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Priority Queueing System with a Single Server Serving Two Queues $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ with Balking and Optional Server Vacation

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Abstract

In this paper we study a vacation queueing system with a single server simultaneously dealing with an $M^{[X_1]}/G_1/1$ and an $M^{[X_2]}/G_2/1$ queues. Two classes of units, priority and non-priority, arrive at the system in two independent compound Poisson streams. Under a non-preemptive priority rule, the server provides a general service to the priority and non-priority units. We further assume that the server may take a vacation of random length just after serving the last customer in the priority unit present in the system. If the server is busy or on vacation, an arriving non-priority customer either join the queue with probability b or balks(does not join the queue) with probability $(1-b)$. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results are obtained explicitly. Also the average number of customer in the priority and the non-priority queue and the average waiting time are derived. Numerical results are computed.

Keywords: Non-Preemptive Priority Queueing systems; Batch Arrival; Modified Server Vacations Transient Solution; Average Queue Size; Average Waiting Time

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1. Introduction

The study on queuing models has become an indispensable area due to its wide applicability in real-life situations; all the models considered have had the property that units proceed to service on a first-come, first-served basis. This is obviously not only the manner of service, and there are many alternatives, such as last-come, first-served, selection in random order, and selection by priority. In order to offer different qualities of service for different kinds of customers, we often control a queueing system by priority mechanism. This phenomenon is common in practice. For example, in telecommunication transfer protocol, for guaranteeing different layers of service for different customers, priority classes control may appear in the header of an IP package or in an ATM cell. Priority control is also widely used in production practice, transportation management, etc.

A few papers appear on bulk arrival priority queueing system. Hawkes (1956) considered the time dependent solution of a priority queue with bulk arrivals. Haghighi and Mishev (2006) have studied a parallel priority queueing system with finite buffers. Vacation queues have been studied by several authors including Doshi (1986), Takagi (1990), and Chae et al. (2001). Ayyappan and Muthu Ganapathi Subramanian (2009) have studied single server retrial queueing system with non-pre-emptive priority service and single vacation exhaustive service type. Madan (2011) studied a Non-preemptive priority queueing system with a single server serving two queues $M/G/1$ and $M/D/1$ with optional server vacations based on the exhaustive service of the priority units. Haghighi and Mishev (2013) have studied a Stochastic Three-stage Hiring Model as a Tandem Queueing Process with Bulk Arrivals and Erlang Phase-Type Selection. Jain and Charu Bhargava (2008) have studied bulk arrival retrial queue with unreliable server and priority subscribers. Jinbio and Lian (2013) have studied a single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. Thangaraj and Vanitha (2010) have studied an $M/G/1$ queue with two-stage heterogeneous service compulsory server vacation and random breakdowns, and Jau-Chauan and Fu-Min (2009) have studied modified vacation policy for $M/G/1$ retrial queue with balking and feedback.

In this paper we consider a priority queueing system with a single server serving two queues $M^{[X_1]}/G_1/1$ and $M^{[X_2]}/G_2/1$ with balking and optional server vacation based on exhaustive service of the priority units. The service time of the priority and non-priority customers follows general (arbitrary) distribution. We assume that the server may take a vacation of random length but no vacation is allowed if there is even a single priority unit present in the system. Thus the server may take an optional vacation of random length just after completing the service of the last customer in the priority unit present in the system with probability θ or else may just continue serving the non-priority units if present in the system with probability $(1-\theta)$. If the server is busy or on vacation, an arriving non-priority customer either joins the queue with probability b or balks (does not join the queue) with probability $(1-b)$ and the priority units are not allowed to balk the queue.

Here we derive time dependent probability generating functions for both priority and non-priority units in terms of Laplace transforms. We also derive the average queue size and average waiting time in the queue for both priority and non-priority units. Some particular cases and numerical results are also discussed.

The rest of the paper is organized as follows: Mathematical description of our model in Section (2). Definitions, equations governing of our model and the time dependent solution have been obtained in Sections (3) and (4). The corresponding steady state results have been derived explicitly in Section (5). Average queue size and the average waiting time are computed in Sections (6) and (7). Some particular cases are discussed in Section (8). In Section (9), we consider a numerical example to illustrate application of our results.

2. Mathematical description of our model

(1) Priority and non-priority units arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda_1 c_i\ dt(i = 1, 2, 3, ...)$ and $\lambda_2 c_j\ dt(j = 1, 2, 3, ...)$ be the first order probability that a batch of i and j customers arrives at the system during a short interval of time $(t, t+dt)$, where $0 \leq c_i \leq 1,\ \sum_{i=1}^{\infty} c_i = 1, 0 \leq c_j \leq 1,\ \sum_{j=1}^{\infty} c_j = 1$ and $\lambda_1 > 0,\ \lambda_2 > 0$ are the average arrival rates for priority and non-priority customers entering into the system and forming two queues. The server must serve all the priority units present in the system before taking up non-priority unit for service. In other words, there is no priority unit present in the system at the time of starting service of a non-priority unit. Further, we assume that the server follows a non-preemptive priority rule, which means that if one or more priority units arrive during the service time of a non-priority unit, the current service of a non-priority units is not stopped and a priority unit will be taken up for service only after the current service of a non-priority unit is complete.

(2) Each customer under priority and non-priority units service provided by a single server on a first come - first served basis. The service time for both priority and non-priority units follows general (arbitrary) distributions with distribution functions $B_i(s)$ and the density functions $b_i(s),\ i = 1, 2$.

(3) Let $\mu_i(x)dx$ be the conditional probability of completion of the priority and non-priority units service during the interval $(x, x + dx]$, given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)},$$

and, therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}.$$

(4) We further assume that as soon as the service of the last priority unit present in the system is completed, the server has the option to take a vacation of random length with probability θ, in which case the vacation starts immediately or else with probability

$(1-\theta)$ he may decide to continue serving the non-priority units present in the system, if any. In the later case, if there is no non-priority unit present in the system, the server remains idle in the system waiting for the new units to arrive.

(5) The vacation time follow general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1-V(x)},$$

and, therefore,

$$v(s) = \gamma(s)e^{-\int_0^s \gamma(x)dx}.$$

(6) If the server is busy or on vacation, an arriving non-priority customer either joins the queue with probability b or balks (does not join the queue) with probability $(1-b)$.

3. Definitions and Notations

We define the following notations:

(1) $P^{(1)}_{m,n}(x,t)$ = Probability that at time t, the server is active providing service and there are m $(m \geq 0)$ priority units and n $(n \geq 0)$ non-priority units in the queue excluding the one priority unit in service with elapsed service time for this customer is x. Accordingly,

$$P^{(1)}_{m,n}(t) = \int_0^\infty P^{(1)}_{m,n}(x,t)dx$$

denotes the probability that at time t there are m $(m \geq 0)$ priority units and n $(n \geq 0)$ non-priority units in the queue excluding one priority unit in service without regard to the elapsed service time x of a priority unit.

(2) $V_{m,n}(x,t)$ = Probability that at time t, the server is on vacation with elapsed vacation time x and there are $m(m \geq 0)$ priority units and n $(n \geq 0)$ non-priority units in the queue. Accordingly,

$$V_{m,n}(t) = \int_0^\infty V_{m,n}(x,t)dx$$

denotes the probability that at time t there are m $(m \geq 0)$ priority units and n $(n \geq 0)$ non-priority units in the queue, without regard to the elapsed vacation time x .

(3) $P^{(2)}_{m,n}(x,t)$ = Probability that at time t, the server is active providing service and there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ non-priority units in the queue excluding the one non-priority unit in service with elapsed service time for this customer is x. Accordingly,

$$P^{(2)}_{m,n}(t) = \int_0^\infty P^{(2)}_{m,n}(x,t)dx$$

denotes the probability that at time t there are m $(m \geq 0)$ priority units in the queue and n $(n \geq 0)$ non-priority units in the queue excluding one non-priority unit in service without regard to the elapsed service time x of a non-priority unit.

(4) $Q(t) =$ Probability that at time t, there are no priority and non-priority customers in the system and the server is idle but available in the system.

