Available at
http://pvamu.edu.edu/pages/398.asp
ISSN: 1932-9466
Vol. 3, Issue 1 (June 2008) pp. 18 – 29
(Previously Vol. 3, No. 1)

Applications and Applied Mathematics (AAM):
An International Journal

On a-ary Subdivision for Curve Design: I. 4-Point and 6-Point Interpolatory Schemes

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Received April 27, 2008; accepted May 27, 2008

Abstract

The classical binary 4-point and 6-point interpolatery subdivision schemes are generalized to a-ary setting for any integer $a \geq 3$. These new a-ary subdivision schemes for curve design are derived easily from their corresponding two-scale scaling functions, a notion from the context of wavelets.

Key Words: Subdivision; Curve design; Stationary; Refinable functions; a-ary

MSC 2000: 14H50; 17A42; 65D17; 68U07

1. Introduction

THE classical 4-point binary interpolatory subdivision scheme for curve design was introduced more than twenty years ago in (Dyn, et al. [6]), and was given by

$$\lambda_{2k}^{(n+1)} = \lambda_k^{(n)}, \tag{1}$$

$$\lambda_{2k+1}^{(n+1)} = -\frac{1}{16}\left(\lambda_{k+2}^{(n)} + \lambda_{k-1}^{(n)}\right) + \frac{9}{16}\left(\lambda_{k+1}^{(n)} + \lambda_k^{(n)}\right), \quad k \in \mathbb{Z}. \tag{2}$$

About two years later, it was extended to the 6-point scheme (Weissman [15]):

$$\lambda_{2k}^{(n+1)} = \lambda_k^{(n)}, \tag{3}$$

$$\lambda_{2k+1}^{(n+1)} = \frac{3}{256}\left(\lambda_{k+3}^{(n)} + \lambda_{k-2}^{(n)}\right) - \frac{25}{256}\left(\lambda_{k+2}^{(n)} + \lambda_{k-1}^{(n)}\right) + \frac{75}{128}\left(\lambda_{k+1}^{(n)} + \lambda_k^{(n)}\right), \quad k \in \mathbb{Z}. \tag{4}$$

Due to the development of the wavelet theory, both schemes (1)–(2) and (3)–(4) can be easily re-discovered by the *scaling functions* ϕ_4 and ϕ_6 satisfying the *two-scale equations*

$$\phi_4(t) = \phi_4(2t) - \frac{1}{16}\left[\phi_4(2t+3) + \phi_4(2t-3)\right] + \frac{9}{16}\left[\phi_4(2t+1) + \phi_4(2t-1)\right], \quad t \in \mathbb{R}, \tag{5}$$

and

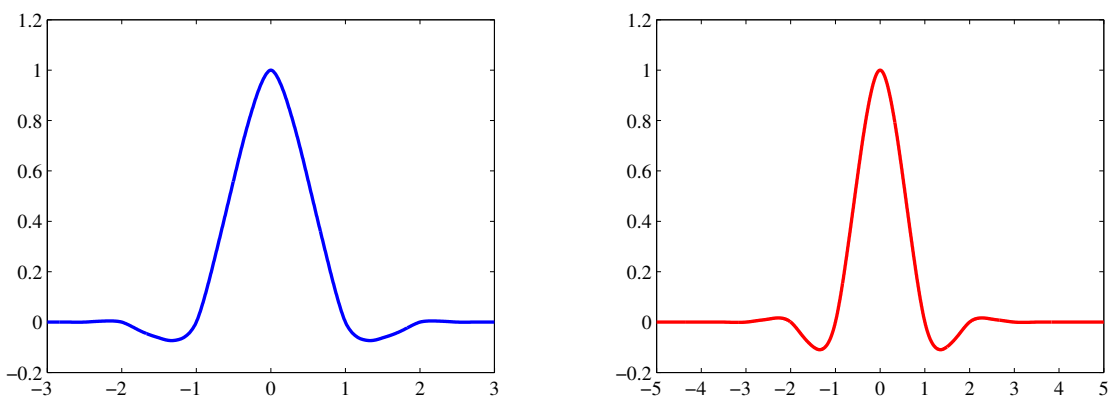
$$\begin{aligned}\phi_6(t) = \phi_6(2t) &+ \frac{3}{256}\left[\phi_6(2t+5) + \phi_6(2t-5)\right] - \frac{25}{256}\left[\phi_6(2t+3) + \phi_6(2t-3)\right] \\ &+ \frac{75}{128}\left[\phi_6(2t+1) + \phi_6(2t-1)\right], \quad t \in \mathbb{R},\end{aligned} \tag{6}$$

respectively. Here, again, the notion of scaling function is from the context of *wavelets*, a relatively new subject area that has been heavily studied for the last two decades or so, and found many successful applications. These scaling functions are sometimes also referred as *father wavelets* in the wavelet literature. Due to the fact that there are many research papers and books in wavelet analysis and its applications, we have no intention in this paper to list or elaborate many unnecessary notions of wavelets in detail. The interested reader is referred to the two exemplary books (Chui [1]) and (Daubechies [3]).

It is known that, with initial control net $\left\{\lambda_k^{(0)}\right\}_{k\in\mathbb{Z}}$, both schemes (1)–(2) and (3)–(4) converge to

$$\sum_{k\in\mathbb{Z}} \lambda_k^{(0)} \phi_4(t-k) \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} \lambda_k^{(0)} \phi_6(t-k),$$

respectively. Moreover, it is also known that ϕ_4 provides *polynomial preservation* of order 4, denoted by $\mathbb{PP}_4$ for short, and with the resulting limiting curve being C^1; while $\phi_6 \in \mathbb{PP}_6$ and with the resulting limiting curve being C^2. See Fig. 1 for graphs of ϕ_4 and ϕ_6.



