Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 10, Issue 2 (December 2015), pp. 678 – 693

Applications and Applied Mathematics:
An International Journal
(AAM)

An M/G/1 Queue with Server Breakdown and Multiple Working Vavation

[1]S. Pazhani Bala Murugan and [2]K. Santhi

Mathematics Section, Faculty of Engineering and Technology
Annamalai University, Annamalainagar-608 002, India
[1]spbmaths@yahoo.co.in, [2]santhimano3169@gmail.com

Received: February 14, 2015; Accepted: October 2, 2015

Abstract

This paper deals with the steady state behavior of an $M/G/1$ multiple working vacation queue with server breakdown. The server works with different service times rather than completely stopping service during a vacation. Both service times in a vacation period and in a regular service period are assumed to be generally distributed random variables. The system may breakdown at random and repair time is arbitrary. Further, just after completion of a customer's service the server may take a multiple working vacation. Supplementary variable technique is employed to find the probability generating function for the number of customers in the system. The mean number of customers in the system is calculated. Some particular cases of interest are discussed. Numerical results are also presented.

Keywords: Poisson arrivals, Random breakdown, Repair time, Working Vacation, Supplementary Variable Technique

MSC 2010 No.: 60K25, 60K30

1. Introduction

In most of the queueing literature it is assumed that the server is available in the service station on a permanent basis and service station never fails. However, these assumptions are unrealistic. In practical situations we often meet the case where service stations may fail or slow down,

during the time, at which the repairing works are carried out. Such phenomenon always occur in the areas of computer communication networks and flexible manufacturing systems. Vacation queueing models subject to breakdowns have been studied by many authors including Gaver (1959), Levy and Yechilai (1976), Fuhrman (1981), Doshi (1986), Shanthikumar (1988), Kramer (1989), Madan (1999) and Madan and Saleh (2001) to mention a few. Sengupta (1990), Takine and Sengupta (1997), Li et al. (1997), Madan (2003), Choudhury and Tadj (2009), and Thangaraj and Vanitha (2010) studied M/G/1 queue with breakdowns and vacations.

Recently a class of semi-vacation policies called working vacation (WV) have been introduced. During this period the server works with a lower rate rather than completely stopping service. Servi and Finn (2002) studied an M/M/1 queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works are by Wu and Takagi (2006), Tian et al. (2008), Begum and Parveen (2011) and Santhi and Pazhani Bala Murugan (2013, 2014).

In this paper we study a non-Markovian queue with multiple working vacation and random breakdown. The organization of the paper is as follows. In Section 2, we describe the model. In Section 3, we obtain the steady state probability generating function. Particular cases are dicussed in Section 4. Some performance measures are obtained in Section 5, and in Section 6 numerical study is presented.

2. The Model description

We assume the following to describe the queueing model under study. Customers arrive at the system one by one according to a Poisson stream with arrival rate $\lambda(>0)$. The service discipline is FCFS. The service time follows a general distribution. Let $S_b(x), s_b(x)$ and $S_b^*(\theta)$ be the distribution function, the probability density function and the Laplace Stieltjes Transform (LST) of the service time S_b.

Whenever the system becomes empty at a service completion instant the working vacation of the server is begun. The duration of the vacation time is assumed to follow an exponential distribution with rate η. At a vacation completion instant, if there are customers in the system a new busy period will start. Otherwise, it stays in working vacation. This type of vacation is called multiple working vacation. During the working vacation, the server provides service with a different service time S_v which follows a general distribution with distribution function $S_v(x)$. Let $s_v(x)$ and $S_v^*(\theta)$ denote the corresponding probability density function, and Laplace Stieltjes Transform respectively.

The system may breakdown at random and it is assumed to occur according to a Poisson stream with mean breakdown rate $\alpha_1(>0)$ during the regular service period and $\alpha_2(>0)$ during the WV period, respectively. Further, we assume that once the system breaks down, the customer whose service is interrupted comes back to the head of the queue and the system enters a repair process immediately. The repair time is also assumed to follow a general distribution. Let the repair time distribution functions be $S_{r_1}(x)$ and $S_{r_2}(x)$ during the regular service period and WV

period, respectively. Let $s_{r_1}(x)$, $s_{r_2}(x)$, $S^*_{r_1}(\theta)$, and $S^*_{r_2}(\theta)$ denoted the corresponding densities and LSTs respectively.

Various stochastic processes involved in the system are assumed to be independent of each other.

3. The System Size Distribution

The system size distribution at an arbitrary time can be obtained by using the supplementary variable technique, that is, from the joint distribution of the queue length and the remaining service time of the customer in service if the server is busy/working vacation. We define the following random variables.

$N(t)$ – the system size at time t
$S^0_b(t)$ – the remaining service time in regular service period.
$S^0_v(t)$ – the remaining service time in working vacation period.
$S^0_{r_1}(t)$ – the remaining repair time in regular service period.
$S^0_{r_2}(t)$ – the remaining repair time in working vacation period.

$$Y(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy at time } t, \\ 2 & \text{if the server is busy on working vacation period at time } t, \\ 3 & \text{if the server is waiting for completion of repairing work during the busy period at time } t, \\ 4 & \text{if the server is waiting for completion of repairing work during the working vacation period at time } t. \end{cases}$$

Supplementary variables $S^0_b(t), S^0_v(t), S^0_{r_1}(t)$ and $S^0_{r_2}(t)$ are introduced in order to obtain bivariate Markov process $\{(N(t), \partial(t)); t \geq 0\}$ where

$$\partial(t) = \begin{cases} S^0_b(t) & \text{if } Y(t) = 1, \\ S^0_v(t) & \text{if } Y(t) = 2, \\ S^0_{r_1}(t) & \text{if } Y(t) = 3, \\ S^0_{r_2}(t) & \text{if } Y(t) = 4. \end{cases}$$

We define the following limiting probabilities:

$$\begin{aligned} Q_0 &= \lim_{t\to\infty} Pr\{N(t) = 0, Y(t) = 0\} , \\ P_n(x) &= \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 1, x < S^0_b(t) \leq x + dx\} ; \quad n \geq 1, \\ Q_n(x) &= \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 2, x < S^0_v(t) \leq x + dx\} ; \quad n \geq 1, \\ R_{1,n}(x) &= \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 3, x < S^0_{r_1}(t) \leq x + dx\} ; \quad n \geq 1, \\ \text{and} \quad R_{2,n}(x) &= \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 4, x < S^0_{r_2}(t) \leq x + dx\} ; \quad n \geq 1. \end{aligned}$$

