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Soliton and Periodic Solutions for (3+1)-Dimensional Nonlinear Evolution Equations by Exp-function Method

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Abstract

In this paper, (3+1)-dimensional Jimbo-Miwa and (3+1)-dimensional potential-YTSF equations are considered and the Exp-Function method is employed to compute the exact solutions. The solutions obtained by this method are compared with the exact solutions obtained through other methods. These equations play a very important role in mathematical physics and engineering sciences. It is shown that the Exp-Function method, with the help of symbolic computation, provides a powerful mathematical tool for solving nonlinear evolution equations in mathematical physics.

Keywords: Exp-function method; Potential YTSF equation; Jimbo-Miwa equation; Periodic solution; Soliton.

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1. Introduction

Nonlinear phenomena play important roles in applied mathematics, physics and also in engineering problems in which each parameter varies depending on different factors. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts that are not simply understood through common observations. Moreover, obtaining exact solutions for these problems is a great purpose that has been quite untouched.

Recently several authors introduced some new method such as the variation iteration method (VIM), homotopy perturbation method (HPM) and Exp-Function method to solve these equations, [see for instance, He (2000, 2004, 2005, 2006), He and Abdou (2007), and He and Wu (2006, 2007), He and Zhang (2008)]. Exp-function method is very strong for solving high nonlinearity of nonlinear equations. Other authors such as Zhu see Zhu (2007) and Zhang (2006) have been working in this field. Other applications of this method for solving nonlinear evolution equations arising in mathematical physics can be found in Borhanifar and Kabir (2009), Borhanifar et al. (2009), Kabir and Khajeh (2009), for example.

2. Basic Idea of Exp-function Method

We first consider a nonlinear equation of the form

$$N(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \cdots) = 0, \tag{1}$$

where N is a nonlinear function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformation. Introducing a complex variation η defined as

$$u = u(\eta), \quad \eta = kx + ly + sz + \omega t, \tag{2}$$

where k, l, s and ω are constants to be determined later. Then, equation (1) reduces to the ODE:

$$N(u, \omega u', k u', k^2 u'', \omega^2 u'', k\omega u'', \cdots) = 0, \tag{3}$$

Hence, solution of $u(\eta)$ is

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}, \tag{4}$$

where c, d, p, and q are positive integers, which may be chosen freely; a_n and b_m are unknown constants to be determined.

3. Application of Exp-function Method

3.1. (3+1)-Dimensional Potential-YTSF Equation

To illustrate the basic idea of the Exp-function method, we first consider the potential-YTSF equation [see Bai and Zhao (2006), and Wazwaz (2007)] that was recently derived by Yu et al. (1998) as

$$u_{xxxz} + 4u_x u_{xz} + 2u_{xx}u_z - 4u_{xt} + 3u_{yy} = 0. \tag{5}$$

Introducing a complex variation η, defined in equation (2), equation (5) becomes an ordinary differential equation as

$$k^3 s\, u'''' + 6k^2 s\, u'u'' + (3l^2 - 4k\omega)\, u'' = 0. \tag{6}$$

In order to determine values of c and p, we balance the linear term of the highest order u'''' with the highest order nonlinear term $u'u''$ in equation (6) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] as

$$u'''' = \frac{c_1 \exp[(15p + c)\eta] + \ldots}{c_2 \exp[16p\eta] + \ldots}, \tag{7}$$

and

$$u'u'' = \frac{c_3 \exp[(c+p)\eta] + \ldots}{c_4 \exp[2p\eta] + \ldots} \times \frac{c_5 \exp[(c+3p)\eta] + \ldots}{c_6 \exp[4p\eta] + \ldots} = \frac{c_7 \exp[(2c+14p)\eta] + \ldots}{c_8 \exp[16p\eta] + \ldots}, \tag{8}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in equations (7) and (8) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] as

$$15p + c = 2c + 14p, \tag{9}$$

which leads to

$$p = c. \tag{10}$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in equation (6) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] as

$$u'''' = \frac{\dots + d_1 \exp[-(15q+d)\eta]}{\dots + d_2 \exp[-16q\eta]}, \tag{11}$$

and

$$u'u'' = \frac{\dots + d_3 \exp[-(d+q)\eta]}{\dots + d_4 \exp[-2q\eta]} \times \frac{\dots + d_5 \exp[-(d+3q)\eta]}{\dots + d_6 \exp[-4q\eta]} = \frac{\dots + d_7 \exp[-(2d+14q)\eta]}{\dots + d_8 \exp[-16q\eta]}, \tag{12}$$

where d_i are determined coefficients only for simplicity. Balancing lowest order of Exp-function in equations (11) and (12) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] as

$$15q + d = 2d + 14q, \tag{13}$$

which leads to

$$q = d. \tag{14}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$. Thus, equation (4) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{15}$$