(a) Graph of $\phi_4(\cdot)$ (b) Graph of $\phi_6(\cdot)$

Fig. 1. The interpolatory scaling functions $\phi_4(\cdot)$ and $\phi_6(\cdot)$ determined from the two-scale equations in (5) and (6), and with supp $\phi_4 = [-3, 3]$ and supp $\phi_6 = [-5, 5]$, respectively.

A natural question is: What if the dilation factor of $a = 2$ in (5) and (6) is replaced by any integer $a \geq 3$? Compactly supported orthonormal scaling functions and wavelets for the situation $a = 3$ has been considered in Lian's Ph.D thesis (Lian [10]) and consequently published in (Chui & Lian [2]). Ternary subdivision schemes were also considered in (Lian [10]). Later, the M-channel

filter bank design, meaning the scale factor $a = M$, was also studied, cf., e.g., the two distinct books (Vaidyanathanm [14], p. 223–271) and (Strang & Nguyen [13], p. 299–336).

One of the main objectives of this paper is to extend both schemes (1)–(2) and (3)–(4) to the a-ary setting for any $a \in \mathbb{Z}_+$ with $a \geq 3$. Our main results are listed in Section 2, with proofs given in Section 3. Some applications to curve design are demonstrated in Section 4. Some remarks and future work constitute Section 5.

2. Main Results

By taking the Fourier transforms of (5) and (6) we arrive at the two-scale equations of ϕ_4 and ϕ_6, namely,

$$\widehat{\phi}_4(\omega) = P_4(z)\widehat{\phi}_4\left(\frac{\omega}{2}\right),$$
$$\widehat{\phi}_6(\omega) = P_6(z)\,\widehat{\phi}_6\left(\frac{\omega}{2}\right),$$

where $z = \exp(-i\omega/2)$, with the two finite Laurent polynomials P_4 and P_6 being given by

$$P_4(z) = \left(\frac{1+z}{2}\right)^4 \frac{-1+4z-z^2}{2z^3}, \tag{7}$$
$$P_6(z) = \left(\frac{1+z}{2}\right)^6 \frac{3-18z+38z^2-18z^3+3z^4}{8z^5}, \tag{8}$$

which are called the *two-scale symbols* of ϕ_4 and ϕ_6, respectively. It is easy to see that both ϕ_4 and ϕ_6 are interpolatory:

$$\phi(k) = \delta_{k,0}, \quad k \in \mathbb{Z}, \tag{9}$$

where $\delta_{k,0}$ is the usual Kronecker delta, meaning $\delta_{k,0}$ is 1 when $k = 0$ and 0 otherwise. Generally speaking, any binary subdivision scheme corresponds to a scaling function. Observe that the inperpolatory property of a subdivision scheme for curve design is equivalent to the interpolatory property of its corresponding scaling function. Observe also that a scaling function ϕ satisfying (9) is equivalent to that its two-scale symbol P satisfies

$$P(z) + P(-z) = 1, \quad |z| = 1.$$

Now, if we allow the scaling factor, denoted by a, to be ≥ 3, and denote such a scaling function by ${}^a\phi$, then the two-scale equation of ${}^a\phi$ becomes

$$\widehat{{}^a\phi}(\omega) = {}^aP(z)\,\widehat{{}^a\phi}\left(\frac{\omega}{a}\right), \tag{10}$$

where $z = \exp(-i\omega/a)$, and, again, ${}^aP$ is its two-scale symbol. The interpolatory property of ${}^a\phi$ is then equivalent to ${}^aP$ satisfies

$$\sum_{\ell=0}^{a-1} {}^aP(w_\ell z) = 1, \quad |z| = 1, \tag{11}$$

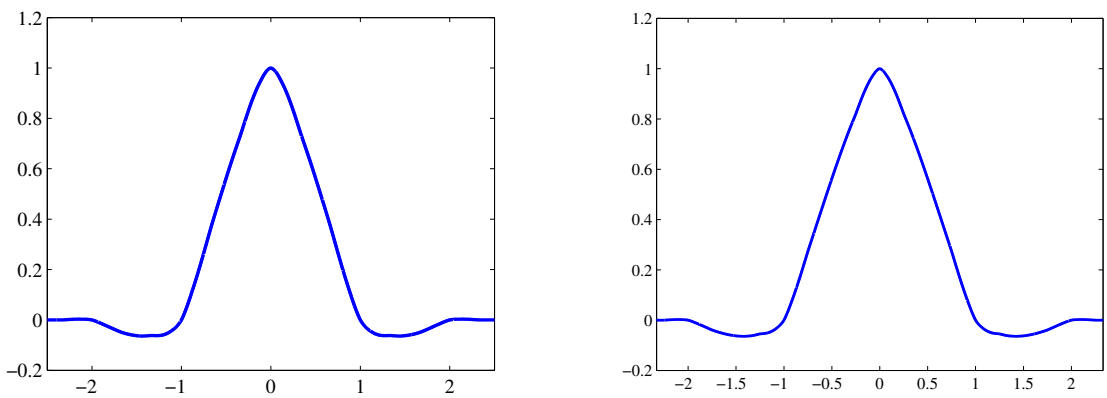
where $\{w_\ell\}_{\ell=0}^{a-1}$ are the a distinct roots of $z^a = 1$, namely,

$$w_\ell = \exp\left(\frac{-2\ell\pi i}{a}\right), \quad \ell = 0, \ldots, a-1.$$

For ${}^a\phi_4 \in \mathbb{PP}_4$, we have the following.