4. Equations Governing the System

The Kolmogorov forward equations to govern the model:

$$\frac{\partial}{\partial t}P^{(1)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,n}(x,t) = -(\lambda_1 + \lambda_2 + \mu_1(x))P^{(1)}_{m,n}(x,t) + \lambda_1 \sum_{i=1}^{m} C_i P^{(1)}_{m-i,n}(x,t) + \lambda_2 b \sum_{j=1}^{n} C_j P^{(1)}_{m,n-j}(x,t) + \lambda_2(1-b)P^{(1)}_{m,n}(x,t);\ m \geq 1,\ n \geq 1, \tag{1}$$

$$\frac{\partial}{\partial t}P^{(1)}_{m,0}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,0}(x,t) = -(\lambda_1 + \lambda_2 + \mu_1(x))P^{(1)}_{m,0}(x,t) + \lambda_1 \sum_{i=1}^{m} C_i P^{(1)}_{m-i,0}(x,t) + \lambda_2(1-b)P^{(1)}_{m,0}(x,t);\ m \geq 1,\ n = 0, \tag{2}$$

$$\frac{\partial}{\partial t}P^{(1)}_{0,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{0,n}(x,t) = -(\lambda_1 + \lambda_2 + \mu_1(x))P^{(1)}_{0,n}(x,t) + \lambda_2 b \sum_{j=1}^{n} C_j P^{(1)}_{0,n-j}(x,t) + \lambda_2(1-b)P^{(1)}_{0,n}(x,t);\ m = 0,\ n \geq 1, \tag{3}$$

$$\frac{\partial}{\partial t}P^{(1)}_{0,0}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{0,0}(x,t) = -(\lambda_1 + \lambda_2 + \mu_1(x))P^{(1)}_{0,0}(x,t) + \lambda_2(1-b)P^{(1)}_{0,0}(x,t);\ m,\ n = 0, \tag{4}$$

$$\frac{\partial}{\partial t}V_{m,n}(x,t) + \frac{\partial}{\partial x}V_{m,n}(x,t) = -(\lambda_1 + \lambda_2 + \gamma(x))V_{m,n}(x,t) + \lambda_1 \sum_{i=1}^{m} C_i V_{m-i,n}(x,t) + \lambda_2 b \sum_{j=1}^{n} C_j V_{m,n-j}(x,t) + \lambda_2(1-b)V_{m,n}(x,t);\ m \geq 1,\ n \geq 1, \tag{5}$$

$$\frac{\partial}{\partial t}V_{m,0}(x,t) + \frac{\partial}{\partial x}V_{m,0}(x,t) = -(\lambda_1 + \lambda_2 + \gamma(x))V_{m,0}(x,t) + \lambda_1 \sum_{i=1}^{m} C_i V_{m-i,0}(x,t) + \lambda_2(1-b)V_{m,0}(x,t);\ m \geq 1,\ n = 0, \tag{6}$$

$$\frac{\partial}{\partial t}V_{0,n}(x,t) + \frac{\partial}{\partial x}V_{0,n}(x,t) = -(\lambda_1 + \lambda_2 + \gamma(x))V_{0,n}(x,t) + \lambda_2 b \sum_{j=1}^{n} C_j V_{0,n-j}(x,t) + \lambda_2(1-b)V_{0,n}(x,t);\ m = 0,\ n \geq 1, \tag{7}$$

$$\frac{\partial}{\partial t}V_{0,0}(x,t) + \frac{\partial}{\partial x}V_{0,0}(x,t) = -(\lambda_1 + \lambda_2 + \gamma(x))V_{0,0}(x,t) + \lambda_2(1-b)V_{0,0}(x,t);\ m,\ n = 0, \tag{8}$$

$$\frac{\partial}{\partial t}P^{(2)}_{m,n}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{m,n}(x,t)=-(\lambda_1+\lambda_2+\mu_2(x))P^{(2)}_{m,n}(x,t)+\lambda_1\sum_{i=1}^{m}C_iP^{(2)}_{m-i,n}(x,t)$$
$$+\lambda_2 b\sum_{j=1}^{n}C_jP^{(2)}_{m,n-j}(x,t)+\lambda_2(1-b)P^{(2)}_{m,n}(x,t);\ m\geq 1,n\geq 1, \tag{9}$$
$$\frac{\partial}{\partial t}P^{(2)}_{m,0}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{m,0}(x,t)=-(\lambda_1+\lambda_2+\mu_2(x))P^{(2)}_{m,0}(x,t)+\lambda_1\sum_{i=1}^{m}C_iP^{(2)}_{m-i,0}(x,t)$$
$$+\lambda_2(1-b)P^{(2)}_{m,0}(x,t);\ m\geq 1,\ n=0, \tag{10}$$
$$\frac{\partial}{\partial t}P^{(2)}_{0,n}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{0,n}(x,t)=-(\lambda_1+\lambda_2+\mu_2(x))P^{(2)}_{0,n}(x,t)+\lambda_2 b\sum_{j=1}^{n}C_jP^{(2)}_{0,n-j}(x,t)$$
$$+\lambda_2(1-b)P^{(2)}_{0,n}(x,t);\ m=0,\ n\geq 1, \tag{11}$$
$$\frac{\partial}{\partial t}P^{(2)}_{0,0}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{0,0}(x,t)=-(\lambda_1+\lambda_2+\mu_2(x))P^{(2)}_{0,0}(x,t)+\lambda_2(1-b)P^{(2)}_{0,0}(x,t);$$
$$m,\ n=0, \tag{12}$$
$$\frac{d}{dt}Q(t)=-(\lambda_1+\lambda_2)Q(t)+(1-\theta)\int_0^\infty P^{(1)}_{0,0}(x,t)\mu_1(x)dx+\int_0^\infty P^{(2)}_{0,0}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty V_{0,0}(x,t)\gamma(x)dx. \tag{13}$$

The above set of equations are to be solved under the following boundary conditions at $x=0$.

$$P^{(1)}_{m,n}(0,t)=\int_0^\infty P^{(1)}_{m+1,n}(x,t)\mu_1(x)dx+\int_0^\infty P^{(2)}_{m+1,n}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty V_{m+1,n}(x,t)\gamma(x)dx;\ m\geq 1,\ n\geq 1, \tag{14}$$
$$P^{(1)}_{m,0}(0,t)=\lambda_1C_{m+1}Q(t)+\int_0^\infty P^{(1)}_{m+1,0}(x,t)\mu_1(x)dx+\int_0^\infty P^{(2)}_{m+1,0}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty V_{m+1,0}(x,t)\gamma(x)dx;\ m\geq 1,\ n=0, \tag{15}$$
$$P^{(1)}_{0,n}(0,t)=\int_0^\infty P^{(1)}_{1,n}(x,t)\mu_1(x)dx+\int_0^\infty P^{(2)}_{1,n}(x,t)\mu_2(x)dx+\int_0^\infty V_{1,n}(x,t)\gamma(x)dx;$$
$$m=0,\ n\geq 1, \tag{16}$$
$$P^{(1)}_{0,0}(0,t)=\lambda_1C_1Q(t)+\int_0^\infty P^{(1)}_{1,0}(x,t)\mu_1(x)dx+\int_0^\infty P^{(2)}_{1,0}(x,t)\mu_2(x)dx$$
$$+\int_0^\infty V_{1,0}(x,t)\gamma(x)dx;\ m,\ n=0, \tag{17}$$
$$V_{0,n}(0,t)=\theta\int_0^\infty P^{(1)}_{0,n}(x,t)\mu_1(x)dx, n\geq 0, \tag{18}$$

$$P_{0,0}^{(2)}(0,t) = \lambda_2 C_1 Q(t) + (1-\theta)\int_0^\infty P_{0,1}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{0,1}^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_{0,1}(x,t)\gamma(x)dx;\ m,\ n=0, \tag{19}$$

$$P_{0,n}^{(2)}(0,t) = \lambda_2 C_{n+1} Q(t) + (1-\theta)\int_0^\infty P_{0,n+1}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{0,n+1}^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_{0,n+1}(x,t)\gamma(x)dx;\ m=0,\ n\geq 1. \tag{20}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$\left.\begin{array}{l} P_{m,n}^{(1)}(0) = P_{m,0}^{(1)}(0) = P_{0,n}^{(1)}(0) = P_{0,0}^{(1)}(0) = 0, \\ P_{m,n}^{(2)}(0) = P_{m,0}^{(2)}(0) = P_{0,n}^{(2)}(0) = P_{0,0}^{(2)}(0) = 0, \\ V_{m,n}(0) = V_{m,0}(0) = V_{0,n}(0) = V_{0,0}(0) = 0,\ \text{and } Q(0)=1. \end{array}\right\} \tag{21}$$

