation method and homotopy perturbation method. It is apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wide classes of linear problems. The results show that HAM is powerful mathematical tool for solving linear partial differential equations having wide applications in engineering.

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