Theorem 1: The scaling function ${}^a\phi_4 \in \mathbb{PP}_4$ satisfying both (10) and (11), and with the smallest support, is determined from the two-scale symbol ${}^aP_4$ of the form

$${}^aP_4(z) = z^{1-2a}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^4\left(\frac{1-a^2}{6} + \frac{2+a^2}{3}z + \frac{1-a^2}{6}z^2\right). \tag{12}$$



(a) Graph of ${}^3\phi_4(\cdot)$ (b) Graph of ${}^4\phi_4(\cdot)$

Fig. 2. The interpolatory scaling functions ${}^3\phi_4(\cdot)$ and ${}^4\phi_4(\cdot)$ determined from the two-scale equations in (12) when $a = 3$ and 4, where supp ${}^3\phi_4 = [-5/2, 5/2]$ and supp ${}^4\phi_4 = [-7/3, 7/3]$, respectively.

See Fig. 2 for the graphs of ${}^3\phi_4$ and ${}^4\phi_4$. For ${}^a\phi_6 \in \mathbb{PP}_6$, we have the following.

Theorem 2: The scaling function ${}^a\phi_6 \in \mathbb{PP}_6$ satisfying both (10) and (11), and with the smallest support, is determined from the two-scale symbol ${}^aP_6$ of the form

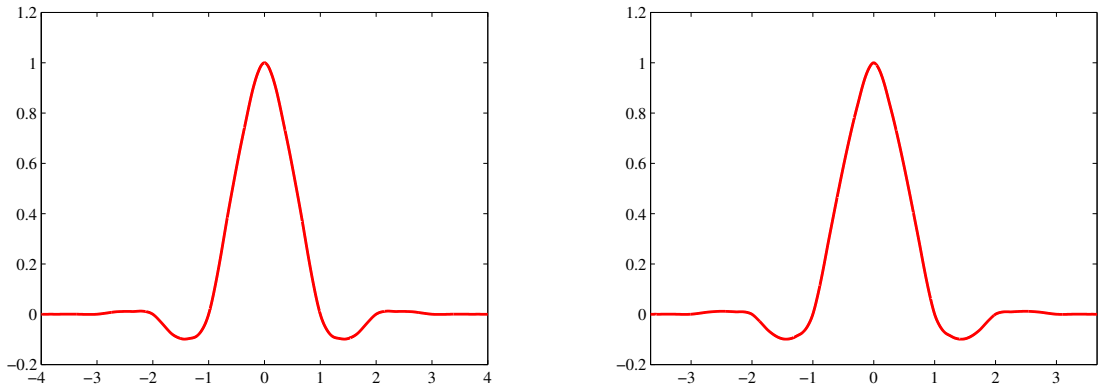
$$\begin{aligned}{}^aP_6(z) = z^{1-3a}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^6 \Bigg[&\frac{(a-1)(a+1)(2a-1)(2a+1)}{120} \\ &-\frac{(a-1)(a+1)(8a^2+13)}{60}z + \frac{11+5a^2+4a^4}{20}z^2 \\ &-\frac{(a-1)(a+1)(8a^2+13)}{60}z^3 + \frac{(a-1)(a+1)(2a-1)(2a+1)}{120}z^4\Bigg].\end{aligned} \tag{13}$$

See Fig. 3 for the graphs of ${}^3\phi_6$ and ${}^4\phi_6$. Notice that, when $a = 2$, ${}^2P_4$ in (12) and ${}^2P_6$ in (13) are exactly the P_4 and P_6 in (7) and (8). It is also easy to verify that

$$\text{supp}\,{}^a\phi_4 = \left[-\frac{2a-1}{a-1}, \frac{2a-1}{a-1}\right], \qquad \text{supp}\,{}^a\phi_6 = \left[-\frac{3a-1}{a-1}, \frac{3a-1}{a-1}\right].$$

If we write ${}^aP_4$ in (12) and ${}^aP_6$ in (13) by

$${}^aP_4(z) = \frac{1}{a}\sum_{k=-2a+1}^{2a-1} {}^a_4p_k z^k, \qquad {}^aP_6(z) = \frac{1}{a}\sum_{k=-3a+1}^{3a-1} {}^a_6p_k z^k,$$



(a) Graph of ${}^3\phi_6(\cdot)$ (b) Graph of ${}^4\phi_6(\cdot)$

Fig. 3. The interpolatory scaling functions ${}^3\phi_6(\cdot)$ and ${}^4\phi_6(\cdot)$ determined from the two-scale equations in (13) when $a = 3$ and 4, where supp ${}^3\phi_6 = [-4, 4]$ and supp ${}^4\phi_6 = [-11/3, 11/3]$, respectively.