Under the assumption that steady state conditions are reached, we have the following system of differential difference equations:

$$\lambda Q_0 = P_1(0) + Q_1(0), \tag{1}$$

$$-\frac{d}{dx}Q_1(x) = -(\lambda+\alpha_2+\eta)Q_1(x) + Q_2(0)s_v(x) + \lambda Q_0 s_v(x) + R_{2,1}(0)s_v(x), \tag{2}$$

$$-\frac{d}{dx}Q_n(x) = -(\lambda+\alpha_2+\eta)Q_n(x) + Q_{n+1}(0)s_v(x) + \lambda Q_{n-1}(x) + R_{2,n}(0)s_v(x)\ ;\ n \geq 2, \tag{3}$$

$$-\frac{d}{dx}P_1(x) = -(\lambda+\alpha_1)P_1(x) + P_2(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_1(y)dy + R_{1,1}(0)s_b(x), \tag{4}$$

$$-\frac{d}{dx}P_n(x) = -(\lambda+\alpha_1)P_n(x) + P_{n+1}(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_n(y)dy + \lambda P_{n-1}(x), \\ +R_{1,n}(0)s_b(x)\ ;\quad n \geq 2, \tag{5}$$

$$-\frac{d}{dx}R_{2,1}(x) = -(\lambda+\eta)R_{2,1}(x) + \alpha_2 s_{r_2}(x)\int_0^\infty Q_1(x)dx, \tag{6}$$

$$-\frac{d}{dx}R_{2,n}(x) = -(\lambda+\eta)R_{2,n}(x) + \lambda R_{2,n-1}(x) + \alpha_2 s_{r_2}(x)\int_0^\infty Q_n(x)dx\ ;\quad n \geq 2, \tag{7}$$

$$-\frac{d}{dx}R_{1,1}(x) = -\lambda R_{1,1}(x) + \alpha_1 s_{r_1}(x)\int_0^\infty P_1(x)dx + \eta s_{r_1}(x)\int_0^\infty R_{2,1}(y)dy, \tag{8}$$

$$-\frac{d}{dx}R_{1,n}(x) = -\lambda R_{1,n}(x) + \lambda R_{1,n-1}(x) + \alpha_1 s_{r_1}(x)\int_0^\infty P_n(x)dx \\ +\eta s_{r_1}(x)\int_0^\infty R_{2,n}(y)dy\ ;n \geq 2. \tag{9}$$

We define the Laplace Stieltjes transforms and the probability generating functions as follows. For $i = 1, 2$,

$$S_b^*(\theta) = \int_0^\infty e^{-\theta x}s_b(x)dx\ ; \qquad S_v^*(\theta) = \int_0^\infty e^{-\theta x}s_v(x)dx\ ; \qquad S_{r_i}^*(\theta) = \int_0^\infty e^{-\theta x}s_{r_i}(x)dx\ ;$$

$$Q_n^*(\theta) = \int_0^\infty e^{-\theta x}Q_n(x)dx\ ; \quad Q_n^*(0) = \int_0^\infty Q_n(x)dx\ , \qquad P_n^*(\theta) = \int_0^\infty e^{-\theta x}P_n(x)dx;$$

$$P_n^*(0) = \int_0^\infty P_n(x)dx; \qquad R_{i,n}^*(\theta) = \int_0^\infty e^{-\theta x}R_{i,n}(x)dx\ ; R_{i,n}^*(0) = \int_0^\infty R_{i,n}(x)dx\ ;$$

$$Q^*(z,\theta) = \sum_{n=1}^{\infty} Q_n^*(\theta)z^n, \quad Q(z,0) = \sum_{n=1}^{\infty} Q_n(0)z^n \quad Q^*(z,0) = \sum_{n=1}^{\infty} Q_n^*(0)z^n$$
$$P^*(z,\theta) = \sum_{n=1}^{\infty} P_n^*(\theta)z^n \quad P(z,0) = \sum_{n=1}^{\infty} P_n(0)z^n, \quad P^*(z,0) = \sum_{n=1}^{\infty} P_n^*(0)z^n \ ;$$
$$R_i^*(z,\theta) = \sum_{n=1}^{\infty} R_{i,n}^*(\theta)z^n, \quad R_i(z,0) = \sum_{n=1}^{\infty} R_{i,n}(0)z^n, \quad R_i^*(z,0) = \sum_{n=1}^{\infty} R_{i,n}^*(0)z^n.$$

Taking LST of (2) to (9), we get

$$\theta Q_1^*(\theta) - Q_1(0) = (\lambda+\alpha_2+\eta)Q_1^*(\theta) - Q_2(0)S_v^*(\theta) - \lambda Q_0 S_v^*(\theta) - R_{2,1}(0)S_v^*(\theta), \tag{10}$$
$$\theta Q_n^*(\theta) - Q_n(0) = (\lambda+\alpha_2+\eta)Q_n^*(\theta) - Q_{n+1}(0)S_v^*(\theta) - \lambda Q_{n-1}^*\theta - R_{2,n}(0)S_v^*(\theta) \ ; \quad n \geq 2, \tag{11}$$
$$\theta P_1^*(\theta) - P_1(0) = (\lambda+\alpha_1)P_1^*(\theta) - P_2(0)S_b^*(\theta) - \eta S_b^*(\theta)Q_1^*(0) - R_{1,1}(0)S_b^*(\theta), \tag{12}$$

$$\theta P_n^*(\theta) - P_n(0) = (\lambda+\alpha_1)P_n^*(\theta) - P_{n+1}(0)S_b^*(\theta) - \eta S_b^*(\theta)Q_n^*(0) - \lambda P_{n-1}^*(\theta) - R_{1,n}(0)S_b^*(\theta) \ ; \ n \geq 2, \tag{13}$$
$$\theta R_{2,1}^*(\theta) - R_{2,1}(0) = (\lambda+\eta)R_{2,1}^*(\theta) - \alpha_2 S_{r_2}^*(\theta)Q_1^*(0), \tag{14}$$
$$\theta R_{2,n}^*(\theta) - R_{2,n}(0) = (\lambda+\eta)R_{2,n}^*(\theta) - \lambda R_{2,n-1}^*(\theta) - \alpha_2 S_{r_2}^*(\theta)Q_n^*(0) \ ; \ n \geq 2, \tag{15}$$
$$\theta R_{1,1}^*(\theta) - R_{1,1}(0) = \lambda R_{1,1}^*(\theta) - \alpha_1 S_{r_1}^*(\theta)P_1^*(0) - \eta S_{r_1}^*(\theta)R_{2,1}^*(0), \tag{16}$$
$$\theta R_{1,n}^*(\theta) - R_{1,n}(0) = \lambda R_{1,n}^*(\theta) - \lambda R_{1,n-1}^*(\theta) - \alpha_1 S_{r_1}^*(\theta)P_n^*(0) - \eta S_{r_1}^*(\theta)R_{2,n}^*(0) \ ; \quad n \geq 2. \tag{17}$$

z^n times (11) summed over n from 2 to ∞ and added up with z times (10) yields

$$[\theta - (\lambda - \lambda z + \alpha_2 + \eta)]Q^*(z,\theta) = \left[\frac{z - S_v^*(\theta)}{z}\right]Q(z,0) - S_v^*(\theta)[\lambda z Q_0 + R_2(z,0) - Q_1(0)]. \tag{18}$$

Inserting $\theta = (\lambda - \lambda z + \alpha_2 + \eta) = (a(z) + \alpha_2)$ in (18), we get

$$Q(z,0) = \frac{zS_v^*(a(z)+\alpha_2)[\lambda z Q_0 + R_2(z,0) - Q_1(0)]}{z - S_v^*(a(z)+\alpha_2)}. \tag{19}$$

z^n times (15) summed over n from 2 to ∞, added up with z times (14), gives

$$[\theta - (\lambda - \lambda z + \eta)]R_2^*(z,\theta) = R_2(z,0) - \alpha_2 S_{r_2}^*(\theta)Q^*(z,0). \tag{20}$$