Next, we define the following probability generating functions:

$$\left.\begin{array}{l} \sum_{m=0}^\infty \sum_{n=0}^\infty z_1^m z_2^n P_{m,n}^{(1)}(x,t) = P^{(1)}(x,z_1,z_2),\ \sum_{m=0}^\infty z_1^m P_m^{(1)}(x,t) = P^{(1)}(x,z_1), \\ \sum_{n=0}^\infty z_2^n P_n^{(1)}(x,t) = P^{(1)}(x,z_2),\ \sum_{m=0}^\infty \sum_{n=0}^\infty z_1^m z_2^n P_{m,n}^{(2)}(x,t) = P^{(2)}(x,z_1,z_2), \\ \sum_{m=0}^\infty z_1^m P_m^{(2)}(x,t) = P^{(2)}(x,z_1),\ \sum_{n=0}^\infty z_2^n P_n^{(2)}(x,t) = P^{(2)}(x,z_2), \\ \sum_{m=0}^\infty \sum_{n=0}^\infty z_1^m z_2^n V_{m,n}(x,t) = V(x,z_1,z_2), \\ \sum_{m=0}^\infty z_1^m V_m(x,t) = V(x,z_1),\ and\ \sum_{n=0}^\infty z_2^n V_n(x,t) = V(x,z_2), \end{array}\right\} \tag{22}$$

which are convergent inside the circle given by $|z_1| \leq 1, |z_2| \leq 1$, and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt.$$

$$\frac{\partial}{\partial x}\overline{P}_{m,n}^{(1)}(x,s) + (s+\lambda_1+\lambda_2 b+\mu_1(x))\overline{P}_{m,n}^{(1)}(x,s) = \lambda_1\sum_{i=1}^m C_i\overline{P}_{m-i,n}^{(1)}(x,s) + \lambda_2 b\sum_{j=1}^n C_j\overline{P}_{m,n-j}^{(1)}(x,s); m\geq 1,\ n\geq 1, \tag{23}$$

$$\frac{\partial}{\partial x}\overline{P}_{m,0}^{(1)}(x,s) + (s+\lambda_1+\lambda_2 b+\mu_1(x))\overline{P}_{m,0}^{(1)}(x,s) = \lambda_1\sum_{i=1}^m C_i\overline{P}_{m-i,0}^{(1)}(x,s);\ m\geq 1, \tag{24}$$

$$\frac{\partial}{\partial x}\overline{P}^{(1)}_{0,n}(x,s)+(s+\lambda_1+\lambda_2 b+\mu_1(x))\overline{P}^{(1)}_{0,n}(x,s)=\lambda_2 b\sum_{j=1}^{n}C_j\overline{P}^{(1)}_{0,n-j}(x,s);\ \ n\geq 1, \tag{25}$$

$$\frac{\partial}{\partial x}\overline{P}^{(1)}_{0,0}(x,s)+(s+\lambda_1+\lambda_2 b+\mu_1(x))\overline{P}^{(1)}_{0,0}(x,s)=0, \tag{26}$$

$$\begin{aligned}\frac{\partial}{\partial x}\overline{V}_{m,n}(x,s)+(s+\lambda_1+\lambda_2 b+\gamma(x))\overline{V}_{m,n}(x,s)&=\lambda_1\sum_{i=1}^{m}C_i\overline{V}_{m-i,n}(x,s)\\&+\lambda_2 b\sum_{j=1}^{n}C_j\overline{V}_{m,n-j}(x,s);\ \ m\geq 1,\ \ n\geq 1,\end{aligned} \tag{27}$$

$$\frac{\partial}{\partial x}\overline{V}_{m,0}(x,s)+(s+\lambda_1+\lambda_2 b+\gamma(x))\overline{V}_{m,0}(x,s)=\lambda_1\sum_{i=1}^{m}C_i\overline{V}_{m-i,0}(x,s);m\geq 1, \tag{28}$$

$$\frac{\partial}{\partial x}\overline{V}_{0,n}(x,s)+(s+\lambda_1+\lambda_2 b+\gamma(x))\overline{V}_{0,n}(x,s)=\lambda_2 b\sum_{j=1}^{n}C_j\overline{V}_{0,n-j}(x,s);\ \ n\geq 1, \tag{29}$$

$$\frac{\partial}{\partial x}\overline{V}_{0,0}(x,s)+(s+\lambda_1+\lambda_2 b+\gamma(x))\overline{V}_{0,0}(x,s)=0, \tag{30}$$

$$\begin{aligned}\frac{\partial}{\partial x}\overline{P}^{(2)}_{m,n}(x,s)+(s+\lambda_1+\lambda_2 b+\mu_2(x))\overline{P}^{(2)}_{m,n}(x,s)&=\lambda_1\sum_{i=1}^{m}C_i\overline{P}^{(2)}_{m-i,n}(x,s)\\&+\lambda_2 b\sum_{j=1}^{n}C_j\overline{P}^{(2)}_{m,n-j}(x,s);\ \ m\geq 1,\ \ n\geq 1,\end{aligned} \tag{31}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}_{m,0}(x,s)+(s+\lambda_1+\lambda_2 b+\mu_2(x))\overline{P}^{(2)}_{m,0}(x,s)=\lambda_1\sum_{i=1}^{m}C_i\overline{P}^{(1)}_{m-i,0}(x,s);m\geq 1, \tag{32}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}_{0,n}(x,s)+(s+\lambda_1+\lambda_2 b+\mu_2(x))\overline{P}^{(2)}_{0,n}(x,s)=\lambda_2 b\sum_{j=1}^{n}C_j\overline{P}^{(2)}_{0,n-j}(x,s);\ \ n\geq 1, \tag{33}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}_{0,0}(x,s)+(s+\lambda_1+\lambda_2 b+\mu_2(x))\overline{P}^{(2)}_{0,0}(x,s)=0, \tag{34}$$

$$\begin{aligned}(s+\lambda_1+\lambda_2)\overline{Q}(s)-1=(1-\theta)\int_0^\infty\overline{P}^{(1)}_{0,0}(x,s)\mu_1(x)dx&+\int_0^\infty\overline{P}^{(2)}_{0,0}(x,s)\mu_2(x)dx\\&+\int_0^\infty\overline{V}_{0,0}(x,s)\gamma(x)dx,\end{aligned} \tag{35}$$

$$\begin{aligned}\overline{P}^{(1)}_{m,n}(0,s)=\int_0^\infty\overline{P}^{(1)}_{m+1,n}(x,s)\mu_1(x)dx&+\int_0^\infty\overline{P}^{(2)}_{m+1,n}(x,s)\mu_2(x)dx\\&+\int_0^\infty\overline{V}_{m+1,n}(x,s)\gamma(x)dx;\ \ m\geq 1,\ \ n\geq 1,\end{aligned} \tag{36}$$

$$\begin{aligned}\overline{P}^{(1)}_{m,0}(0,s)=\lambda_1 C_{m+1}\overline{Q}(s)+\int_0^\infty\overline{P}^{(1)}_{m+1,0}(x,s)\mu_1(x)dx&+\int_0^\infty\overline{P}^{(2)}_{m+1,0}(x,s)\mu_2(x)dx\\&+\int_0^\infty\overline{V}_{m+1,0}(x,s)\gamma(x)dx;\ \ m\geq 1,\end{aligned} \tag{37}$$

$$\overline{P}^{(1)}_{0,n}(0,s) = \int_0^\infty \overline{P}^{(1)}_{1,n}(x,s)\mu_1(x)dx + \int_0^\infty \overline{P}^{(2)}_{1,n}(x,s)\mu_2(x)dx + \int_0^\infty \overline{V}_{1,n}(x,s)\gamma(x)dx; \quad n \geq 1, \tag{38}$$

$$\overline{P}^{(1)}_{0,0}(0,s) = \lambda_1 C_1 \overline{Q}(s) + \int_0^\infty \overline{P}^{(1)}_{1,0}(x,s)\mu_1(x)dx + \int_0^\infty \overline{P}^{(2)}_{1,0}(x,s)\mu_2(x)dx + \int_0^\infty \overline{V}_{1,0}(x,s)\gamma(x)dx, \tag{39}$$

$$\overline{V}_{0,n}(0,s) = \theta \int_0^\infty \overline{P}^{(1)}_{0,n}(x,s)\mu_1(x)dx; \; n \geq 0, \tag{40}$$

$$\overline{P}^{(2)}_{0,0}(0,s) = \lambda_2 C_1 \overline{Q}(s) + (1-\theta)\int_0^\infty \overline{P}^{(1)}_{0,1}(x,s)\mu_1(x)dx + \int_0^\infty \overline{P}^{(2)}_{0,1}(x,s)\mu_2(x)dx + \int_0^\infty \overline{V}_{0,1}(x,s)\gamma(x)dx; \; m, \; n = 0, \tag{41}$$

$$\overline{P}^{(2)}_{0,n}(0,s) = \lambda_2 C_{n+1} \overline{Q}(s) + (1-\theta)\int_0^\infty \overline{P}^{(1)}_{0,n+1}(x,s)\mu_1(x)dx + \int_0^\infty \overline{P}^{(2)}_{0,n+1}(x,s)\mu_2(x)dx + \int_0^\infty \overline{V}_{0,n+1}(x,s)\gamma(x)dx; \; m = 0, \; n \geq 1. \tag{42}$$