TABLE I

WEIGHTS OF a-ARY 4-POINT SUBDIVISION SCHEME

	$\lambda_{k+2}^{(n)}$	$\lambda_{k+1}^{(n)}$	$\lambda_{k}^{(n)}$	$\lambda_{k-1}^{(n)}$
$\lambda_{ak}^{(n+1)}$		${}_4^a p_{-a}$	${}_4^a p_0$	${}_4^a p_a$
$\lambda_{ak+1}^{(n+1)}$	${}_4^a p_{-2a+1}$	${}_4^a p_{-a+1}$	${}_4^a p_1$	${}_4^a p_{a+1}$
$\lambda_{ak+2}^{(n+1)}$	${}_4^a p_{-2a+2}$	${}_4^a p_{-a+2}$	${}_4^a p_2$	${}_4^a p_{a+2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\lambda_{ak+a-1}^{(n+1)}$	${}_4^a p_{-a-1}$	${}_4^a p_{-1}$	${}_4^a p_{a-1}$	${}_4^a p_{2a-1}$

the a-ary 4- and 6-point interpolatory subdivision schemes for curve design can be given by Table I and Table II, i.e., the 4-point scheme is given by

$$\lambda_{ak+\ell}^{(n+1)} = \sum_{j=-2}^{1} {}_4^a p_{aj+\ell} \lambda_{k-j}^{(n)}, \quad \ell = 0, \ldots, a-1; \quad n \in \mathbb{Z}_+, \tag{14}$$

while the 6-point a-ary scheme is given by

$$\lambda_{ak+\ell}^{(n+1)} = \sum_{j=-3}^{2} {}_6^a p_{aj+\ell} \lambda_{k-j}^{(n)}, \quad \ell = 0, \ldots, a-1, \quad n \in \mathbb{Z}_+. \tag{15}$$

Here, $\{{}_4^a p_k\}_{k\in\mathbb{Z}}$ and $\{{}_6^a p_k\}_{k\in\mathbb{Z}}$ are called *two-scale sequences* in wavelet literature and *weights* in CAGD, which are listed explicitly in the following,

$${}_4^a p_{-k} = {}_4^a p_k = \frac{1}{2a^3}(a+k)(a-k)(2a-k), \quad k = 0, \ldots, a-1; \tag{16}$$

$${}_4^a p_{-k} = {}_4^a p_k = \frac{1}{6a^3}(a-k)(2a-k)(3a-k), \quad k = a, \ldots, 2a-1; \tag{17}$$

$${}_4^a p_k = 0, \quad |k| \geq 2a, \tag{18}$$

and

$$ {}^a_6p_{-k} = {}^a_6p_k = \frac{1}{12a^5}(a-k)(a+k)(2a-k)(2a+k)(3a-k), \quad k = 0, \ldots, a-1; \tag{19} $$

$$ {}^a_6p_{-k} = {}^a_6p_k = \frac{1}{24a^5}(a-k)(a+k)(2a-k)(3a-k)(4a-k), \quad k = a, \ldots, 2a-1; \tag{20} $$

$$ {}^a_6p_{-k} = {}^a_6p_k = \frac{1}{120a^5}(a-k)(2a-k)(3a-k)(4a-k)(5a-k), \; k = 2a, \ldots, 3a-1; \tag{21} $$

$$ {}^a_6p_k = 0, \quad |k| \geq 3a. \tag{22} $$

The interpolatory property of both schemes in (14) and (15) is now clear from (16)–(18) and (19)–(22).

TABLE II

WEIGHTS OF a-ARY 6-POINT SUBDIVISION SCHEME

	$\lambda^{(n)}_{k+3}$	$\lambda^{(n)}_{k+2}$	$\lambda^{(n)}_{k+1}$	$\lambda^{(n)}_{k}$	$\lambda^{(n)}_{k+1}$	$\lambda^{(n)}_{k+2}$
$\lambda^{(n+1)}_{ak}$		${}^a_6p_{-2a}$	${}^a_6p_{-a}$	${}^a_6p_0$	${}^a_6p_a$	${}^a_6p_{2a}$
$\lambda^{(n+1)}_{ak+1}$	${}^a_6p_{-3a+1}$	${}^a_6p_{-2a+1}$	${}^a_4p_{-a+1}$	${}^a_4p_1$	${}^a_4p_{a+1}$	${}^a_4p_{2a+1}$
$\lambda^{(n+1)}_{ak+2}$	${}^a_6p_{-3a+2}$	${}^a_6p_{-2a+2}$	${}^a_4p_{-a+2}$	${}^a_4p_2$	${}^a_4p_{a+2}$	${}^a_4p_{2a+2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\lambda^{(n+1)}_{ak+a-1}$	${}^a_6p_{-2a-1}$	${}^a_6p_{-a-1}$	${}^a_4p_{-1}$	${}^a_4p_{a-1}$	${}^a_4p_{2a-1}$	${}^a_4p_{3a-1}$

3. Proofs of Main Results

Proof of Theorem 1.

First, an a-ary 4-point scheme needs at most $4a$ weights, i.e., the two-scale sequence $\{{}^a_4p_k\}_{k\in\mathbb{Z}}$ of ${}^a\phi_4$ has at most $4a$ consecutive nontrivial entries. For ${}^a\phi_4$ to be symmetric and interpolatory, the length of $\{{}^a_4p_k\}_{k\in\mathbb{Z}}$ has to be reduced by 1. Secondly, for ${}^a\phi_4$ to have the highest possible m of $\mathbb{PP}_m$, its two-scale symbol ${}^aP_4$ has to have the highest possible order of factor of $(1+z+\cdots+z^{a-1})$. This leads to both $m = 4$ and ${}^aP_4$ must have the form

$$ {}^aP_4(z) = z^{1-2a}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^4\left(s_0 + s_1z + s_2z^2\right) $$

for some constant s_0, s_1, and s_2 satisfying $s_2 = s_0$ and $s_0 + s_1 + s_2 = 1$. By using $(1-z)^{-4} = \sum_{\ell=0}^{\infty}\binom{3+\ell}{3}z^\ell$ we have

$$ \left(s_0 + s_1z + s_2z^2\right)(1-z)^{-4} = \sum_{\ell=0}^{\infty} g_\ell z^\ell, \quad \text{where} $$

$$ g_\ell = \binom{\ell+3}{3}s_0 + \binom{\ell+2}{3}s_1 + \binom{\ell+1}{3}s_2, \quad \ell \in \mathbb{Z}_+. \tag{23} $$