Substituting (15) into (6) and using Maple, we arrive at

$$\frac{1}{A}[c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta)] = 0, \tag{16}$$

where

$$A = \left(\exp(\eta) + b_0 + b_{-1} \exp(-\eta)\right)^5, \tag{17}$$

and c_n are the coefficients of $\exp(n\eta)$. Vanishing the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, k, l, s$ and ω. Solving the system with the aid of Maple 11, we obtain:

$$\begin{aligned}
&\omega = \frac{3l^2 + k^3 s}{4k}, \; a_1 = a_1, \; b_0 = b_0, \; a_0 = a_0, \\
&a_{-1} = -\frac{4a_1 b_0^2 k^2 - 4a_0 b_0 k^2 - 4a_1^2 b_0^2 k + 6a_1 b_0 a_0 k + a_1 a_0^2 + a_1^3 b_0^2 - 2a_1^2 b_0 a_0 - 2a_0^2 k}{4k^2}, \\
&b_{-1} = \frac{2ka_1 b_0^2 - 2ka_0 b_0 - a_0^2 - a_1^2 b_0^2 + 2a_1 b_0 a_0}{4k^2}, \; k = k, \; l = l, \; s = s.
\end{aligned} \tag{18}$$

Inserting (18) into (15) admits to the generalized solitary wave solution of equation (5) as follows

$$\begin{aligned}
u(x,y,z,t) &= \frac{a_1 \exp\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right) + a_0}{\exp\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right) + b_0} \\
&\quad \frac{-\frac{4a_1 b_0^2 k^2 - 4a_0 b_0 k^2 - 4a_1^2 b_0^2 k + 6a_1 a_0 b_0 k + a_1 a_0^2 + a_1^3 b_0^2 - 2a_0 b_0 a_1^2 - 2ka_0^2}{4k^2} \exp\left[-\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right)\right]}{+\frac{2a_1 b_0^2 k - 2a_0 b_0 k - a_0^2 - a_1^2 b_0^2 + 2a_1 b_0 a_0}{4k^2} \exp\left[-\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right)\right]} \\
&= a_1 - \frac{4k^2 a_1 b_0 - 4k^2 a_0 + 2k(2a_1 b_0^2 k - 2a_0 b_0 k - a_1^2 b_0^2 + 2a_1 a_0 b_0 - a_0^2) \exp\left[-\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right)\right]}{4k^2 \exp\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right) + 4b_0 k^2 + (2ka_1 b_0^2 - 2ka_0 b_0 - a_0^2 - a_1^2 b_0^2 + 2a_1 a_0 b_0) \exp\left[-\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right)\right]}.
\end{aligned} \tag{19}$$

In case k, l, s and ω are complex numbers, the obtained solitary solution (19) reduces to the periodic solution. We write $k = iK,\, l = iL,\, s = iS$ and use the transformation

$$\begin{aligned}
&\exp\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right) = \cos\left(K\,x + L\,y + S\,z + \frac{3L^2 - K^3 S}{4K} t\right) + \mathrm{i}\sin\left(K\,x + L\,y + S\,z + \frac{3L^2 - K^3 S}{4K} t\right), \\
&\exp\left[-\left(k\,x + l\,y + s\,z + \frac{3l^2 + k^3 s}{4k} t\right)\right] = \cos\left(K\,x + L\,y + S\,z + \frac{3L^2 - K^3 S}{4K} t\right) - \mathrm{i}\sin\left(K\,x + L\,y + S\,z + \frac{3L^2 - K^3 S}{4K} t\right).
\end{aligned} \tag{20}$$