Inserting $\theta = (\lambda - \lambda z + \eta) = a(z)$ in (20), we get

$$R_2(z,0) = \alpha_2 S_{r_2}^*(a(z))Q^*(z,0)(21). \tag{21}$$

Substituting (21) in (20) and putting $\theta = 0$, we get

$$R_2^*(z,0) = \frac{\alpha_2 Q^*(z,0)(1 - S_{r_2}^*(a(z)))}{(a(z))}. \tag{22}$$

Substituting (21) in (19), we get

$$Q(z,0) = \frac{zS_v^*(a(z)+\alpha_2)[\lambda z Q_0 + \alpha_2 S_{r_2}^*(a(z))Q^*(z,0) - Q_1(0)]}{z - S_v^*(a(z)+\alpha_2)}. \tag{23}$$

Substituting (21) and (23) in (18), we get

$$[\theta - (a(z)+\alpha_2)]Q^*(z,\theta) = \frac{\left[z(S_v^*(a(z)+\alpha_2) - S_v^*(\theta))[\alpha_2 S_{r_2}^*(a(z))Q^*(z,0) + \lambda z Q_0 - Q_1(0)]\right]}{z - S_v^*(a(z)+\alpha_2)}.$$

Putting $\theta = 0$, we get

$$Q^*(z,0) = \frac{z(1 - S_v^*(a(z)+\alpha_2))(\lambda z Q_0 - Q_1(0))}{\left[(a(z)+\alpha_2)(z - S_v^*(a(z)+\alpha_2)) - \alpha_2 z(1 - S_v^*(a(z)+\alpha_2))S_{r_2}^*(\lambda - \lambda z + \eta)\right]}. \tag{24}$$

The denominator of the above equation has a unique root z_1 in (0,1) and thus $Q_1(0) = \lambda z_1 Q_0$. Substituting this in (24), we get

$$Q^*(z,0) = \frac{\lambda z(z - z_1)(1 - S_v^*(a(z)+\alpha_2))Q_0}{\left[(a(z)+\alpha_2)(z - S_v^*(a(z)+\alpha_2)) - \alpha_2 z(1 - S_v^*(a(z)+\alpha_2))S_{r_2}^*(\lambda - \lambda z + \eta)\right]}. \tag{25}$$

Substituting (25) in (22), we get

$$R_2^*(z,0) = \frac{Q_0\alpha_2(1 - S_{r_2}^*(a(z)))\lambda z(z - z_1)(1 - S_v^*(a(z)+\alpha_2))}{a(z)\left[(a(z)+\alpha_2)(z - S_v^*(a(z)+\alpha_2)) - \alpha_2 z(1 - S_v^*(a(z)+\alpha_2))S_{r_2}^*(a(z))\right]}. \tag{26}$$

z^n times (13) summed over n from 2 to ∞, is added up with z times (12) yields

$$[\theta - (\lambda - \lambda z + \alpha_1)]P^*(z,\theta) = \left[\frac{z - S_b^*(\theta)}{z}\right]P(z,0) - S_b^*(\theta)[\eta Q^*(z,0) + R_1(z,0) - P_1(0)]. \tag{27}$$

Inserting $\theta = (\lambda - \lambda z + \alpha_1) = a_1(z)$ and substituting $\lambda(1 - z_1)Q_0 = P_1(0)$ in (27), we get

$$P(z,0) = \frac{zS_b^*(a_1(z))[\eta Q^*(z,0) + R_1(z,0) - \lambda(1 - z_1)Q_0]}{z - S_b^*(a_1(z))}. \tag{28}$$

z^n times (17) summed over n from 2 to ∞, added with z times (16), results in

$$[\theta - (\lambda - \lambda z)]R_1^*(z,\theta) = R_1(z,0) - \alpha_1 S_{r_1}^*(\theta)P^*(z,0) - \eta S_{r_1}^*(\theta)R_2^*(z,0). \tag{29}$$

Inserting $\theta = (\lambda - \lambda z)$ in (29), we get

$$R_1(z,0) = S_{r_1}^*(\lambda - \lambda z)\left[\alpha_1 P^*(z,0) + \eta R_2^*(z,0)\right]. \tag{30}$$

Substituting (30) in (29) and putting $\theta = 0$ in (29), we get

$$R_1^*(z,0) = \frac{(1 - S_{r_1}^*(\lambda - \lambda z))[\alpha_1 P^*(z,0) + \eta R_2^*(z,0)]}{(\lambda - \lambda z)}. \tag{31}$$

Substituting (22), (25), (28), (30) and $\lambda(1-z_1)Q_0 = P_1(0)$ and putting $\theta = 0$ in (27), we get

$$P^*(z,0) = \frac{Nr(z)}{Dr(z)}, \tag{32}$$

where

$$\begin{aligned} Nr(z) &= Q_0\lambda z(1-S_b^*(a_1(z)))\big\{\eta z(z-z_1)(1-S_v^*(a(z)+\alpha_2))\big[a(z)+\alpha_2 S_{r_1}^*(\lambda-\lambda z) \\ &\quad \times(1-S_{r_2}^*(a(z)))\big] - (a(z))(1-z_1)\big[(a(z)+\alpha_2)(z-S_v^*(a(z)+\alpha_2)) \\ &\quad -\alpha_2 z(1-S_v^*(a(z)+\alpha_2))S_{r_2}^*(a_1(z))\big]\big\}, \\ Dr(z) &= a(z)\big\{(a_1(z))(z-S_b^*(a_1(z))) - z\alpha_1(1-S_b^*(a_1(z)))S_{r_1}^*(\lambda-\lambda z)\big\} \\ &\quad \times\big\{(a(z)+\alpha_2)(z-S_v^*(a(z)+\alpha_2)) - \alpha_2 z(1-S_v^*(a(z)+\alpha_2))S_{r_2}^*(a(z))\big\}. \end{aligned}$$

Substituting (26) and (32) in (31), we get

$$\begin{aligned} R_1^*(z,0) &= \frac{Q_0(1-S_{r_1}^*(\lambda-\lambda z))}{D_1(z)D_2(z)D_3(z)}\Big\{\alpha_1\lambda z(1-S_b^*(a_1(z)))\big\{\eta z(z-z_1)(1-S_v^*(a(z)+\alpha_2))) \\ &\quad \times\big[(a(z)+\alpha_2 S_{r_1}^*(\lambda-\lambda z)(1-S_{r_2}^*(a(z)))\big] - (a(z))\times(1-z_1)\big[(a(z)+\alpha_2) \\ &\quad \times(z-S_v^*(a(z)+\alpha_2)) + \alpha_2\eta\lambda z(z-z_1)(1-S_v^*(a(z)+\alpha_2))\times(1-S_{r_2}^*(a(z))) \\ &\quad \times\big\{(a_1(z))\times(z-S_b^*(a_1(z))) - \alpha_1 z(1-S_b^*(a_1(z)))\times S_{r_1}^*(\lambda-\lambda z)\big\}\Big\}, \end{aligned} \tag{33}$$