Hence, by defining $g_\ell = 0$ for all $\ell < 0$ and multiplying by the expansion of $(1-z^a)^4$ we obtain the explicit expressions for $\{{}^a_4p_k\}_{k\in\mathbb{Z}}$ in terms of $\{g_\ell\}$, namely,

$$\begin{aligned} {}^a_4p_k = \frac{1}{a^3}(&g_{2a-1+k} - 4g_{a-1+k} + 6g_{-1+k} \\ &-4g_{-a-1+k} + g_{-2a-1+k}), \qquad k = -2a+1, \dots, 2a-1. \end{aligned} \tag{24}$$

Next, the three identities ${}^a_4p_{-a} = 0, {}^a_4p_0 = 1$, and ${}^a_4p_a = 0$, in turn, become

$$\begin{aligned} &\frac{1}{a^3}g_{a-1} = 0, \\ &\frac{1}{a^3}(g_{2a-1} - 4g_{a-1}) = 1, \\ &\frac{1}{a^3}(g_{3a-1} - 4g_{2a-1} + 6g_{a-1}) = 0, \end{aligned}$$

or simply $g_{a-1} = 0, g_{2a-1} = a^3, g_{3a-1} = 4a^3$, or, equivalently,

$$\begin{aligned} &\binom{a+2}{3}s_0 + \binom{a+1}{3}s_1 + \binom{a}{3}s_2 = 0, \\ &\binom{2a+2}{3}s_0 + \binom{2a+1}{3}s_1 + \binom{2a}{3}s_2 = a^3, \\ &\binom{3a+3}{3}s_0 + \binom{3a+1}{3}s_1 + \binom{3a}{3}s_2 = 4a^3. \end{aligned}$$

By solving this linear system, s_0, s_1, and s_2 are given by

$$s_0 = s_2 = \frac{1-a^2}{6}, \quad s_1 = \frac{a^2+2}{3},$$

as they were in (12). Substituting s_0, s_1, and s_2 into (23) leads to

$$g_\ell = \frac{1}{6}(\ell+1)((\ell+1)^2 - a^2), \qquad \ell \in \mathbb{Z}_+.$$

Finally, by substituting g_ℓ's into (24) we arrive at the explicit expressions for ${}^a_4p_k$'s in (16)–(18). This completes the proof of Theorem 1. $\square$

Proof of Theorem 2.

Similar to the proof of Theorem 1, the two-scale symbol ${}^aP_6$ of ${}^a\phi_6$ must have the form

$${}^aP_6(z) = z^{1-3a}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^6\left(s_0 + s_1z + s_2z^2 + s_3z^3 + s_4z^4\right)$$

for some constants $s_0, \dots, s_4$ satisfying $s_4 = s_0, s_3 = s_1$, and $s_0 + \cdots + s_4 = 1$. First, multiply $s_0 + s_1z + s_2z^2 + s_3z^3 + s_4z^4$ and $(1-z)^{-6} = \sum_{\ell=0}^\infty \binom{5+\ell}{5}z^\ell$ to get

$$\begin{aligned} &\left(s_0 + s_1z + s_2z^2 + s_3z^3 + s_4z^4\right)(1-z)^{-6} = \sum_{\ell=0}^\infty h_\ell z^\ell, \quad \text{where} \\ &h_\ell = \binom{\ell+5}{5}s_0 + \binom{\ell+4}{5}s_1 + \binom{\ell+3}{5}s_2 + \binom{\ell+2}{5}s_3 + \binom{\ell+1}{5}s_4, \quad \ell \in \mathbb{Z}_+. \end{aligned} \tag{25}$$

Secondly, multiply by the expansion of $(1-z^a)^6$, $\{{}^a_6p_k\}_{k\in\mathbb{Z}}$ can be expressed in terms of $\{h_\ell\}$ in (25). Then, with $h_\ell = 0$ for all $\ell < 0$, all coefficients of ${}^aP_6(z)$ are now in terms of $s_0, \ldots, s_4$, namely,

$$\begin{aligned} {}^a_6p_k = \frac{1}{a^5}(&h_{3a-1+k} - 6h_{2a-1+k} + 15h_{a-1+k} - 20h_{-1+k} + 15h_{-a-1+k} \\ &-6h_{-2a-1+k} + h_{-3a-1+k}), \qquad k = -3a+1, \ldots, 3a-1. \end{aligned} \tag{26}$$