Now we multiply equations (23), (25), (27), (29), (31) and (33) by z_2^n summing over n from 1 to ∞, adding to equations (24), (26), (28), (30), (32) and (34) and using the generating function defined in equation (22), we get

$$\frac{\partial}{\partial x}\overline{P}^{(1)}_m(x,s,z_2) + (s + \lambda_1 + \lambda_2 b[1 - C(z_2)] + \mu_1(x))\overline{P}^{(1)}_m(x,s,z_2) = \lambda_1 \sum_{i=1}^{m} C_i \overline{P}^{(1)}_{m-i,0}(x,s,z_2); \; m \geq 1, \; n = 0, \tag{43}$$

$$\frac{\partial}{\partial x}\overline{P}^{(1)}_0(x,s,z_2) + (s + \lambda_1 + \lambda_2 b[1 - C(z_2)] + \mu_1(x))\overline{P}^{(1)}_0(x,s,z_2) = 0, \tag{44}$$

$$\frac{\partial}{\partial x}\overline{V}_m(x,s,z_2) + (s + \lambda_1 + \lambda_2 b[1 - C(z_2)] + \gamma(x))\overline{V}_m(x,s,z_2) = \lambda_1 \sum_{i=1}^{m} C_i \overline{V}_{m-i,0}(x,s,z_2); \; m \geq 1, \; n = 0, \tag{45}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s,z_2) + (s + \lambda_1 + \lambda_2 b[1 - C(z_2)] + \gamma(x))\overline{V}_0(x,s,z_2) = 0, \tag{46}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}_m(x,s,z_2) + (s + \lambda_1 + \lambda_2 b[1 - C(z_2)] + \mu_2(x))\overline{P}^{(2)}_m(x,s,z_2) = \lambda_1 \sum_{i=1}^{m} C_i \overline{P}^{(2)}_{m-i,0}(x,s,z_2); \; m \geq 1, \; n = 0, \tag{47}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}_0(x,s,z_2) + (s + \lambda_1 + \lambda_2 b[1 - C(z_2)] + \mu_2(x))\overline{P}^{(2)}_0(x,s,z_2) = 0. \tag{48}$$

Now we multiply equations (43), (45) and (47) by z_1^m summing over m from 1 to ∞, adding to equations (44), (46) and (48) and using the generating function defined in equation (22), we get

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,s,z_1,z_2)+(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]+\mu_1(x))\overline{P}^{(1)}(x,s,z_1,z_2)=0, \tag{49}$$

$$\frac{\partial}{\partial x}\overline{V}(x,s,z_1,z_2)+(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]+\gamma(x))\overline{V}(x,s,z_1,z_2)=0, \tag{50}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}(x,s,z_1,z_2)+(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]+\mu_2(x))\overline{P}^{(2)}(x,s,z_1,z_2)=0. \tag{51}$$

For the boundary conditions, we multiply both sides of equations (36) and (37) by z_1^{m+1} summing over m from 1 to ∞, adding to equations $z_1\times(38)$ and $z_1\times(39)$ and use the the equation (22), we get

$$\begin{aligned}z_1\overline{P}_n^{(1)}(0,s,z_1)=&\int_0^\infty \overline{P}_n^{(1)}(x,s,z_1)\mu_1(x)dx-\int_0^\infty \overline{P}_{0,n}^{(1)}(x,s)\mu_1(x)dx\\&+\int_0^\infty \overline{P}_n^{(2)}(x,s,z_1)\mu_2(x)dx-\int_0^\infty \overline{P}_{0,n}^{(2)}(x,s)\mu_2(x)dx+\int_0^\infty \overline{V}_n(x,s,z_1)\gamma(x)dx\\&-\int_0^\infty \overline{V}_{0,n}(x,s)\gamma(x)dx,\end{aligned} \tag{52}$$

$$\begin{aligned}z_1\overline{P}_0^{(1)}(0,s,z_1)=&\lambda_1C(z_1)\overline{Q}(s)+\int_0^\infty \overline{P}_0^{(1)}(x,s,z_1)\mu_1(x)dx-\int_0^\infty \overline{P}_{0,0}^{(1)}(x,s)\mu_1(x)dx\\&+\int_0^\infty \overline{P}_0^{(2)}(x,s,z_1)\mu_2(x)dx-\int_0^\infty \overline{P}_{0,0}^{(2)}(x,s)\mu_2(x)dx+\int_0^\infty \overline{V}_0(x,s,z_1)\gamma(x)dx\\&-\int_0^\infty \overline{V}_{0,0}(x,s)\gamma(x)dx.\end{aligned} \tag{53}$$

Now multiply equations (52) and (42) by z_2^n summing over n from 1 to ∞, adding to equations (53) and $z_2\times(41)$ and using the equation (22), we get

$$\begin{aligned}z_1\overline{P}^{(1)}(0,s,z_1,z_2)=&\lambda_1C(z_1)\overline{Q}(s)+\int_0^\infty \overline{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx\\&-\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx+\int_0^\infty \overline{P}^{(2)}(x,s,z_1,z_2)\mu_2(x)dx-\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx\\&+\int_0^\infty \overline{V}(x,s,z_1,z_2)\gamma(x)dx-\int_0^\infty \overline{V}_0(x,s,z_2)\gamma(x)dx,\end{aligned} \tag{54}$$

$$\begin{aligned}z_2\overline{P}_0^{(2)}(0,s,z_2)=&\lambda_2C(z_2)\overline{Q}(s)+(1-\theta)\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx\\&-(1-\theta)\int_0^\infty \overline{P}_{0,0}^{(1)}(x,s)\mu_1(x)dx+\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx-\int_0^\infty \overline{P}_{0,0}^{(2)}(x,s)\mu_2(x)dx\\&+\int_0^\infty \overline{V}_0(x,s,z_2)\gamma(x)dx-\int_0^\infty \overline{V}_{0,0}(x,s)\gamma(x)dx.\end{aligned} \tag{55}$$

Now multiply equation (40) by z_2^n summing over n from 0 to ∞, use the equation (22), we get

$$\overline{V}_0(0,s,z_2) = \theta\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx. \tag{56}$$

Integrate equation (49) between 0 to x, we obtain

$$\overline{P}^{(1)}(x,s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2) \times e^{\{-(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])x - \int_0^x \mu_1(t)dt\}}. \tag{57}$$

Again integrate (57) by parts with respect to x, and we get

$$\overline{P}^{(1)}(s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B_1}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}]. \tag{58}$$

Multiply equation (57) by $\mu_1(x)$ and integrate with respect to x, we get

$$\int_0^\infty \overline{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx = \overline{P}^{(1)}(0,s,z_1,z_2)\overline{B_1}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]). \tag{59}$$

Now equation (44) can be rewritten as,

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x,s,z_2) + (s+\lambda_1+\lambda_2 b[1-C(z_2)]+\mu_1(x))\overline{P}_0^{(1)}(x,s,z_2) = 0,$$

which on integration gives

$$\overline{P}_0^{(1)}(x,s,z_2) = \overline{P}_0^{(1)}(0,s,z_2)e^{\{-(s+\lambda_1+\lambda_2 b[1-C(z_2)])x - \int_0^x \mu_1(t)dt\}}. \tag{60}$$

Again integrate by parts with respect to x

$$\overline{P}_0^{(1)}(s,z_2) = \overline{P}_0^{(1)}(0,s,z_2)[\frac{1-\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1+\lambda_2 b[1-C(z_2)])}]. \tag{61}$$