where

$$D_1(z) = a(z)\big\{a_1(z)(z-S_b^*(a_1(z))) - z\alpha_1(1-S_b^*(a_1(z)))S_{r_1}^*(\lambda-\lambda z)\big\}, \tag{34}$$

$$D_2(z) = \big\{(a(z)+\alpha_2)(z-S_v^*(a(z)+\alpha_2)) - \alpha_2 z(1-S_v^*(a(z)+\alpha_2))S_{r_2}^*(a(z))\big\}, \tag{35}$$

$$D_3(z) = (\lambda-\lambda z). \tag{36}$$

We define

$$P_B(z) = P^*(z,0) + R_1^*(z,0) \tag{37}$$

as the probability generating function for the number of customers in the system when the server is in regular service period,

$$P_V(z) = Q^*(z,0) + R_2^*(z,0) + Q_0 \tag{38}$$

as the probability generating function for the number of customers in the system when the server is on working vacation period, and

$$P(z) = P_B(z) + P_V(z) \tag{39}$$

as the probability generating function for the number of customers in the system. We now use the normalizing condition $P(1) = 1$ to determine the only unknown, Q_0, which appears in (39). Substituting $z = 1$ in (39) and using L'Hôpital's rule, we obtain

$$Q_0 = \frac{1-\rho_b}{\left[\dfrac{\eta+\lambda(1-z_1)}{\eta}\right] - \left[\dfrac{C_1}{C_2}\right]}, \tag{40}$$

where

$$\begin{aligned} C_1 &= \lambda(1-z_1)\big\{(1-S_b^*(\alpha_1))S_v^*(\eta+\alpha_2)\times[\alpha_2+\eta(1+\alpha_1 E(S_{r_1}))]+\alpha_2\alpha_1 E(S_{r_1}) \\ &\quad \times[S_v^*(\eta+\alpha_2)-S_b^*(\alpha_1)[1-S_{r_2}^*(\eta)(1-S_v^*(\eta+\alpha_2))]]\big\}, \\ C_2 &= \alpha_1 S_b^*(\alpha_1)(1-S_v^*(\eta+\alpha_2))(\eta+\alpha_2(1-S_{r_2}^*(\eta))), \end{aligned}$$

and $\rho_b = \dfrac{\lambda(1-S_b^*(\alpha_1))(1+\alpha_1 E(S_{r_1}))}{\alpha_1 S_b^*(\alpha_1)}$, $E(S_{r_1})$ is the mean repair time in regular service period. From (40) we obtain the system stability condition $\rho_b < 1$.

4. Particular Cases

Case (i): If the system suffers no breakdowns, then letting $\alpha_1 = 0$ and $\alpha_2 = 0$ in (39), we have

$$P(z) = \frac{Q_0\lambda z(1-S_b^*(\lambda-\lambda z))\times Nr_1(z)}{(\lambda-\lambda z)(a(z))(z-S_b^*(\lambda-\lambda z))(z-S_v^*(a(z)))}, \tag{41}$$

where

$$\begin{aligned} Nr_1(z) &= \Big\{\eta z(z-z_1)(1-S_v^*(a(z)))-(1-z_1)(a(z))(z-S_v^*(a(z)))\Big\}+(\lambda-\lambda z), \\ &\quad \times(z-S_b^*(\lambda-\lambda z))\big\{\lambda z(z-z_1)(1-S_v^*(a(z)))+(a(z))(z-S_v^*(a(z)))\big\}\Big\}, \end{aligned}$$

where $Q_0 = \dfrac{1-\rho_b}{\left[\dfrac{\lambda-\lambda z_1+\eta}{\eta}-\dfrac{\rho_b(1-z_1)S_v^*(\eta)}{1-S_v^*(\eta)}\right]}$, $\qquad \rho_b = \lambda E(S_b)$.

Equation (41) is a well-known probability generating function of the steady state system length distribution of an $M/G/1$ queue with multiple working vacation (Takagi (2006)) irrespective of the notations.

Case (ii): If the server never does the work during vacation period then setting $S_v^*(\lambda-\lambda z+\eta+\alpha_2)=0, \alpha_2=0$ and $S_{r_2}^*(\lambda-\lambda z+\eta)=0$ in (39) and by taking the repair time to be exponentially distributed, we get

$$P(z) = P_V(z) + P_B(z), \tag{42}$$

where

$$P_V(z) = \frac{Q_0(\lambda(1-z_1)+\eta)}{\lambda-\lambda z+\eta}$$

$$P_B(z) = \frac{Q_0 z[S_b^*(a_1(z))-1][\lambda(1-z_1)+\eta][(\lambda-\lambda z)(\beta+\lambda-\lambda z)+\alpha_1(\lambda-\lambda z)]}{\left\{\begin{array}{c}(a(z))[(\lambda-\lambda z)(z-S_b^*(a_1(z)))(\beta+\lambda-\lambda z) \\ +\alpha_1 z(\lambda-\lambda z)-\alpha_1 S_b^*(a_1(z))(\beta(1-z)+\lambda-\lambda z)]\end{array}\right\}}$$

$$Q_0 = \frac{1-\rho_b}{\left[\dfrac{\eta+\lambda(1-z_1)}{\eta}\right]}, \quad \rho_b = \frac{\lambda(1-S_b^*(\alpha_1))(\alpha_1+\beta)}{\alpha_1\beta S_b^*(\alpha_1)}.$$

Equation (42) is a well-known probability generating function of the steady state system length distribution of an $M/G/1$ queue with Server Vacation and Random Breakdown (Thangaraj (2010) no second stage service) irrespective of the notations.

Case (iii): If the system suffers no breakdowns and the server never takes a vacation then on setting $\alpha_1 = 0, \alpha_2 = 0$ and taking limit $\eta \to \infty$ in (39) we get

$$P(z) = \frac{(1 - \lambda E(S_b))(1 - z)S_b^*(\lambda - \lambda z)}{S_b^*(\lambda - \lambda z) - z}. \tag{43}$$

Equation (43) is a well-known probability generating function of the steady state system length distribution of an $M/G/1$ queue (Medhi (1982)) irrespective of the notations where $E(S_{r_1})$ is the mean repair time in regular service period.