Substituting equation (20) into (19) results in a periodic solution

$$u(x,y,z,t) = a_1 + \frac{2K\left[-2a_0 K + 2a_1 b_0 K + (2Ka_1 b_0^2 - 2Ka_0 b_0)\cos(\eta) + (a_1^2 b_0^2 + a_0^2 - 2a_1 a_0 b_0)\sin(\eta) + (a_1^2 b_0^2 + a_0^2 - 2a_1 a_0 b_0) i\cos(\eta) + (-2Ka_1 b_0^2 + 2Ka_0 b_0) i\sin(\eta)\right]}{(-4K^2 - a_0^2 - a_1^2 b_0^2 + 2a_1 a_0 b_0)\cos(\eta) - 4b_0 K^2 + (2Ka_1 b_0^2 - 2Ka_0 b_0)\sin(\eta) + (2Ka_1 b_0^2 - 2Ka_0 b_0) i\cos(\eta) + (-4K^2 + a_0^2 + a_1^2 b_0^2 - 2a_1 a_0 b_0) i\sin(\eta)}, \tag{21}$$

where in this case $\eta = Kx + Ly + Sz + \frac{3L^2 - K^3 S}{4K} t$ and a_1, a_0 and b_0 are free parameters that can be determined by the related initial and boundary conditions. Setting $b_0 = 0$ and $a_0 = \pm 2K$ in equation (21), it yields

$$u(x,y,z,t) = a_1 - Ki \pm K\sec\left(Kx + Ly + Sz + \frac{3L^2 - K^3 S}{4K} t\right) - K\tan\left(Kx + Ly + Sz + \frac{3L^2 - K^3 S}{4K} t\right), \tag{22}$$

whereas, $k = iK, l = iL, s = iS$, we write $K = -ik, L = -il, S = -is$, and with substituting into equation (22), we obtain

$$u(x,y,z,t) = (a_1 - k) \mp ki \operatorname{sech}(kx + ly + sz + \frac{3l^2 + k^3 s}{4k} t) + k \tanh(kx + ly + sz + \frac{3l^2 + k^3 s}{4k} t). \quad (23)$$

Bai and Zhao (2006a) obtained the solitary solutions of equation (5) by generalized extended tanh function method. Here, we consider one of the cases expressed by equation (17a) of Bai and Zhao (2006a, 2006b) as follows

$$u = A_0(y,z,t) + pR \tanh R\left(px + q(y,z,t) + c_0\right) \pm \sqrt{\mu_1} B_1(y,z,t) \operatorname{sec}h\, R\left(px + q(y,z,t) + c_0\right)$$
$$\pm \frac{1}{\sqrt{\mu_1}} \frac{D_1(y,z,t)}{\operatorname{sec}h\, R\left(px + q(y,z,t) + c_0\right)},$$

where R and p are arbitrary nonzero constants, c_0 is an arbitrary constant, $\mu_1 = \pm 1$, and $A_0(y,z,t), B_1(y,z,t), D_1(y,z,t)$ and $q(y,z,t)$ are determined by equation (10c) of Bai and Zhao (2006a and 2006b). To compare our result, equation (23), with Bai and Zhao's solution, we set

$$p = k,\ R = 1,\ c_0 = 0,\ F_9 = 0,\ F_{10} = l,\ F_{11} = s,\ F_{12} = \left(\frac{3l^2 + k^3 s}{4k}\right) t,\ F_{13} = a_1,\ F_{14} = k$$

(see Bai and Zhao's solution). Then, we see that results are the same.

3.2. (3+1)-dimensional Jimbo-Miwa Equation

In this case, let us consider the Jimbo-Miwa equation [see Xu (2006), Ma et al. (2007)] in the form

$$u_{xxxy} + 3u_x u_{xy} + 3u_{xx} u_y + 2u_{yt} - 3u_{xz} = 0. \quad (24)$$

Making the transformation (2), equation (24) becomes

$$k^3 l u'''' + 6k^2 l u' u'' + (2l\omega - 3ks) u'' = 0. \quad (25)$$

In order to determine values of c and p, we balance the linear term of the highest order u'''' with the highest order nonlinear term $u'u''$ in equation (25) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] and obtain

$$u'''' = \frac{c_1 \exp[(15p + c)\eta] + ...}{c_2 \exp[16p\eta] + ...}, \tag{26}$$

and

$$u'u'' = \frac{c_3 \exp[(c + p)\eta] + ...}{c_4 \exp[2p\eta] + ...} \times \frac{c_5 \exp[(c + 3p)\eta] + ...}{c_6 \exp[4p\eta] + ...} = \frac{c_7 \exp[(2c + 14p)\eta] + ...}{c_8 \exp[16p\eta] + ...}, \tag{27}$$

where c_i are determined coefficients only for simplicity. Balancing the highest order of Exp-function see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007) in equations (26) and (27), we have

$$15p + c = 2c + 14p, \tag{28}$$

which leads to

$$p = c. \tag{29}$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in equation (25) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] and obtain

$$u'''' = \frac{... + d_1 \exp[-(15q + d)\eta]}{... + d_2 \exp[-16q\eta]}, \tag{30}$$