The five requirements

$${}^a_6p_{-2a} = {}^a_6p_{-a} = 0, \quad {}^a_6p_0 = 1, \quad {}^a_6p_a = {}^a_6p_{2a} = 0$$

yield

$$\begin{aligned} &\frac{1}{a^5}h_{a-1} = 0, \\ &\frac{1}{a^5}(h_{2a-1} - 6h_{a-1}) = 0, \\ &\frac{1}{a^5}(h_{3a-1} - 6h_{2a-1} + 15h_{a-1}) = 1, \\ &\frac{1}{a^5}(h_{4a-1} - 6h_{3a-1} + 15h_{2a-1} - 20h_{a-1}) = 0, \\ &\frac{1}{a^5}(h_{5a-1} - 6h_{4a-1} + 15h_{3a-1} - 20h_{2a-1} + 15h_{a-1}) = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\binom{a+4}{5}s_0 + \binom{a+3}{5}s_1 + \binom{a+2}{5}s_2 + \binom{a+1}{5}s_3 + \binom{a}{5}s_4 = 0, \\ &\binom{2a+4}{5}s_0 + \binom{2a+3}{5}s_1 + \binom{2a+2}{5}s_2 + \binom{2a+1}{5}s_3 + \binom{2a}{5}s_4 = 0, \\ &\binom{3a+4}{5}s_0 + \binom{3a+3}{5}s_1 + \binom{3a+2}{5}s_2 + \binom{3a+1}{5}s_3 + \binom{3a}{5}s_4 = a^5, \\ &\binom{4a+4}{5}s_0 + \binom{4a+3}{5}s_1 + \binom{4a+2}{5}s_2 + \binom{4a+1}{5}s_3 + \binom{4a}{5}s_4 = 6a^5, \\ &\binom{5a+4}{5}s_0 + \binom{5a+3}{5}s_1 + \binom{5a+2}{5}s_2 + \binom{5a+1}{5}s_3 + \binom{5a}{5}s_4 = 21a^5. \end{aligned}$$

Solving this linear system we have $s_0, \ldots, s_4$ in (13), i.e.,

$$\begin{aligned} s_0 = s_4 &= \frac{1}{120}(a^2-1)(4a^2-1), \\ s_1 = s_3 &= -\frac{1}{60}(a^2-1)(8a^2+13), \\ s_2 &= \frac{1}{20}(11+5a^2+4a^4). \end{aligned}$$

By substituting $s_0, \ldots, s_4$ into (25) to get

$$h_\ell = \frac{1}{120}(\ell+1)((\ell+1)^2-a^2)((\ell+1)^2-4a^2), \qquad \ell \in \mathbb{Z}_+,$$

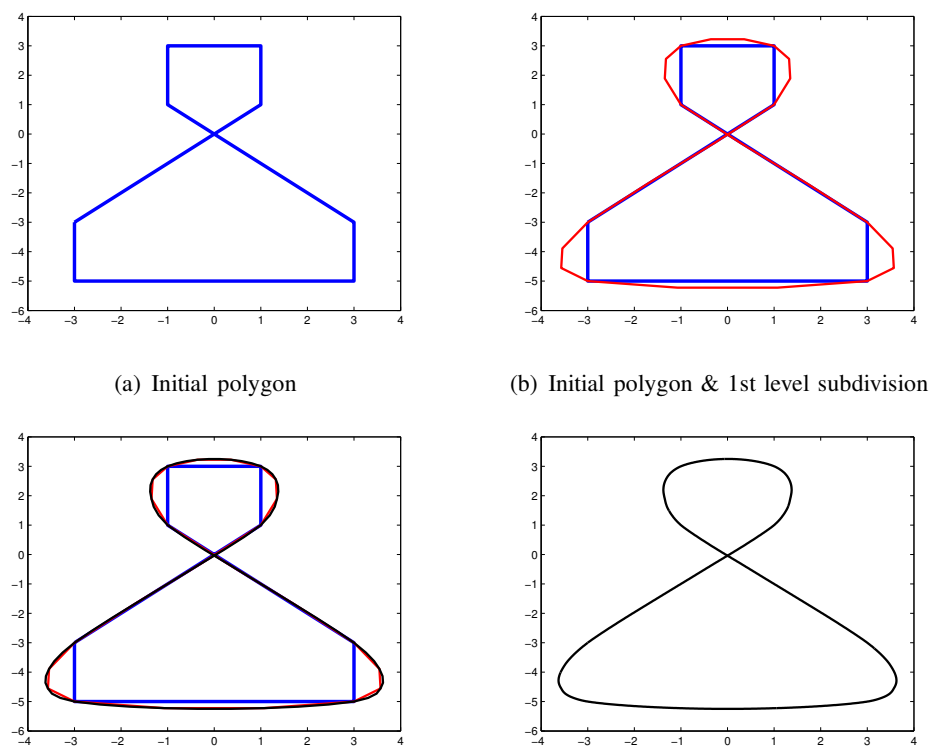
and subsequently substituting h_ℓ's into (26), we have ${}^{a}_{6}p_k$'s in (19)–(22). This completes the proof of Theorem 2. $\square$

4. Applications to Curve Design

The 4- and 6-point schemes can be applied to any curve design in any dimension. In Fig. 4 and Fig. 5 we demonstrate a planar curve and a 3D curve by using the 4-point ternary interpolatory subdivision scheme from Table I and (16)–(18) with $a = 3$, namely,

$$\begin{aligned}
\lambda_{3k}^{(n+1)} &= \lambda_k^{(n)}, \\
\lambda_{3k+1}^{(n+1)} &= -\frac{4}{81}\lambda_{k+2}^{(n)} + \frac{10}{27}\lambda_{k+1}^{(n)} + \frac{20}{27}\lambda_k^{(n)} - \frac{5}{81}\lambda_{k-1}^{(n)}, \\
\lambda_{3k+2}^{(n+1)} &= -\frac{5}{81}\lambda_{k+2}^{(n)} + \frac{20}{27}\lambda_{k+1}^{(n)} + \frac{10}{27}\lambda_k^{(n)} - \frac{4}{81}\lambda_{k-1}^{(n)}, \quad k \in \mathbb{Z}_+.
\end{aligned}$$

The planar curve in Fig. 4(d) was from a closed polygon formed by nine initial control points in Fig. 4(a), and the 3D curve in Fig. 5(d) was from a closed polygon formed by nine initial control points on corners of a cube, as shown in Fig. 5(a).