5. Performance Measures

Let L_v and L_b denote the mean system size during the working vacation and regular service period respectively and let W_v and W_b be the mean waiting time of the customers in the system during working vacation period and regular service period respectively.

$$L_v = \frac{d}{dz}\big[P_V(z)\big]\big|_{z=1} = \frac{d}{dz}\left[\frac{A(z)}{D_2(z)} + \frac{B(z)}{a(z)D_2(z)}\right]Q_0\big|_{z=1},$$

where

$$\begin{aligned} A(z) &= \lambda z(z - z_1)(1 - S_v^*(a(z) + \alpha_2)), \\ B(z) &= \alpha_2 \lambda z(z - z_1)(1 - S_{r_2}^*(a(z)))(1 - S_v^*(a(z) + \alpha_2)). \end{aligned}$$

$D_2(z)$ is given in (35). Therefore

$$L_v = Q_0\left[\frac{D_2(1)A'(1) - A(1)D_2'(1)}{(D_2(1))^2} + \frac{\eta(D_2(1)B'(1) - B(1)D_2'(1)) + \lambda B(1)D_2(1)}{(\eta D_2(1))^2}\right],$$

and applying Little's formula $W_v = \dfrac{L_v}{\lambda}$, we have

$$\begin{aligned} A(1) &= \lambda(1 - z_1)(1 - S_v^*(\eta + \alpha_2)), \\ A'(1) &= (1 - S_v^*(\eta + \alpha_2))(\lambda + \lambda(1 - z_1)) + \lambda^2(1 - z_1)S_v^{*'}(\eta + \alpha_2), \\ D_2(1) &= (1 - S_v^*(\eta + \alpha_2))[\eta + \alpha_2(1 - S_{r_2}^*(\eta))], \\ B(1) &= \alpha_2\lambda(1 - z_1)(1 - S_{r_2}^*(\eta)(1 - S_v^*(\eta + \alpha_2)), \\ B'(1) &= \alpha_2\big[\lambda(1 - z_1)[(1 - S_v^*(\eta + \alpha_2))(1 - S_{r_2}^*(\eta) + \lambda S_{r_2}^*(\eta)) + \lambda(1 - S_{r_2}^*(\eta))S_v^{*'}(\eta + \alpha_2)] \\ &\quad + \lambda(1 - S_{r_2}^*(\eta)(1 - S_v^*(\eta + \alpha_2)))\big], \\ L_b &= \frac{d}{dz}\big[P_B(z)\big]\big|_{z=1} = \frac{d}{dz}\left[\frac{N_1(z)N_2(z)}{D_1(z)D_2(z)} + \frac{N_3(z)N_4(z)}{D_1(z)D_2(z)D_3(z)}\right]Q_0\Big|_{z=1}, \end{aligned}$$

where

$$
\begin{aligned}
N_1(z) &= \lambda z(1-S_b^*(a_1(z))),\\
N_2(z) &= \eta z(z-z_1)(1-S_v^*(a(z)+\alpha_2))[(a(z))+\alpha_2 S_{r_1}^*(\lambda-\lambda z)(1-S_{r_2}^*(a(z)))]-(a(z))\\
&\quad\times(1-z_1)\big[(a(z)+\alpha_2)(z-S_v^*(a(z)+\alpha_2))-\alpha_2 z S_{r_2}^*(a(z))(1-S_v^*(a(z)+\alpha_2))\big],\\
N_3(z) &= (1-S_{r_1}^*(\lambda-\lambda z)),\\
N_4(z) &= \alpha_1\lambda z(1-S_b^*(a_1(z)))\big\{\eta z(z-z_1)(1-S_v^*(a(z)+\alpha_2))[(a(z))+\alpha_2 S_{r_1}^*(\lambda-\lambda z)\\
&\quad\times(1-S_{r_2}^*(a(z)))]-(a(z))(1-z_1)\big((a(z)+\alpha_2)(z-S_v^*(a(z)+\alpha_2))-\alpha_2 z\\
&\quad\times(1-S_v^*(a(z)+\alpha_2))S_{r_2}^*(a(z))\big)\big\}+\alpha_2\eta\lambda z(z-z_1)(1-S_v^*(a(z)+\alpha_2))\\
&\quad\times(1-S_{r_2}^*(a(z)))\big\{(\lambda-\lambda z+\alpha_1)(z-S_b^*(a_1(z)))-\alpha_1 z(1-S_b^*(a_1(z)))S_{r_1}^*(\lambda-\lambda z)\big\},
\end{aligned}
$$

$D_1(z), D_2(z)$ and $D_3(z)$ are given in equations (34), (35), and (36), respectively and

$$
\begin{aligned}
L_b &= Q_0\frac{\left[\begin{array}{rl} 2D_1'(1)N_2'(1)(D_2(1)N_1'(1)-N_1(1)D_2'(1))+ & D_2(1)N_1(1)(D_1'(1)N_2''(1)\\ & -N_2'(1)D_1''(1))\end{array}\right]}{4(D_1'(1)D_2(1))^2}\\
&\quad+Q_0\frac{\left[\begin{array}{rl} D_1'(1)D_2(1)D_3'(1)(N_3''(1)N_4'(1)+ & N_3'(1)N_4''(1))-D_3'(1)N_3'(1)N_4'(1)\\ & \times(D_1''(1)D_2(1)+2D_1'(1)D_2'(1))\end{array}\right]}{2(D_1'(1)D_2(1)D_3'(1))^2},\\
W_b &= \frac{L_b}{\lambda},\\
N_1(1) &= \lambda(1-S_b^*(\alpha_1)),\\
N_1'(1) &= \lambda(1-S_b^*(\alpha_1)+\lambda^2 S_b^{*'}(\alpha_1)),\\
N_2'(1) &= (\eta+\alpha_2(1-S_{r_2}^*(\eta)))(1-S_v^*(\eta+\alpha_2))(\eta+\lambda(1-z_1))-\eta(1-z_1)\\
&\quad\times\big[(\eta+\alpha_2)S_v^*(\eta+\alpha_2)-\lambda\alpha_2 E(S_{r_1})(1-S_v^*(\eta+\alpha_2))(1-S_{r2}^*(\eta))\big]\\
N_2''(1) &= 2\eta(\eta+\alpha_2(1-S_{r_2}^*(\eta)))(1-S_v^*(\eta+\alpha_2)+\lambda(1-z_1)S_v^{*'}(\eta+\alpha_2)+\lambda S_v^{*'}(\eta+\alpha_2))\\
&\quad+2\eta(-\lambda+\lambda\alpha_2 E(S_{r_1})(1-S_{r_2}^*(\eta)))[(1-S_v^*(\eta+\alpha_2))+(1-z_1)+\lambda(1-z_1)\\
&\quad\times S_v^{*'}(\eta+\alpha_2)]+2\eta\lambda(1-S_v^*(\eta+\alpha_2))\alpha_2 S_{r_2}^{*'}(\eta)+\eta(1-z_1)\lambda^2 S_v^{*'}(\eta+\alpha_2)\\
&\quad\times\alpha_2 S_{r_2}^{*'}(\eta)+\lambda^2\alpha_2\eta(1-z_1)(1-S_v^*(\eta+\alpha_2))[E(S_{r_1})^2(1-S_{r_2}^*(\eta))+2E(S_{r_1})\\
&\quad\times S_{r_2}^{*'}(\eta)]+2\lambda(1-z_1)[(1-S_v^*(\eta+\alpha_2))(-\lambda-\alpha_2 S_{r_2}^*(\eta)+\lambda\alpha_2 S_{r_2}^{*'}(\eta))\\
&\quad+\alpha_2(1+\lambda S_v^{*'}(\eta+\alpha_2))-\lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)S_{r_2}^*(\eta)]-\eta(1-z_1)[\lambda^2\alpha_2 S_v^{*''}(\eta+\alpha_2)\\
&\quad\times S_{r_2}^*(\eta)-2\lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)S_{r_2}^*(\eta)],\\
N_3'(1) &= -\lambda E(S_{r_1}),\\
N_3''(1) &= -\lambda^2 E(S_{r_1})^2,
\end{aligned}
$$