Now multiply (60) by $\mu_1(x)$, and integrate over x, we get

$$\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx = \overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)]). \tag{62}$$

Performing similar operations on equations (50) and (51), we get

$$\overline{V}(x,s,z_1,z_2)=\overline{V}(0,s,z_1,z_2)e^{\{-(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])x-\int_0^x\gamma(t)dt\}}, \tag{63}$$

$$\overline{V}(s,z_1,z_2)=\overline{V}(0,s,z_1,z_2)[\frac{1-\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{64}$$

$$\int_0^\infty \overline{V}(x,s,z_1,z_2)\gamma(x)dx=\overline{V}(0,s,z_1,z_2)\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]), \tag{65}$$

$$\overline{V}_0(x,s,z_2)=\overline{V}_0(0,s,z_2)e^{\{-(s+\lambda_1+\lambda_2 b[1-C(z_2)])x-\int_0^x\gamma(t)dt\}}, \tag{66}$$

$$\int_0^\infty \overline{V}_0(x,s,z_2)\gamma(x)dx=\overline{V}_0(0,s,z_2)\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)]). \tag{67}$$

However, by its definition

$$\overline{V}(0,s,z_1,z_2)=\overline{V}_0(0,s,z_2), \tag{68}$$

and

$$\overline{P}^{(2)}(x,s,z_1,z_2)=\overline{P}^{(2)}(0,s,z_1,z_2)\times e^{\{-(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])x-\int_0^x\mu_2(t)dt\}}, \tag{69}$$

$$\overline{P}^{(2)}(s,z_1,z_2)=\overline{P}^{(2)}(0,s,z_1,z_2)[\frac{1-\overline{B_2}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{70}$$

$$\int_0^\infty \overline{P}^{(2)}(x,s,z_1,z_2)\mu_2(x)dx=\overline{P}^{(2)}(0,s,z_1,z_2)\overline{B_2}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]), \tag{71}$$

$$\overline{P}_0^{(2)}(x,s,z_2)=\overline{P}_0^{(2)}(0,s,z_2)e^{\{-(s+\lambda_1+\lambda_2 b[1-C(z_2)])x-\int_0^x\mu_2(t)dt\}}, \tag{72}$$

$$\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx=\overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(s+\lambda_1+\lambda_2 b[1-C(z_2)]). \tag{73}$$

However, by its definition

$$\overline{P}^{(2)}(0,s,z_1,z_2)=\overline{P}_0^{(2)}(0,s,z_2). \tag{74}$$

Using equation (68) in (65), we get

$$\int_0^\infty \overline{V}(x,s,z_1,z_2)\gamma(x)dx=\overline{V}_0(0,s,z_2)\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]). \tag{75}$$

By using (62) in (56), we get

$$\overline{V}_0(0,s,z_2)=\theta\overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)]). \tag{76}$$

Now substitute equations (59), (62), (65), (67), (71) and (73) in (54), we get

$$\begin{aligned}
&\{z_1-\overline{B_1}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\}\overline{P}^{(1)}(0,s,z_1,z_2)=\lambda_1 C(z_1)\overline{Q}(s)\\
&\quad-\overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\{1-\theta\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\\
&\quad+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}+\overline{P}_0^{(2)}(0,s,z_2)\{\overline{B_2}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\\
&\quad-\overline{B_2}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}.
\end{aligned}\tag{77}$$

Using equations (35), (62), (67), (73) in (55), we get

$$\begin{aligned}
\overline{P}_0^{(2)}(0,s,z_2)&=\frac{\{1-(s+\lambda_1+\lambda_2 b[1-C(z_2)])\overline{Q}(s)\}}{\{z_2-\overline{B_2}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}}\\
&+\frac{\overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\{1-\theta+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}}{\{z_2-\overline{B_2}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}}.
\end{aligned}\tag{78}$$

By applying Rouchy's theorem, $\{z_1-\overline{B_1}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\}$ has one and only one zero inside the circle, $|z_1|=1$ for Re $(s)>0$, $|z_2|\le 1$. Then equation (77) gives

$$\overline{P}_0^{(1)}(0,s,z_2)=\frac{\begin{aligned}&\Big[\lambda_1 C(g(z_2))\overline{Q}(s)+\overline{P}_0^{(2)}(0,s,z_2)\{\overline{B_2}(s+\lambda_1[1-C(g(z_2))]\\&+\lambda_2 b[1-C(z_2)])-\overline{B_2}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\Big]\end{aligned}}{\begin{aligned}&\big[\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\{1-\theta\overline{V}(s+\lambda_1[1-C(g(z_2)]\\&+\lambda_2 b[1-C(z_2)])+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\big]\end{aligned}}.\tag{79}$$

Substitute (79) in (78) and (77), we get

$$\overline{P}_0^{(2)}(0,s,z_2)=\frac{\begin{aligned}&\big[\{1-(s+\lambda_1+\lambda_2 b[1-C(z_2)]))\overline{Q}(s)\}\\&\times\{1-\theta\overline{V}(s+\lambda_1[1-C(g(z_2)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\\&+\{1-\theta+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\lambda_1 C(g(z_2)\overline{Q}(s)\big]\end{aligned}}{\begin{aligned}&\big[\{z_2-\overline{B_2}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\{1-\theta\overline{V}(s+\lambda_1[1-C(g(z_2)]\\&+\lambda_2 b[1-C(z_2)])+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\\&-\{\overline{B_2}(s+\lambda_1[1-C(g(z_2))]+\lambda_2 b[1-C(z_2)])-\overline{B_2}(s+\lambda_1+\lambda_2 b\\&\times[1-C(z_2)])\}\{1-\theta+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\big]\end{aligned}},\tag{80}$$

$$\overline{P}^{(1)}(0,s,z_1,z_2)=\frac{\begin{aligned}&\big[\{1-(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]))\overline{Q}(s)\}\\&\times\{1-\theta\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\\&-\overline{P}_0^{(2)}(0,s,z_2)\{z_2-\overline{B_2}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\}\\&\times\{1-\theta\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\big]\end{aligned}}{\begin{aligned}&\big[\{z_1-\overline{B_1}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\}\\&\times\{1-\theta\overline{V}(s+\lambda_1[1-C(g(z_2)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(s+\lambda_1+\lambda_2 b[1-C(z_2)])\}\big]\end{aligned}}.\tag{81}$$

Substitute equations (81), (79) and (80) in (58), (64) and (70), respectively,

$$\overline{P}^{(1)}(s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B_1}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{82}$$

$$\overline{V}(s,z_1,z_2) = \theta\overline{P}_0^{(1)}(0,s,z_2)\overline{B_1}(s+\lambda_1+\lambda_2 b[1-C(z_2)]) \times [\frac{1-\overline{V}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{83}$$

$$\overline{P}^{(2)}(s,z_1,z_2) = \overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B_2}(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}]. \tag{84}$$

Thus $\overline{P}^{(1)}(s,z_1,z_2), \overline{V}(s,z_1,z_2)$, and $\overline{P}^{(2)}(s,z_1,z_2)$ are completely determined from equations (82) to (84).

5. Steady state Analysis: Limiting Behavior

In this section, we derive the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t).$$