and

$$u'u'' = \frac{... + d_3 \exp[-(d + q)\eta]}{... + d_4 \exp[-2q\eta]} \times \frac{... + d_5 \exp[-(d + 3q)\eta]}{... + d_6 \exp[-4q\eta]} = \frac{... + d_7 \exp[-(2d + 14q)\eta]}{... + d_8 \exp[-16q\eta]}, \tag{31}$$

where d_i are determined coefficients only for simplicity. Balancing the lowest order of Exp-function in equations (30) and (31) [see Borhanifar and Kabir (2009), Borhanifar et al. (2009), He and Zhang (2008), Kabir and Khajeh (2009), Wu and He (2007)] and obtain

$$15q + d = 2d + 14q, \tag{32}$$

which leads to

$$q = d. \tag{33}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so equation (4) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{34}$$

Substituting (34) into (25) and by making use of Maple, we arrive at

$$\begin{aligned}\frac{1}{A}[c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1}\exp(-\eta) \\ + c_{-2}\exp(-2\eta) + c_{-3}\exp(-3\eta) + c_{-4}\exp(-4\eta)] = 0,\end{aligned} \tag{35}$$

where

$$A = (\exp(\eta) + b_0 + b_{-1}\exp(-\eta))^5 \tag{36}$$

and c_n are the coefficients of $\exp(n\eta)$. Vanishing the coefficients of all powers of $\exp(n\eta)$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_{-1}, k, l, s$ and ω. Solving the system with the aid of Maple 11, we obtain:

$$\begin{aligned}&\omega = \frac{k(3s - k^2 l)}{2l}\ ,\ k = k\ ,\ l = l\ ,\ s = s\ ,\ a_1 = a_1\ ,\\ &a_{-1} = -\frac{4a_1 b_0^2 k^2 - 4a_0 b_0 k^2 - 4a_1^2 b_0^2 k + 6a_1 b_0 a_0 k + a_1 a_0^2 + a_1^3 b_0^2 - 2a_1^2 b_0 a_0 - 2a_0^2 k}{4k^2},\\ &b_0 = b_0\ ,\ a_0 = a_0\ ,\ b_{-1} = \frac{2ka_1 b_0^2 - 2ka_0 b_0 - a_0^2 - a_1^2 b_0^2 + 2a_1 b_0 a_0}{4k^2}.\end{aligned} \tag{37}$$

Inserting equation (37) into (34) admits to the generalized solitary wave solution of equation (24) as follows

$$u(x,y,z,t) = \frac{a_1 \exp\left(k\,x + l\,y + s\,z + \frac{k(3s - k^2 l)}{2l}\,t\right) + a_0}{\exp\left(k\,x + l\,y + s\,z + \frac{k(3s - k^2 l)}{2l}\,t\right) + b_0} \tag{38}$$

$$\frac{-\frac{\left(4a_1 b_0^2 k^2 - 4a_0 b_0 k^2 - 4a_1^2 b_0^2 k + 6a_1 a_0 b_0 k + a_1 a_0^2 + a_1^3 b_0^2 - 2a_0 b_0 a_1^2 - 2ka_0^2\right)}{4k^2}\exp\left[-\left(k\,x + l\,y + s\,z + \frac{k(3s - k^2 l)}{2l}\,t\right)\right]}{+\frac{2a_1 b_0^2 k - 2a_0 b_0 k - a_0^2 - a_1^2 b_0^2 + 2a_1 b_0 a_0}{4k^2}\exp\left[-\left(k\,x + l\,y + s\,z + \frac{k(3s - k^2 l)}{2l}\,t\right)\right]}$$

$$= a_1 - \frac{4k^2 a_1 b_0 - 4k^2 a_0 + 2k\left(2a_1 b_0^2 k - 2a_0 b_0 k - a_1^2 b_0^2 + 2a_1 a_0 b_0 - a_0^2\right)\exp\left[-\left(k\,x + l\,y + s\,z + \dfrac{k(3s-k^2 l)}{2l}\,t\right)\right]}{4k^2\exp\left(k\,x + l\,y + s\,z + \dfrac{k(3s-k^2 l)}{2l}\,t\right) + 4b_0 k^2 + \left(2ka_1 b_0^2 - 2ka_0 b_0 - a_0^2 - a_1^2 b_0^2 + 2a_1 a_0 b_0\right)\exp\left[-\left(k\,x + l\,y + s\,z + \dfrac{k(3s-k^2 l)}{2l}\,t\right)\right]}.$$