$$
\begin{aligned}
N_4'(1) &= \lambda\alpha_1(1-S_b^*(\alpha_1))\big\{-\eta(1-z_1)S_v^*(\eta+\alpha_2)(\eta+\alpha_2)+(1-S_v^*(\eta+\alpha_2))\\
&\quad\times(\eta+\alpha_2(1-S_{r_2}^*(\eta)))(\eta+\lambda(1-z_1))\big\}+\eta\alpha_2(1-S_{r_2}^*(\eta))\lambda(1-z_1)\\
&\quad\times(1-S_v^*(\eta+\alpha_2))[\alpha_1 S_b^*(\alpha_1)-\lambda(1-S_b^*(\alpha_1))],
\end{aligned}
$$

$$
\begin{aligned}
N_4''(1) &= 2(\lambda\alpha_1(1-S_b^*(\alpha_1))+\alpha_1\lambda^2 S_b^{*'}(\alpha_1))\big\{\big((1-S_v^*(\eta+\alpha_2))(\eta+\eta(1-z_1))\\
&\quad+\eta(1-z_1)\lambda S_v^{*'}(\eta+\alpha_2)\big)(\eta+\alpha_2(1-S_r^*(\eta)))+\eta(1-z_1)(1-S_v^*(\eta+\alpha_2))\\
&\quad\times[-\lambda+\lambda\alpha_2 S_{r_2}^{*'}(\eta)+\lambda\alpha_2 E(S_{r_1})(1-S_{r_2}^*(\eta))]+\lambda(1-z_1)(1-S_v^*(\eta+\alpha_2))\\
&\quad\times\alpha_2(1-S_{r_2}^*(\eta))+\eta(1-z_1)[\eta+\alpha_2+\lambda S_v^{*'}(\eta+\alpha_2)(\eta+\alpha_2(1-S_{r_2}^*(\eta)))]\\
&\quad-\alpha_2(1-S_v^*(\eta+\alpha_2))(S_{r_2}^*(\eta)-\lambda S_{r_2}^{*'}(\eta))\big\}+\lambda\alpha_1(1-S_b^*(\alpha_1))\\
&\quad\times\big\{\big(2\eta(1-S_v^*(\eta+\alpha_2))+2\eta(1-z_1)\lambda S_v^{*'}(\eta+\alpha_2)+2\eta\lambda S_v^{*'}(\eta+\alpha_2)\\
&\quad-\eta(1-z_1)\lambda^2 S_v^{*''}(\eta+\alpha_2)\big)(\eta+\alpha_2(1-S_{r_2}^*(\eta)))+2\big((1-S_v^*(\eta+\alpha_2))\\
&\quad\times(\eta+\eta(1-z_1))+\eta\lambda(1-z_1)S_v^{*'}(\eta+\alpha_2)\big)\big(-\lambda+\lambda\alpha_2 E(S_{r_1})(1-S_{r_2}^*(\eta))\\
&\quad+\lambda\alpha_2 S_{r_2}^{*'}(\eta)\big)+\eta(1-z_1)(1-S_v^*(\eta+\alpha_2))[2\lambda^2\alpha_2 E(S_{r_1})S_{r_2}^{*'}(\eta)-\lambda^2\alpha_2 S_{r_2}^{*''}(\eta)\\
&\quad+\lambda^2\alpha_2 E(S_{r_1})^2(1-S_{r_2}^*(\eta))]+2\lambda(1-z_1)\big[(1-S_v^*(\eta+\alpha_2))[-\lambda-\alpha_2 S_{r_2}^*(\eta)\\
&\quad+\lambda\alpha_2 S_{r_2}^{*'}(\eta)]+\eta+\alpha_2+\lambda S_v^{*'}(\eta+\alpha_2)(\eta+\alpha_2(1-S_{r_2}^*(\eta)))\big]\big\}\\
&\quad+2\big\{\eta\lambda\alpha_2(1-S_v^*(\eta+\alpha_2))\big((1-S_{r_2}^*(\eta))(1+(1-z_1))+\lambda S_{r_2}^{*'}(\eta)\big)\\
&\quad+\lambda^2(1-z_1)\eta\alpha_2(1-S_{r_2}^*(\eta))S_v^{*'}(\eta+\alpha_2)\big\}\\
&\quad\times\big\{\lambda(1-S_b^*(\alpha_1))(-1-\alpha_1 E(S_{r_1}))+\alpha_1 S_b^*(\alpha_1)\big\}+\eta\alpha_2\lambda(1-z_1)(1-S_{r_2}^*(\eta))\\
&\quad\times(1-S_v^*(\eta+\alpha_2))\big\{-2\lambda(1+\lambda S_b^{*'}(\alpha_1))-2\lambda\alpha_1 S_b^{*'}(\alpha_1)-2\lambda\alpha_1(1-S_b^*(\alpha_1))E(S_{r_1})\\
&\quad-2\lambda^2\alpha_1 S_b^{*'}(\alpha_1)E(S_{r_1})-\lambda^2\alpha_1(1-S_b^*(\alpha_1))E(S_{r_1})^2\big\},
\end{aligned}
$$

$$
D_1'(1) = \eta\{\alpha_1-(1-S_b^*(\alpha_1))(\lambda+\alpha_1(1+\lambda E(S_{r_1})))\},
$$

$$
\begin{aligned}
D_1''(1) &= -2\lambda(\alpha_1-(1-S_b^*(\alpha_1))(\lambda+\alpha_1+\lambda E(S_{r_1})))\\
&\quad+\eta\big\{-2\lambda(1+\lambda S_b^{*'}(\alpha_1))-2\lambda\alpha_1 S_b^{*'}(\alpha_1)-2\lambda\alpha_1(1-S_b^*(\alpha_1))E(S_{r_1})\\
&\quad-2\lambda^2\alpha_1 S_b^{*'}(\alpha_1)E(S_{r_1})-\lambda^2\alpha_1(1-S_b^*(\alpha_1))E(S_{r_1})^2\big\},
\end{aligned}
$$

$$
D_2(1) = (1-S_v^*(\eta+\alpha_2))[\eta+\alpha_2(1-S_{r_2}^*(\eta))],
$$

$$
\begin{aligned}
D_2'(1) &= (1-S_v^*(\eta+\alpha_2))[-\lambda-\alpha_2 S_{r_2}^*(\eta)+\lambda\alpha_2 S_{r_2}^{*'}(\eta)]+\alpha_2+\eta(1+\lambda S_v^{*'}(\eta+\alpha_2))\\
&\quad+\lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)(1-S_{r_2}^*(\eta)),
\end{aligned}
$$

$$
D_3'(1) = -\lambda.
$$

6. Numerical Result

Assuming that the service time distribution for both regular service period and working vacation period as exponentially distributed and using the fact that