In order to determine Q, we use the normalizing condition

$$P^{(1)}(1,1)+V(1,1)+P^{(2)}(1,1)+Q=1.$$

The steady state probability for an priority queueing system with a single server serving two queues $M^{[X_1]}/G_1/1$ and $M^{[X_2]}/G_2/1$ with balking and optional server vacation based on exhaustive service of the priority units are given by

$$P^{(1)}(z_1,z_2) = P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B_1}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{85}$$

$$V(z_1,z_2) = \theta P_0(0,z_2)\overline{B_1}(\lambda_1+\lambda_2 b[1-C(z_2)]) \times [\frac{1-\overline{V}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{86}$$

$$P^{(2)}(z_1,z_2) = P_0^{(2)}(0,z_2)[\frac{1-\overline{B_2}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}{(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])}], \tag{87}$$

where

$$P_0^{(2)}(0,z_2) = \frac{\begin{array}{l}[\{1-(\lambda_1+\lambda_2 b[1-C(z_2)]))Q\} \\ \times\{1-\theta\overline{V}(\lambda_1[1-C(g(z_2)]+\lambda_2 b[1-C(z_2)]) \\ +\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\} \\ +\{1-\theta+\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\}\lambda_1 C(g(z_2)Q]\end{array}}{\begin{array}{l}[\{z_2-\overline{B_2}(\lambda_1+\lambda_2 b[1-C(z_2)])\}\{1-\theta\overline{V}(\lambda_1[1-C(g(z_2)] \\ +\lambda_2 b[1-C(z_2)])+\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\} \\ -\{\overline{B_2}(\lambda_1[1-C(g(z_2))]+\lambda_2 b[1-C(z_2)])-\overline{B_2}(\lambda_1+\lambda_2 b[1-C(z_2)])\} \\ \times\{1-\theta+\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\}]\end{array}}, \tag{88}$$

$$P^{(1)}(0,z_1,z_2)=\frac{\begin{aligned}&[\{1-(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]))Q\}\\&\times\{1-\theta\overline{V}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\}-\overline{P}_0^{(2)}(0,z_2)\\&\times\{z_2-\overline{B_2}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\}\\&\times\{1-\theta\overline{V}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\}]\end{aligned}}{\begin{aligned}&\big[\{z_1-\overline{B_1}(\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)])\}\\&\times\{1-\theta\overline{V}(\lambda_1[1-C(g(z_2)]+\lambda_2 b[1-C(z_2)])\\&+\theta\overline{V}(\lambda_1+\lambda_2 b[1-C(z_2)])\}]\end{aligned}}. \tag{89}$$

Let $W_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding equations (85) to (87), we obtain

$$W_q(z_1,z_2)=P^{(1)}(z_1,z_2)+V(z_1,z_2)+P^{(2)}(z_1,z_2), \tag{90}$$

$$W_q(z_1,z_2)=\frac{N_1(z_1,z_2)}{D(z_1,z_2)}+\frac{N_2(z_1,z_2)}{D(z_1,z_2)}, \tag{91}$$

where

$$\begin{aligned}N_1(z_1,z_2)=&-f_1(z_1,z_2)Q\{1-\theta\overline{V}(f_1(z_1,z_2))+\theta\overline{V}(f_3(z_2))\}\{1-\overline{B_1}(f_1(z_1,z_2))\}\\&+\theta f_4(z_2)Q\{1-\overline{V}(f_1(z_1,z_2))\}\{z_1-\overline{B_1}(f_1(z_1,z_2))\},\end{aligned} \tag{92}$$

$$\begin{aligned}N_2(z_1,z_2)=&P_0^{(2)}(0,z_2)[\theta\{\overline{B_2}(f_2(z_2))-\overline{B_2}(f_3(z_2))\}\{1-\overline{V}(f_1(z_1,z_2))\}\{z_1-\overline{B_1}(f_1(z_1,z_2)\\&)\}-\{z_2-\overline{B_2}(f_1(z_1,z_2))\}\{1-\theta\overline{V}(f_1(z_1,z_2))+\theta\overline{V}(f_3(z_2))\}\{1-\overline{B_1}(f_1(z_1,z_2))\}\\&+\{z_1-\overline{B_1}(f_1(z_1,z_2))\}\{1-\overline{B_2}(f_1(z_1,z_2))\}\{1-\theta\overline{V}(f_2(z_2))+\theta\overline{V}(f_3(z_2))\}],\end{aligned} \tag{93}$$

$$D(z_1,z_2)=\{z_1-\overline{B_1}(f_1(z_1,z_2))\}\{1-\theta\overline{V}(f_2(z_2))+\theta\overline{V}(f_3(z_2))\}f_1(z_1,z_2) \tag{94}$$

and
$f_1(z_1,z_2)=\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]),\ f_2(z_2)=\lambda_1[1-C(g(z_2)]+\lambda_2 b[1-C(z_2)],$
$f_3(z_2)=\lambda_1+\lambda_2 b[1-C(z_2)],\ f_4(z_2)=\lambda_1 C(g(z_2).$

We see that for $z_1=1,\ z_2=1,\ W_q(z_1,z_2)$ is indeterminate of the form $\frac{0}{0}$. Therefore, we apply L'Hopitals rule and on simplification, we obtain the result of equation (92) where $\overline{B_1}(0)=1,\ \overline{B_2}(0)=1,\ \overline{B_1'}(0)=-E(B_1)$ is the mean service time of a priority customer, $\overline{B_2'}(0)=-E(B_2)$ is the mean service time of a non-priority customer, $\overline{V}(0)=1,\ \overline{V'}(0)=-E(V)$ is the mean vacation time, $C(1)=1,\ g(1)=1,\ C'(1)=E(I)$ is the mean batch size of the arriving customer for both priority and non-priority units and $g'(1)=E(I_1)$ is the mean batch size of the arriving non-priority units during the busy period of priority units.

$$N_1(1,1) = 2Q(\lambda_1 C'(1) + \lambda_2 bC'(1))\{\overline{B_1'}(0)(\lambda_1 C'(1) + \lambda_2 bC'(1))\}\{1-\theta+\theta\overline{V}(\lambda_1)\}$$
$$+ 2\overline{V'}(0)(\lambda_1 C'(1) + \lambda_2 bC'(1))\{1 + \overline{B_1'}(0)(\lambda_1 C'(1) + \lambda_2 bC'(1))\}\theta Q\lambda_1, \tag{95}$$

$$N_2(1,1) = P_0^{(2)}(0,1)[2\theta\{1-\overline{B_2}(\lambda_1)\}\overline{V'}(0)(\lambda_1 C'(1) + \lambda_2 bC'(1))\{1+\overline{B_1'}(0)(\lambda_1 C'(1)$$
$$+ \lambda_2 bC'(1))\} - 2(\lambda_1 C'(1) + \lambda_2 bC'(1))\{1-\theta+\theta\overline{V}(\lambda_1)\}\{\overline{B_1'}(0) - \overline{B_2'}(0)\}], \tag{96}$$

$$D(1,1) = -2\{1+\overline{B_1'}(0)(\lambda_1 C'(1) + \lambda_2 bC'(1))\}(\lambda_1 C'(1) + \lambda_2 bC'(1))\{1-\theta+\theta\overline{V}(\lambda_1)\} \tag{97}$$

and

$$P_0^{(2)}(0,1) = \frac{\begin{aligned}&\big[\{1-\theta+\theta\overline{V}(\lambda_1)\}Q(\lambda_1 E(I)E(I_1) + \lambda_2 bE(I))\\ &+ \lambda_1 Q\theta E(V)(\lambda_1 E(I)E(I_1) + \lambda_2 bE(I))]\end{aligned}}{\begin{aligned}&\big[\{1-\theta+\theta\overline{V}(\lambda_1)\}\{1-E(B_2)(\lambda_1 E(I)E(I_1) + \lambda_2 bE(I))\}\\ &- \{1-\overline{B_2}(\lambda_1)\}\{\theta E(V)(\lambda_1 E(I)E(I_1) + \lambda_2 bE(I)) + \theta\overline{V'}(\lambda_1)\lambda_2 bE(I)\}\big]\end{aligned}}. \tag{98}$$

We shall use the normalizing condition $W_q(1,1) + Q = 1$, we get

$$Q = \frac{\big[\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\{1-\theta+\theta\overline{V}(\lambda_1)\}\big]}{\begin{aligned}&\big[\{1-\theta+\theta\overline{V}(\lambda_1)\} + \{1-\theta+\theta\overline{V}(\lambda_1)\}\{E(B_2) - E(B_1)\}\\ &+ \lambda_1\theta E(V)\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\\ &+ P_0^{(2)}(0,1)\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\big]\end{aligned}}. \tag{99}$$

The utilization factor is given by

$$\rho = 1 - Q, \tag{100}$$

that is,

$$\rho = \frac{\begin{aligned}&\big[\{E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\{1-\theta+\theta\overline{V}(\lambda_1)\}\\ &+ \{1-\theta+\theta\overline{V}(\lambda_1)\}\{E(B_2) - E(B_1)\}\\ &+ \lambda_1\theta E(V)\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\\ &+ P_0^{(2)}(0,1)\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\big]\end{aligned}}{\begin{aligned}&\big[\{1-\theta+\theta\overline{V}(\lambda_1)\} + \{1-\theta+\theta\overline{V}(\lambda_1)\}\{E(B_2) - E(B_1)\}\\ &+ \lambda_1\theta E(V)\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\\ &+ P_0^{(2)}(0,1)\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\{1-E(B_1)(\lambda_1 E(I) + \lambda_2 bE(I))\}\big]\end{aligned}}, \tag{101}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (100) gives the probability that the server is idle. Substituting equation (99) into (91), we have completely and explicitly determined $W_q(z_1, z_2)$, the probability generating function of the queue size.