(a) Initial polygon

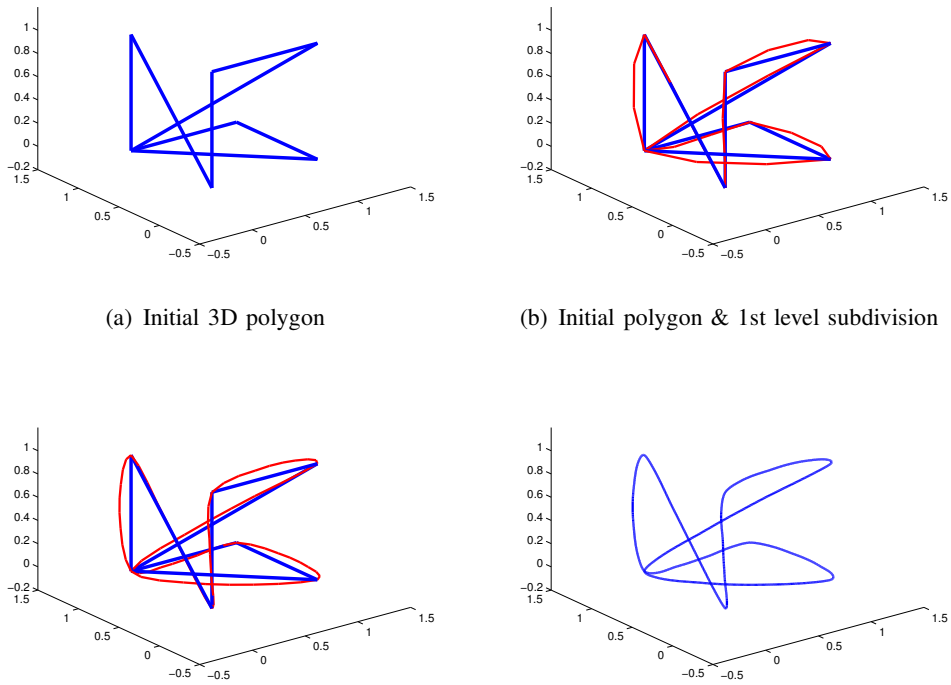
(b) Initial polygon & 1st level subdivision

(c) Initial polygon, & 1st & 2nd level subdivisions

(d) Result after 5th subdivision

Fig. 4. A planar curve with eight initial control points.

In Fig. 6, we demonstrate a space curve by using the 6-point quaternary interpolatory subdivision



(a) Initial 3D polygon

(b) Initial polygon & 1st level subdivision

(c) Initial polygon & 2nd level subdivision

(d) Result after 6th subdivision

Fig. 5. A space curve with initial control points selected from some corners of a cube.

scheme from Table II and (19)–(22) with $a = 4$, namely, for $k \in \mathbb{Z}_+$,

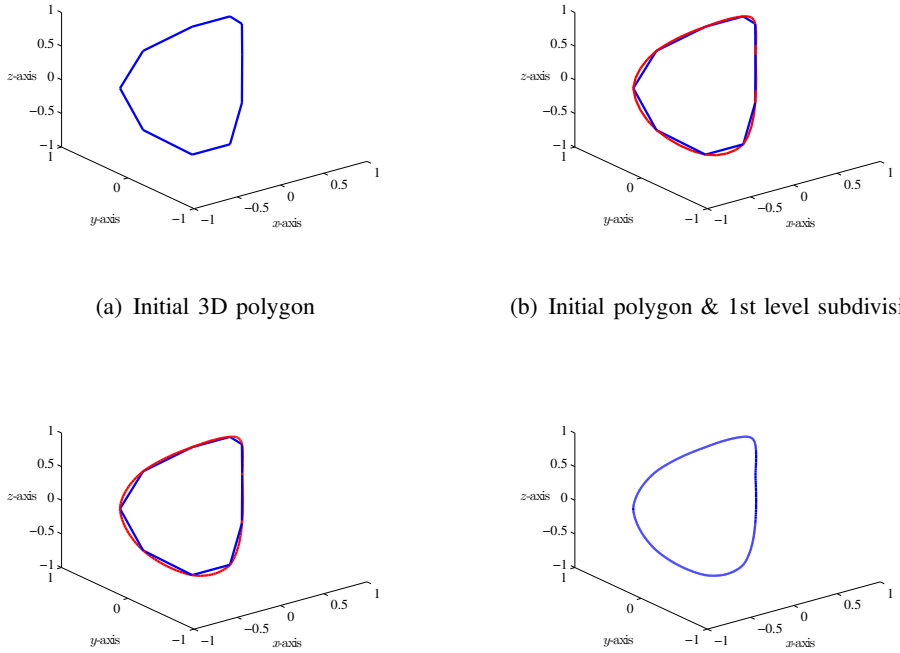
$$\begin{aligned}
\lambda_{4k}^{(n+1)} &= \lambda_k^{(n)}, \\
\lambda_{4k+1}^{(n+1)} &= \frac{63}{8192}\lambda_{k+3}^{(n)} - \frac{495}{8192}\lambda_{k+2}^{(n)} + \frac{1155}{4096}\lambda_{k+1}^{(n)} + \frac{3465}{4096}\lambda_k^{(n)} - \frac{693}{8192}\lambda_{k-1}^{(n)} + \frac{77}{8192}\lambda_{k-2}^{(n)}, \\
\lambda_{4k+2}^{(n+1)} &= \frac{3}{256}\lambda_{k+3}^{(n)} - \frac{25}{256}\lambda_{k+2}^{(n)} + \frac{75}{128}\lambda_{k+1}^{(n)} + \frac{75}{128}\lambda_k^{(n)} - \frac{25}{256}\lambda_{k-1}^{(n)} + \frac{3}{256}\lambda_{k-2}^{(n)}, \\
\lambda_{4k+3}^{(n+1)} &= \frac{77}{8192}\lambda_{k+3}^{(n)} - \frac{693}{8192}\lambda_{k+2}^{(n)} + \frac{3465}{4096}\lambda_{k+1}^{(n)} + \frac{1155}{4096}\lambda_k^{(n)} - \frac{495}{8192}\lambda_{k-1}^{(n)} + \frac{63}{8192}\lambda_{k-2}^{(n)}.
\end{aligned}$$