$$\begin{aligned} S_b^*(\alpha_1) &= \frac{\mu_b}{(\alpha_1+\mu_b)}\ , \quad S_v^*(\eta+\alpha_2) = \frac{\mu_v}{(\eta+\alpha_2+\mu_v)}\ , \quad E(S_{r_1}) = \frac{1}{\mu_{r_1}} \\ S_b^{*\prime}(\alpha_1) &= -\frac{\mu_b}{(\alpha_1+\mu_b)^2}\ , S_v^{*\prime}(\eta+\alpha_2) = -\frac{\mu_v}{(\eta+\alpha_2+\mu_v)^2}\ , \quad E(S_{r_1}^2) = \frac{2}{\mu_{r_1}^2}\ , \\ S_{r_2}^*(\eta) &= \frac{\mu_{r_2}}{(\eta+\mu_{r_2})}\ , \quad S_v^{*\prime\prime}(\eta+\alpha_2) = \frac{2\mu_v}{(\eta+\alpha_2+\mu_v)^3}\ , \quad S_{r_2}^{*\ \prime}(\eta) = -\frac{\mu_{r_2}}{(\eta+\mu_{r_2})^2}\ , \end{aligned}$$

and by fixing the values of $z_1 = 0.6,\ \mu_v = 6, \mu_b = 15,\ \mu_{r_1} = 2,\ \mu_{r_2} = 5,\ \alpha_1 = 2,\ \alpha_2 = 1$ and ranging the values of λ from 3.1 to 3.5 in steps of 0.1 and varying the values of η from 3.1 to 3.9 in steps of 0.2, we calculated the corresponding values of L_b and W_b for multiple working vacation and tabulated in Table 1 and in Table 2, respectively.

Table 1. Arrival rate (λ) versus mean system size (L_b) in regular service period

λ \ η	3.1	3.3	3.5	3.7	3.9
3.1	0.484350	0.475244	0.467617	0.461146	0.455593
3.2	0.566134	0.553773	0.543432	0.534666	0.527152
3.3	0.654603	0.638586	0.625199	0.613862	0.604151
3.4	0.750153	0.730055	0.713273	0.699071	0.686915
3.5	0.853213	0.828583	0.808033	0.790658	0.775796

Table 2. Arrival rate (λ) versus mean waiting time (W_b) in regular service period

λ \ η	3.1	3.3	3.5	3.7	3.9
3.1	0.156242	0.153304	0.150844	0.148757	0.146966
3.2	0.176917	0.173054	0.169823	0.167083	0.164735
3.3	0.198364	0.193511	0.189454	0.186019	0.183076
3.4	0.220633	0.214722	0.209786	0.205609	0.202034
3.5	0.243775	0.236738	0.230867	0.225902	0.221656

The corresponding graphs have been drawn for λ versus L_b and λ versus W_b and are shown in Figure 1 and in Figure 2, respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for various values of η. Again fixing the values of $z_1 = 0.8, \mu_v = 5, \mu_b = 11, \mu_{r_1} = 2, \mu_{r_2} = 4, \alpha_1 = 1, \alpha_2 = 1$ and ranging the values of λ from 1.5 to 1.9 in steps of 0.1 and varying the values of η from 2.10 to 2.30 in steps of 0.05, we calculated the corresponding values of L_v and W_v for multiple working vacation and tabulated in Table 3 and in Table 4, respectively.

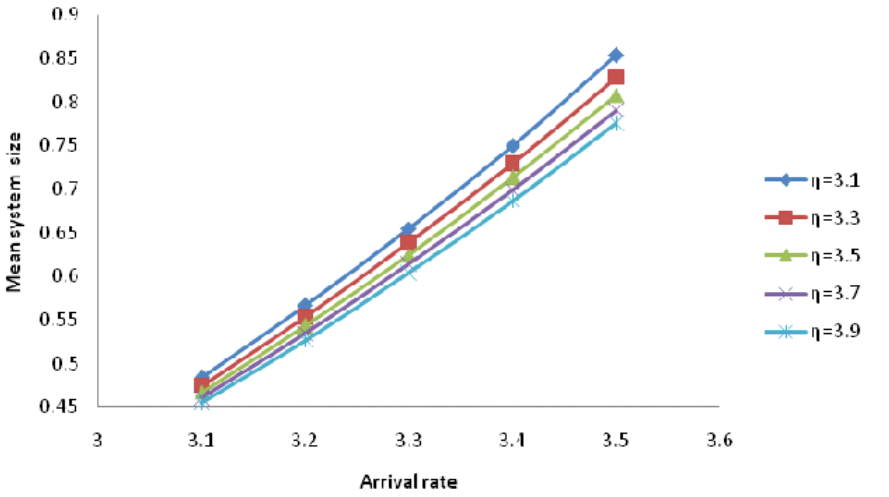

Figure 1. Arrival rate (λ) versus mean system size (L_b) in regular service period

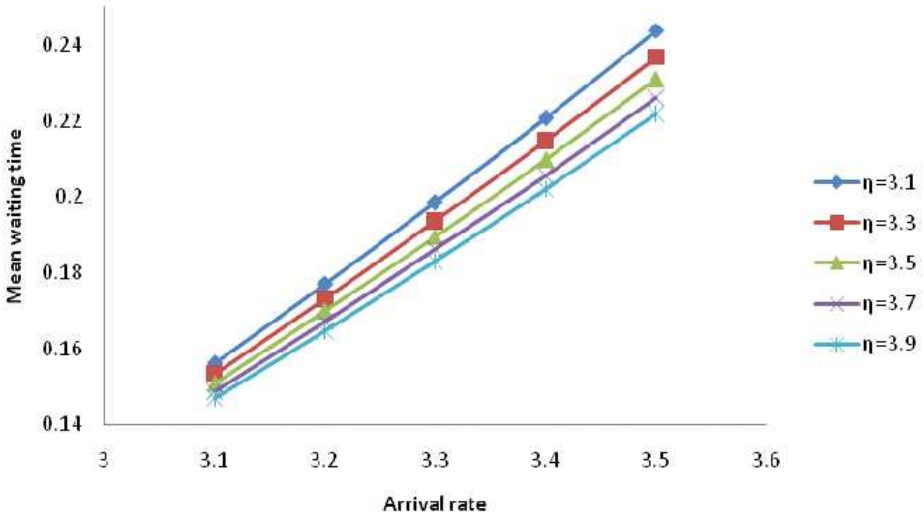

Figure 2. Arrival rate (λ) versus mean waiting time (W_b) in regular service period

Table 3. Arrival rate (λ) versus mean system size (L_v) in WV period

λ \ η	2.10	2.15	2.20	2.25	2.30
1.5	0.310922	0.310464	0.309687	0.308635	0.307349
1.6	0.343920	0.342170	0.340202	0.338051	0.335746
1.7	0.377454	0.374326	0.371091	0.367771	0.364388
1.8	0.411382	0.406801	0.402232	0.397684	0.393166
1.9	0.445566	0.439468	0.433507	0.427679	0.421979

Table 4. Arrival rate (λ) versus mean waiting time (W_v) in WV period

λ \ η	2.10	2.15	2.20	2.25	2.30
1.5	0.207281	0.206976	0.206458	0.205757	0.204899
1.6	0.214950	0.213856	0.212626	0.211282	0.209842
1.7	0.222032	0.220192	0.218289	0.216336	0.214346
1.8	0.228545	0.226000	0.223462	0.220936	0.218426
1.9	0.234509	0.231299	0.228162	0.225094	0.222094