In case k, l, s and ω are imaginary numbers, the obtained solitary solution (38) reduces to the periodic solution. We write $k = iK,\, l = iL,\, s = iS$ and using the transformation

$$\exp\left(k\,x + l\,y + s\,z + \frac{k(3s-k^2 l)}{2l}\,t\right) = \cos\left(K\,x + L\,y + S\,z + \frac{K(3S+K^2 L)}{2L}\,t\right) + \mathrm{i}\sin\left(K\,x + L\,y + S\,z + \frac{K(3S+K^2 L)}{2L}\,t\right),$$
$$\exp\left[-\left(k\,x + l\,y + s\,z + \frac{k(3s-k^2 l)}{2l}\,t\right)\right] = \cos\left(K\,x + L\,y + S\,z + \frac{K(3S+K^2 L)}{2L}\,t\right) - \mathrm{i}\sin\left(K\,x + L\,y + S\,z + \frac{K(3S+K^2 L)}{2L}\,t\right). \quad (39)$$

Substituting equation (39) into (38) results in a periodic solution

$$u(x,y,z,t) = a_1 + \frac{2K\left[-2a_0 K + 2a_1 b_0 K + (2Ka_1 b_0^2 - 2Ka_0 b_0)\cos(\eta) + (a_1^2 b_0^2 + a_0^2 - 2a_1 a_0 b_0)\sin(\eta)\right.}{\left(-4K^2 - a_0^2 - a_1^2 b_0^2 + 2a_1 a_0 b_0\right)\cos(\eta) - 4b_0 K^2 + \left(2Ka_1 b_0^2 - 2Ka_0 b_0\right)\sin(\eta)} \quad (40)$$
$$\frac{\left.+(a_1^2 b_0^2 + a_0^2 - 2a_1 a_0 b_0)i\cos(\eta) + (-2Ka_1 b_0^2 + 2Ka_0 b_0)i\sin(\eta)\right]}{+(2Ka_1 b_0^2 - 2Ka_0 b_0)\, i\cos(\eta) + (-4K^2 + a_0^2 + a_1^2 b_0^2 - 2a_1 a_0 b_0)\, i\sin(\eta)},$$

where in this case $\eta = Kx + Ly + Sz + \dfrac{K(3S+K^2 L)}{2L}t$ and a_1, a_0 and b_0 are arbitrary parameters that can be determined by the related initial and boundary conditions. If we set $b_0 = 0$ and $a_0 = \pm 2K$ in equation (40), we obtain

$$u(x,y,z,t) = (\mathrm{a}_1 - Ki) \pm K\sec\left(Kx + Ly + Sz + \frac{K(3S+K^2 L)}{2L}t\right) - K\tan\left(Kx + Ly + Sz + \frac{K(3S+K^2 L)}{2L}t\right), \quad (41)$$

whereas $k = iK,\, l = iL,\, s = iS$, we write $K = -ik,\, L = -il,\, S = -is$, and with substituting into equation (41), we obtain

$$u(x,y,z,t) = (a_1 - k) \mp ki\,\mathrm{sech}\left(kx + ly + sz + \frac{3ks - k^3 l}{2l}t\right) + k\tanh\left(kx + ly + sz + \frac{3ks - k^3 l}{2l}t\right). \quad (42)$$

Ma and co-workers found the following solitary wave solutions of equation (24) by the improved mapping approach. Here, we consider one of the cases expressed by equation (9) in Ma et al. (2007) as follows

$$u = \frac{-1}{3}\int \frac{aX_x^3 - 3bX_x + aX_{xxx} + 2aX_t}{aX_x}\,dx + X_x\tanh\left(X + ay + bz\right) \pm X_x\sqrt{\tanh^2\left(X + ay + bz\right) - 1},$$

where $X \equiv X(x,t)$ is an arbitrary function of (x,t), a and b are two arbitrary constants. To compare our result, equation (42), with Ma and co-workers' solution [see Ma et al. (2007)], we set $a = l, b = s, X(x,t) = kx + \left(\frac{3ks - k^3 l}{2l}\right)t$ in their solution and we found that the results were the same.

Remark:

We have verified all the obtained solutions by putting them back into the original equations (5) and (24) with the aid of Maple 11.

4. Conclusion

Based on what we did in this paper, we conclude that the Exp-function method is very powerful and efficient technique in finding exact solutions for wide classes of problems such as nonlinear wave equations and systems. The Exp-function method has got many merits and much more advantages than the exact solutions. Calculations in the Exp-function method are simple and straightforward. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. The results show that the Exp-function method is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in engineering.