6. The Average Queue Length

The mean number of customer in priority queue under the steady state is

$$L_{q_1} = \frac{d}{dz_1} W_{q_1}(z_1, 1)|_{z_1=1} \tag{102}$$

and the mean number of customer in the non-priority queue under the steady state is

$$L_{q_2} = \frac{d}{dz_2} W_{q_2}(1, z_2)|_{z_2=1}, \tag{103}$$

then

$$L_{q_1} = \frac{D_1''(1)N_1'''(1) - D_1'''(1)N_1''(1)}{3(D_1''(1))^2} + \frac{D_1''(1)N_2'''(1) - D_1'''(1)N_2''(1)}{3(D_1''(1))^2}, \tag{104}$$

$$L_{q_2} = \frac{D_2''(1)n_1'''(1) - D_2'''(1)n_1''(1)}{3(D_2''(1))^2} + \frac{D_2''(1)n_2'''(1) - D_2'''(1)n_2''(1)}{3(D_2''(1))^2}, \tag{105}$$

where

$$\begin{aligned}
N_1''(1) = {} & -2Q\lambda_1 E(I)E(B_1)\lambda_1 E(I)\{1-\theta+\theta\overline{V}(\lambda_1)\} - 2Q\lambda_1\theta E(V)\lambda_1 E(I) \\
& \times \{1 - E(B_1)\lambda_1 E(I)\}, \\
N_1'''(1) = {} & Q\{-3\lambda_1 E(I[I-1])E(B_1)\lambda_1 E(I) - 3\lambda_1 E(I)\{E(B_1^2)(\lambda_1 E(I))^2 + E(B_1)\lambda_1 \\
& E(I[I-1])\}\}\{1-\theta+\theta\overline{V}(\lambda_1)\} + 6Q\{\lambda_1 E(I)E(B_1)\lambda_1 E(I)\}\theta E(V)\lambda_1 E(I) \\
& - 3\theta\lambda_1 Q\{(E(V^2)(\lambda_1 E(I))^2 + E(V)\lambda_1 E(I[I-1]))(1 - E(B_1)\lambda_1 E(I)) \\
& + (E(V)\lambda_1 E(I))(E(B_1^2)(\lambda_1 E(I))^2 + E(B_1)\lambda_1 E(I[I-1]))\}, \\
N_2''(1) = {} & -2P_0^{(2)}(0,1)[\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\lambda_1 E(I)(1 - E(B_1)\lambda_1 E(I)) + E(B_2)\lambda_1 E(I) \\
& \{1-\theta+\theta\overline{V}(\lambda_1)\}], \\
N_2'''(1) = {} & P_0^{(2)}(0,1)[-3\theta\{1-\overline{B_2}(\lambda_1)\}(E(V^2)(\lambda_1 E(I))^2 + E(V)\lambda_1 E(I[I-1])) \\
& \times (1 - E(B_1)\lambda_1 E(I)) + 3\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\lambda_1 E(I)(E(B_1^2)(\lambda_1 E(I))^2 \\
& + E(B_1)\lambda_1 E(I[I-1])) + 6(E(B_2)\lambda_1 E(I))(E(B_1)\lambda_1 E(I))\theta E(V)\lambda_1 E(I) \\
& - 3[(E(B_2^2)(\lambda_1 E(I))^2 + E(B_2)\lambda_1 E(I[I-1]))]\{1-\theta+\theta\overline{V}(\lambda_1)\}], \\
D_1''(1) = {} & -2\lambda_1 E(I)[1 - E(B_1)\lambda_1 E(I)](1-\theta+\theta\overline{V}(\lambda_1)), \\
D_1'''(1) = {} & [-3\lambda_1 E(I[I-1])[1 - E(B_1)\lambda_1 E(I)] + 3\lambda_1 E(I)[E(B_1^2)(\lambda_1 E(I))^2 + E(B_1)\lambda_1 \\
& E(I[I-1])]](1-\theta+\theta\overline{V}(\lambda_1)), \\
n_1''(1) = {} & -2Q\lambda_2 bE(I)E(B_1)\lambda_2 bE(I)\{1-\theta+\theta\overline{V}(\lambda_1)\} - 2Q\lambda_1\theta E(V)\lambda_2 bE(I)E(B_1)\lambda_2 \\
& bE(I), \\
n_1'''(1) = {} & Q\{-3\lambda_2 bE(I[I-1])E(B_1)\lambda_2 bE(I) - 3\lambda_2 bE(I)\{E(B_1^2)(\lambda_2 bE(I))^2 + E(B_1)\lambda_2 \\
& E(I[I-1])\}\}\{1-\theta+\theta\overline{V}(\lambda_1)\} + 6Q\{\lambda_2 bE(I)E(B_1)\lambda_2 bE(I)\}\theta\{E(V)\lambda_2 bE(I) \\
& + \overline{V'}(\lambda_1)\lambda_2 bE(I)\} + 6\theta\lambda_1 E(I)E(I_1)Q\{E(V)\lambda_2 bE(I)\}\{E(B_1)\lambda_2 bE(I)\} \\
& + 3\theta\lambda_1 Q\{(E(V^2)(\lambda_2 bE(I))^2 + E(V)\lambda_2 bE(I[I-1]))(E(B_1)\lambda_2 bE(I)) \\
& + 3\theta\lambda_1 Q(E(V)\lambda_2 bE(I))(E(B_1^2)(\lambda_2 bE(I))^2 + E(B_1)\lambda_2 bE(I[I-1]))\},
\end{aligned}$$

$$
\begin{aligned}
n_2''(1) = 2P_0^{(2)}(0,1)[&\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\lambda_2 bE(I)(E(B_1)\lambda_2 bE(I)) + E(B_1)\lambda_2 bE(I)\\
&\{1-\theta+\theta\overline{V}(\lambda_1)\}],\\
n_2'''(1) = 6P_0^{(2)'}(0,1)[&\theta\{1-\overline{B_2}(\lambda_1)\}E(V)\lambda_2 bE(I)(E(B_1)\lambda_2 bE(I)) + (E(B_1)\lambda_2 bE(I))\\
&\{1-\theta+\theta\overline{V}(\lambda_1)\}] + 3P_0^{(2)}(0,1)[\theta\{1-\overline{B_2}(\lambda_1)\}(E(B_1)\lambda_2 bE(I))\{(E(V^2)\\
&(\lambda_2 bE(I))^2 + E(V)\lambda_2 bE(I[I-1]))\} + \theta\{1-\overline{B_2}(\lambda_1)\}E(V)\lambda_2 bE(I)(E(B_1^2)\\
&(\lambda_2 bE(I))^2 + E(B_1)\lambda_2 bE(I[I-1])) + (E(B_1^2)(\lambda_2 bE(I))^2 + E(B_1)\lambda_2 b\\
&E(I[I-1]))\{1-\theta+\theta\overline{V}(\lambda_1)\} - 2\theta E(B_1)\lambda_2 bE(I)\{E(V)\lambda_2 bE(I)\\
&+\overline{V}'(\lambda_1)\lambda_2 bE(I)\}],\\
D_2''(1) = 2\lambda_2 bE(I)&[E(B_1)\lambda_2 bE(I)](1-\theta+\theta\overline{V}(\lambda_1)),\\
D_2'''(1) = [3\lambda_2 b&E(I[I-1])[E(B_1)\lambda_2 bE(I)](1-\theta+\theta\overline{V}(\lambda_1)) + 3\lambda_2 bE(I)(E(B_1^2)\\
&(\lambda_2 bE(I))^2 + E(B_1)\lambda_2 bE(I[I-1]))](1-\theta+\theta\overline{V}(\lambda_1)) - 6\theta\lambda_2 bE(I)E(B_1)\lambda_2 bE(I)\\
&[E(V)(\lambda_1 E(I)E(I_1) + \lambda_2 bE(I)) + \overline{V}'(\lambda_1)\lambda_2 bE(I)].
\end{aligned}
$$

7. The Average Waiting Time in the Queue

Average Waiting time of a customer in the priority queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1}. \tag{106}$$

Average Waiting time of a customer in the non-priority queue is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}. \tag{107}$$

where L_{q_1} and L_{q_2} have been found in equations (104) and (105).