The 3D closed polygon in Fig. 6(a) was formed by 11 initial control points selected from the closed space curve $((\cos t)/\sqrt{2}, (\cos^2 t)/2, \sin t), t \in [0, 2\pi]$, which is the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the cylindrical surface $y = x^2$. The 3D closed space curve in Fig. 6(d) was the result after 4 consecutive subdivisions.

We also remark that these scaling functions can also be applied to curve editing. For some early studies of the multiresolution representation of parametric curves, the reader is referred to both (Finkelstein & Salesin [7]) and (Reissell [12]).

5. Conclusion

The classical binary 4- and 6-point interpolatery subdivision schemes for curve design were



(a) Initial 3D polygon

(b) Initial polygon & 1st level subdivision

(c) Initial polygon & 2nd level subdivision

(d) Result after 4th subdivision

Fig. 6. A closed space curve with initial closed control polygon formed by 11 points selected from the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the cylindrical surface $y = x^2$.

extended to a-ary for any integer $a \geq 3$. For both schemes the polynomial preservation order is fixed, namely, either 4 or 6, which is independent of a. The smoothness of the corresponding scaling functions for various values of $a \geq 3$ are needed to and will be studied in detail in the forthcoming paper. Certainly, scaling functions for a-ary approximation subdivision schemes with highest possible polynomial preservation orders are the classical B-splines and are no need for any further study. The interpolatory schemes can be applied to curve editing, data fitting and regression, and image rescaling. It is also expected that interpolatory schemes for curve designs can be combined with subdivision schemes for surface design to make the latter more adaptive and flexible along the boundaries of a 3D polyhedron.

Acknowledgment

This work was partially supported by ARO under Grant W911NF-05-1-0388.

References

[1] Chui, C. K. (1992). An Introduction to Wavelets, Academic Press, San Diego, California.

[2] Chui, C. K. and J.-A. Lian (1995). Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale $= 3$, *Appl. Comp. Harmonic Anal.* **2**, 21-51.

[3] Daubechies, I. (1992). Ten Lectures On Wavelets, SIAM, Philadelphia, Pennsylvania.

[4] Doo, D. and M. Sabin (1978). Behaviour of recursive division surfaces near extraordinary points, Computer-Aided Design **10**, 356–360.

[5] Dubuc, S. (1986). Interpolation through an iterative scheme, J. Math. Anal. Appl. **114**, 185–204.

[6] Dyn, N., D. Levin, and J. Gregory (1987). A 4-point interpolatory subdivision scheme for curve design, *Computer Aided Geometric Design* **4**, 257-268.

[7] Finkelstein, A. and D. Salesin (1994). Multiresolution curves, in: **SIGGRAPH'94**: *Proceedings of the 21st Annual Conference on Computer Graphics and Interactive Techniques*, ACM, New York, NY, 261–268.

[8] Hassan, M. F. and N. A. Dodgson (2002). Ternary and three-point univariate subdivision schemes, in: *Curve and Surface Fitting: Saint-Malo 2002*, A. Cohen, J.-L. Merrien, and L. L. Schumaker (eds.), Nashboro Press, Brentwood, TN, 199–208.

[9] Ivrissimtzisa, I. P., N. A. Dodgson, M. F. Hassan, and M. A. Sabin (2002). The refinability of the four point scheme, *Computer Aided Geometric Design* **19**, 235-238.

[10] Lian, J.-A. (1993). *Compactly Supported Wavelets with Dilation Factor* $a = 3$, Ph. D Dissertation, Texas A&M University, College Station, TX.

[11] Lian, J.-A. (2001). Polynomial identities of Bezout type, in: *Trends in Approximation Theory*, K. Kopotun, T. Lyche, and M. Neamtu (eds.), Vanderbilt University Press, Nashville, TN, 243–252.

[12] Reissell, L.-M. (1996). Wavelet multiresolution representation of curves and surfaces, *Graphical Models and Image Processing* **58**, 198–217.

[13] Strang, G. and T. Nguyen (1996). *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, MA.

[14] Vaidyanathanm, P. P. (1993). *Multirate Systems And Filter Banks*, Prentice Hall PTR, Englewood Cliffs, NY.

[15] Weissman, A. (1989). A 6-point interpolatory subdivision scheme for curve design, Masters Thesis, Tel-Aviv University.