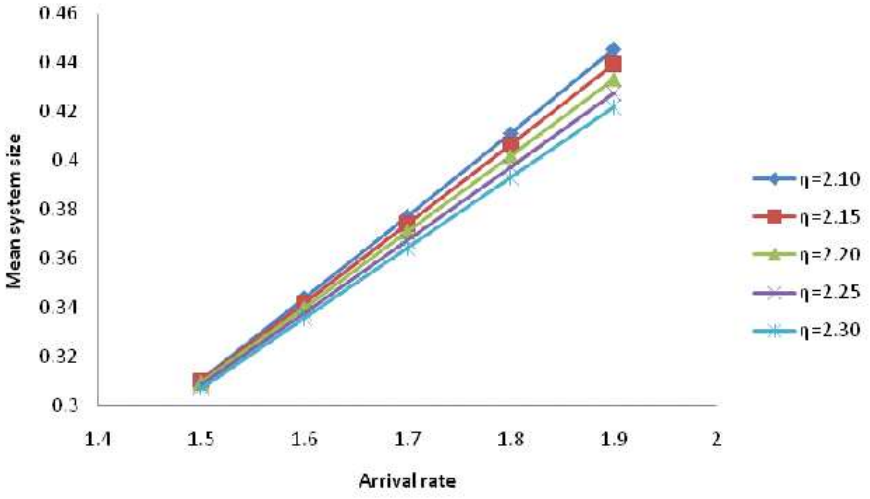

Figure 3. Arrival rate (λ) versus mean system size (L_v) in WV period

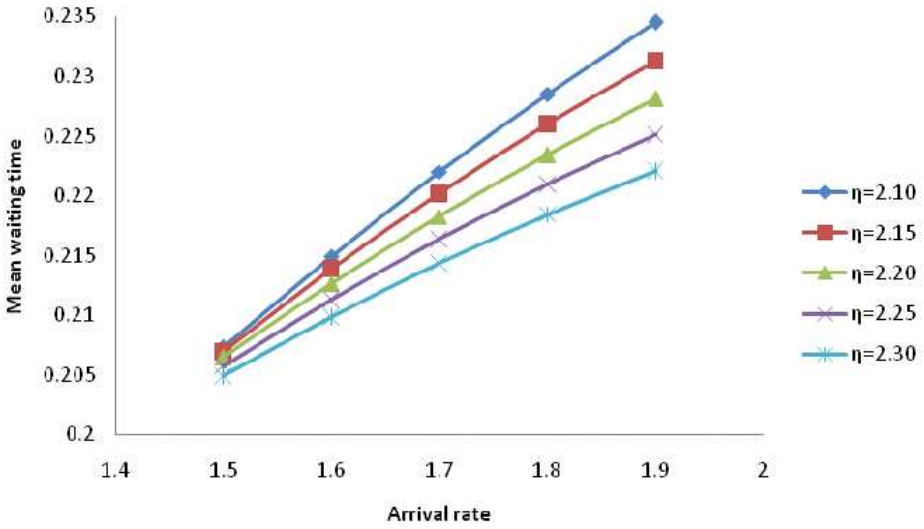

Figure 4. Arrival rate (λ) versus mean waiting time (W_v) in WV period

The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 3 and in Figure 4 respectively. From the graphs it is seen that as λ increases both L_v and

W_v increases for various values of η.

Acknowledgments

We thank both referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for their valuable suggestions and constructive comments which helped us in revising the paper to its present form.

REFERENCES

Afthab Begum, M.I. and Jemila Parveen, M. (2011). Analysis of the batch arrival $M^X/G/1$ queue with exponentially distributed Multiple Working Vacations, International Journal of Mathematical Sciences and Applications, Vol. 1, No. 2, pp. 865-880.

Choudhury, G. and Tadj, L. (2009). An M/G/1 queue with two phases of service subject to the server breakdown and delayed repair, Appl. Math. Modelling, Vol. 31, No. 6, pp. 2699-2709.

Doshi, B.T. (1986). Queueing systems with vacations - A survey, Queueing Systems, Vol. 1, pp. 29-66.

Fuhrman, S. (1981). A note on the $M/G/1$ queue with server vacations, Oper. Res., Vol. 31, p. 1368.

Gaver, D.P. (1959). Imbedded Markov chain analysis of a waiting line process in continuous time, Ann. Math. Stat., Vol. 30, pp. 698-720.

Kramer, M. (1989). Stationary distributions in a queueing system with vacation times and limited service, Queueing Systems, Vol. 4, No. 1, pp. 57-68.

Li., W., Shi, D. and Chao, X. (1997). Reliability analysis of $M/G/1$ queueing system with server breakdowns and vacations, J. Appl. Prob., Vol. 34, pp. 546-555.

Levy, Y. and Yechilai, U., (1976). An $M/M/s$ queue with server vacations, Infor, Vol. 14, No. 2, pp. 153-163.

Madan, K.C. (1999). An $M/G/1$ queue with optional deterministic server vacations, Metron, Vol. LVII, No. 3-4, pp. 83-95.

Madan, K.C. (2003). An $M/G/1$ type queue with time-homogeneous breakdowns and deterministic repair times, Soochow Journal of Mathematics, Vol. 29, No. 1, pp. 103-110.

Madan, K.C. and Saleh, M.F. (2001). On $M/D/1$ queue with server vacations, International Journal of Man. and Inform. Sc., Vol. 12, No. 2, pp. 25-37.

Medhi, J. (1982). *Stochastic Processes,* Wiley Eastern.

Santhi, K. and Pazhani Bala Murugan, S. (2013). An $M/G/1$ queue with two-stage heterogeneous service and single working vacation, Int. Math. Forum, Vol. 8, No. 27, pp. 1323-1336.

Santhi, K. and Pazhani Bala Murugan, S. (2014). A Bulk Input Queueing System with Feedback and Single Working Vacation, International Journal of Scientific Research and Management Studies, Vol. 1, No. 5, pp. 168-176.

Sengupta, B. (1990). A queue with service interruptions in an alternating random environment, Operations Research, Vol. 38, pp. 308-318.

Shanthikumar, J.G. (1988). On stochastic decomposition in the $M/G/1$ type queues with

generalized vacations, Operations Research, Vol. 36, pp. 566-569.

Servi, L.D. and Finn, S.G. (2002). $M/M/1$ queues with working vacations (M/M/1/WV), *Performance Evaluation,* Vol. 50, pp. 41-52.

Takine, T. and Sengupta, B. (1997). A single server queue with service interruptions, Queueing System, Vol. 26, pp. 285-300.

Thangaraj, V. and Vanitha, S. (2010). $M/G/1$ queue with two-stage heterogeneous service compulsary server vacation and random breakdowns, Int. J. Contemp. Math. Sci., Vol. 5, No. 7, pp. 307-322.

Tian, N., Zhao, X. and Wang, K. (2008). The $M/M/1$ queue with single working vacation, Int. J. Infor. and Management Sciences., Vol. 4, pp. 621-634.

Wu, D and Takagi, H. (2006). The $M/G/1$ queue with multiple working vacation, Performance Evaluations, Vol. 63, pp. 654-681.