8. Particular Cases

Case 1:
If there is no vacation, no balking, single arrival and the service time follows exponential $\Big($when $\theta = 0,\ b = 1,\ E(I) = 1,\ E(I_1) = \dfrac{\lambda_2}{(\mu-\lambda_1)},\ E[I(I-1)] = 0$ and $E[I_1(I_1-1)] = \dfrac{2\lambda_2\mu}{(\mu-\lambda_1)^3}\Big)$. Then,

$$
\begin{aligned}
Q &= 1 - E(B_1)(\lambda_1+\lambda_2),\\
\rho &= E(B_1)(\lambda_1+\lambda_2),\\
L_{q_1} &= \frac{D_1''(1)N_1'''(1) - D_1'''(1)N_1''(1)}{3(D_1''(1))^2} + \frac{D_1''(1)N_2'''(1) - D_1'''(1)N_2''(1)}{3(D_1''(1))^2}
\end{aligned}
$$

and

$$L_{q_2} = \frac{D_2''(1)n_1'''(1) - D_2'''(1)n_1''(1)}{3(D_2''(1))^2} + \frac{D_2''(1)n_2'''(1) - D_2'''(1)n_2''(1)}{3(D_2''(1))^2},$$

where

$$
\begin{aligned}
&N_1''(1) = -2Q\lambda_1 E(B_1)\lambda_1,\\
&N_1'''(1) = -3Q\lambda_1 E(B_1^2)(\lambda_1)^2,\\
&N_2''(1) = -2P_0^{(2)}(0,1)[E(B_2)\lambda_1],\\
&N_2'''(1) = -3P_0^{(2)}(0,1)[E(B_2^2)(\lambda_1)^2],\\
&D_1''(1) = -2\lambda_1[1 - E(B_1)\lambda_1],\\
&D_1'''(1) = 3\lambda_1 E(B_1^2)(\lambda_1)^2,\\
&n_1''(1) = -2Q\lambda_2 E(B_1)\lambda_2,\\
&n_1'''(1) = -3Q\lambda_2 E(B_1^2)(\lambda_2)^2,\\
&n_2''(1) = 2P_0^{(2)}(0,1)[E(B_1)\lambda_2],\\
&n_2'''(1) = 6P_0^{(2)'}(0,1)[E(B_1)\lambda_2] + 3P_0^{(2)}(0,1)[E(B_2^1)(\lambda_1)^2],\\
&D_2''(1) = 2\lambda_2[E(B_1)\lambda_2],\\
&D_2'''(1) = 3\lambda_2 E(B_1^2)(\lambda_2)^2.
\end{aligned}
$$

The above result coincides with the results of Gross and Harris (1985).

Case 2:
If there is no low priority queue, no vacation, single arrival, no balking and priority service, then also consider normal service and service time as follows exponentially
$\big($when $\theta = 0,\ b = 1,\ E(I) = 1,\ E(I_1) = 0,\ E[I(I-1)] = 0,\ \lambda_2 = 0$ and $E(B_2) = 0\big)$. Then,

$$
\begin{aligned}
&Q = 1 - E(B_1)(\lambda_1),\\
&\rho = E(B_1)(\lambda_1),\\
&L_{q_1} = \frac{D_1''(1)N_1'''(1) - D_1'''(1)N_1''(1)}{3(D_1''(1))^2},
\end{aligned}
$$

where

$$
\begin{aligned}
&N_1''(1) = -2Q\lambda_1 E(B_1)\lambda_1,\\
&N_1'''(1) = -3Q\lambda_1 E(B_1^2)(\lambda_1)^2,\\
&D_1''(1) = -2\lambda_1[1 - E(B_1)\lambda_1]\\
&D_1'''(1) = 3\lambda_1 E(B_1^2)(\lambda_1)^2.
\end{aligned}
$$

The above result coincides with the results of Gross and Harris (1985).

9. Numerical Results

To numerically illustrate the results obtained in this work, we consider that the service time for priority and non-priority customers and the vacation time are exponentially distributed with rates $\mu_1,\ \mu_2,$ and γ respectively.

We base our numerical example on the result found in equations $(99),\ (101),\ (104),\ (105),$ (118) and (119). For this purpose in Table I, we choose the following arbitrary values $\theta = 0.3,\ b = 0.3,\ \mu_1 = 9,\ \mu_2 = 9,\ E(I) = 1,\ E[I(I-1)] = 0,\ E(I_1) = \dfrac{\lambda_2}{(\mu - \lambda_1)},$ $E[I_1(I_1-1)] = \dfrac{2\lambda_2\mu}{(\mu-\lambda_1)^3},\ \gamma = 6$ and $\lambda_2 = 2$. While λ_1 varies from 0.1 to 1.0 such that the stability condition is satisfied.

Table I: Effect of λ_1 on various queue characteristics

$\lambda 1$	Q	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.9176	0.0824	0.0010	0.0136	0.0102	0.0068
0.2	0.9022	0.0978	0.0025	0.0239	0.0123	0.0119
0.3	0.8869	0.1131	0.0043	0.0344	0.0144	0.0172
0.4	0.8718	0.1282	0.0066	0.0452	0.0166	0.0226
0.5	0.8569	0.1431	0.0094	0.0563	0.0188	0.0282
0.6	0.8421	0.1579	0.0127	0.0678	0.0211	0.0339
0.7	0.8276	0.1724	0.0164	0.0796	0.0235	0.0398
0.8	0.8133	0.1867	0.0208	0.0918	0.0260	0.0459
0.9	0.7991	0.2009	0.0257	0.1045	0.0285	0.0522
1.0	0.7851	0.2149	0.0311	0.1176	0.0311	0.0588

It clearly shows that as long as increasing the arrival rate of priority units the servers idle time decreases while the utilisation factor, average queue length and waiting time of a queue for priority and non-priority units are all increases.

In Table II, we choose the following arbitrary values $\theta = 0.3,\ b = 0.3,\ \mu_1 = 9,\ \mu_2 = 9,$ $E(I) = 1,\ E[I(I-1)] = 0,\ E(I_1) = \dfrac{\lambda_2}{(\mu-\lambda_1)},\ E[I_1(I_1-1)] = \dfrac{2\lambda_2\mu}{(\mu-\lambda_1)^3},\ \gamma = 6$ and $\lambda_1 = 2.$ Also λ_2 varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows that as long as increasing the arrival rate of non-priority units the servers idle time decreases while the utilisation factor, average queue lenth for both priority and non-priority unit and waiting time of a priority queue are all increases and waiting time of a non-priority units is decreases.

Table II: Effect of λ_2 on various queue characteristics

$\lambda 2$	Q	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
0.1	0.7143	0.2857	0.0744	0.0168	0.0372	0.1682
0.2	0.7111	0.2889	0.0768	0.0333	0.0384	0.1666
0.3	0.7080	0.2920	0.0792	0.0495	0.0396	0.1651
0.4	0.7048	0.2952	0.0817	0.0654	0.0408	0.1635
0.5	0.7016	0.2984	0.0842	0.0810	0.0421	0.1620
0.6	0.6984	0.3016	0.0867	0.0963	0.0434	0.1605
0.7	0.6953	0.3047	0.0893	0.1113	0.0447	0.1590
0.8	0.6921	0.3079	0.0920	0.1260	0.0460	0.1575
0.9	0.6889	0.3111	0.0946	0.1405	0.0473	0.1561
1.0	0.6857	0.3143	0.0974	0.1546	0.0487	0.1546

10. Conclusion

In this paper we studied a priority queueing system with balking and optional server vacation based on exhaustive service of the priority units. The server provides two types of service, namely priority and non-priority under non-preemptive priority rule. We derived the probability generating functions of the number of customers in the priority and non-priority units are found by using the supplementary variable technique, average queue size, the average waiting time for the priority and non-priority units and numerical results are also obtained. The above model finds potential application for guaranteeing different layers service for different customers. The Internet Protocol (IP) is not only successful in data communications, but IP emerges as the convergence layer for all forms of communication including voice, video or multimedia in general. The packet switched internet is taking over the circuit switched telephone network. The packet switched paradigm has great advantages in flexibility but it can give the same Quality of Service (QoS) as circuit switching. Therefore, all over the world great effort is put into improving the QoS packet switched systems. Priorities are a key instrument in giving each communication flow the QoS that it asks (and pays) for or Asynchronous Transfer Mode (ATM) was an attempt by telephone companies to design a network architecture that could efficiently transport both voice and data.

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