REFERENCES

Bai, C. L. and Zhao, H. (2006a). Generalized method to construct the solitonic solutions to (3+1)-dimensional nonlinear equation, Physics Letters A, 354, pp. 428-436.

Bai, C. L. and Zhao, H. (2006b). Generalized extended tanh-function method and its application, Chaos, Solitons and Fractals 27, pp. 1026-1035.

Borhanifar, A. and Kabir, M. M. (2009). New periodic and soliton solutions by application of Exp-function method for nonlinear evolution equations. Journal of computational and applied mathematics, 229, pp. 158-167.

Borhanifar, A., Kabir, M. M. and Vahdat Lasemi, M. (2009). New periodic and soliton wave solutions for the generalized Zakharov system and (2 + 1)-dimensional Nizhnik-Novikov-Veselov system. Chaos, Solitons and Fractals, 42, pp. 1646-1654.

He, J. H. (2000). A coupling method of a homotopy technique and a perturbation technique for non-linear problems. International Journal of Non-Linear Mechanics, 35, pp. 37-43.

He, J. H. (2004). Comparison of homotopy perturbation method and homotopy analysis method, Applied Mathematics and Computation, 156, pp. 527-539.

He, J. H. (2004). Homotopy perturbation method: a new nonlinear analytical technique, Applied Mathematics and Computation, 156, pp. 591-596.

He, J. H. (2005). Variational iteration method for autonomous ordinary differential systems, Applied Mathematics and Computation, 114, pp. 115-123.

He, J. H. (2005). Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons & Fractals, 26, pp. 695-700.

He, J. H. (2006). Asymptotology by homotopy perturbation method, Applied Mathematics and Computation, 156, pp. 591-596.

He, J. H. (2006). Non-Perturbative method for strongly nonlinear problems, Berlin: dissertation, De-Verlag im Internet GmbH.

He, J. H. and Abdou, M, A. (2007). New periodic solutions for nonlinear evolution equations using Exp-function method. Chaos, Solitons & Fractals, 34, pp. 1421-1429.

He, J. H. and Wu, X, H. (2006). Exp-function method for nonlinear wave equations, Chaos, Solitons & Fractals, 30, pp. 700-708.

He, J. H. and Zhang, L. N. (2008). Generalized solitary solution and compaction like solution of the Jaulent-Miodek equations using the Exp-function method. Physics Letters, A 372, pp. 1044-1047.

Kabir, M. M. and Khajeh, A. (2009). New explicit solutions for the Vakhnenko and a generalized form of the nonlinear heat conduction equations via Exp-function method, International Journal of Nonlinear Sciences & Numerical Simulation, 10, pp. 1307-1318.

Ma, S. H., Fang, J. P. and Zheng, C. L. (2007). New Exact solutions for the (3+1)-dimensional Jimbo-Miwa system. Chaos, Solitons & Fractals, in press.

Wazwaz, A. M. (2007). New solutions of distinct physical structures to high-dimensional nonlinear evolution equations, Applied Mathematics and Computation, in press.

Wu, X. H. and He, J. H. (2007). Exp-function method and its application to nonlinear equations, Chaos, Solitons & Fractals, doi:10.1016/j.chaos.2007.01.024.

Wu, X. H. and He J. H. (2007). Solitary solutions periodic solutions and compaction like solutions using Exp-function method, Comput. Math. With App., in press.

Xu, G. (2006). The soliton solutions, dominos of the Kadomtsev-Petviashvili and Jimbo-Miwa equations in (3 + 1)-dimensions, Chaos, Solitons and Fractals, 30, PP. 71-76.

Yu, S. J., Toda, K., Sasa, N. and Fukuyama, T. (1998). *N* soliton solutions to the Bogoyavlenskii-Schiff equation and quest for the soliton solution in (3+1) dimensions. J. Phys. A: Math. Gen., 31, pp. 3337-3347.

Zhang, S. (2006). Application of Exp-function method to high-dimensional nonlinear evolution equation, Chaos, solution & fractals, doi:10.1016/j.chaos.2006.11.014.

Zhu, S. D. (2007a). Exp-function Method for the Hybrid-Lattice system, International Journal of Nonlinear Science and Numerical Simulation, 8, pp. 461-464.

Zhu, S. D. (2007b). Exp-function Method for the Discrete mKdV Lattice, International Journal of Nonlinear Science and Numerical Simulation, 8, pp. 465-468.