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On a-ary Subdivision for Curve Design
II. 3-Point and 5-Point Interpolatory Schemes

Jian-ao Lian

Department of Mathematics
Prairie View A&M University
Prairie View, TX 77446-0519 USA
JiLian@pvamu.edu

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Abstract

The a-ary 3-point and 5-point interpolatery subdivision schemes for curve design are introduced for arbitrary odd integer $a \geq 3$. These new schemes further extend the family of the classical 4- and 6-point interpolatory schemes.

Keywords: Subdivision; Curve design; Stationary; Refinable functions; a-ary

MSC 2000: 14H50; 17A42; 65D17; 68U07

1. Introduction

THIS is a continuation of (Lian [4]), where the classical 4- and 6-point binary interpolatory subdivision schemes for curve design in (Dyn, et al. [1]) and (Weissman [5]) were extended to a-ary interpolatory schemes for any $a \geq 3$.

One of the main objectives of the current paper is to introduce and extend both the 4- and 6-point a-ary interpolatory schemes further to the 3- and 5-point a-ary interpolatory schemes for any odd $a \in \mathbb{Z}_+$ with $a \geq 3$. Similar to the 4- and 6-point a-ary schemes, we also require the refinable functions corresponding to the 3- and 5-point a-ary interpolatory schemes have polynomial preservation orders of 3 and 5, respectively, or ${}^a\phi_3 \in \mathbb{PP}_3$ and ${}^a\phi_5 \in \mathbb{PP}_5$ for short. Observe that, when $a \geq 2$ is even, for either ${}^a\phi_3 \in \mathbb{PP}_3$ or ${}^a\phi_5 \in \mathbb{PP}_5$, the interpolatory property and the symmetry on either ${}^a\phi_3$ or ${}^a\phi_5$ are not compatible. That is exactly the reason why the dilation factor a has to be odd now.

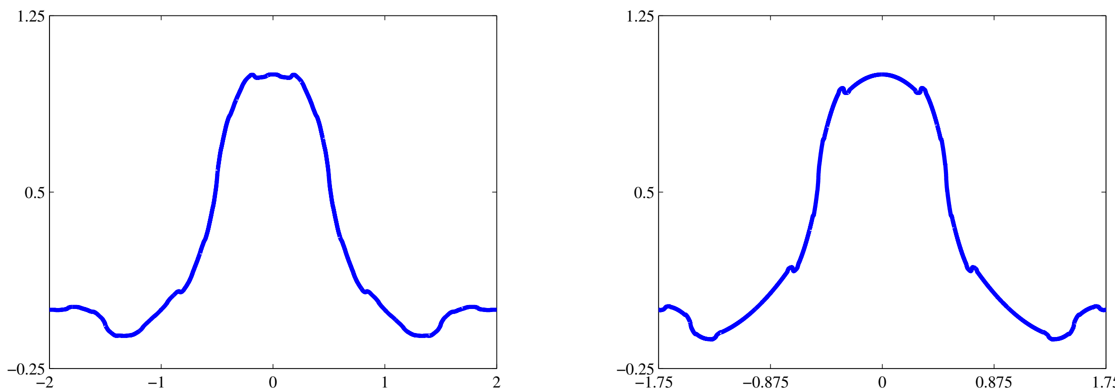
Our main results are, listed in Section 2, the explicit expressions of two-scale symbols of both ${}^a\phi_3 \in \mathbb{PP}_3$ and ${}^a\phi_5 \in \mathbb{PP}_5$. Their proofs are given in Section 3. Some applications to curve design are demonstrated in Section 4. A few remarks and future work constitute Section 5.

2. Main Results

Let ${}^a\phi_3$ and ${}^a\phi_5$ be the scaling functions with odd dilation factor $a \geq 3$, which correspond the 3- and 5-point interpolatory subdivision schemes for curve design. For ${}^a\phi_3$, we have the following.

Theorem 1: The scaling function ${}^a\phi_3 \in \mathbb{PP}_3$ with the smallest support, is determined from the two-scale symbol ${}^aP_3$ of the form

$$ {}^aP_3(z) = z^{(1-3a)/2}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^3\left(\frac{1-a^2}{8}+\frac{3+a^2}{4}z+\frac{1-a^2}{8}z^2\right). \tag{1} $$



(a) Graph of ${}^3\phi_3(\cdot)$ (b) Graph of ${}^5\phi_3(\cdot)$

Fig. 1. The interpolatory scaling functions ${}^3\phi_3(\cdot)$ and ${}^5\phi_3(\cdot)$ determined from the two-scale equations in (1) when $a = 3$ and 5, where supp ${}^3\phi_3 = [-2, 2]$ and supp ${}^5\phi_3 = [-7/4, 7/4]$, respectively.

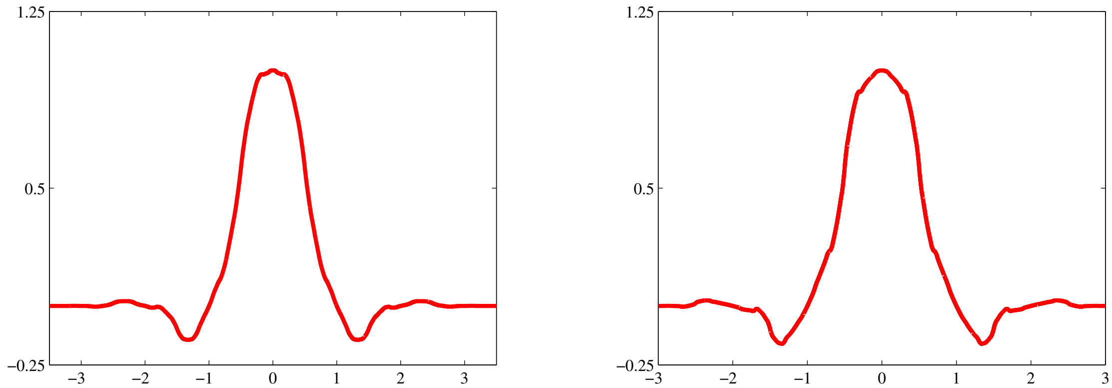
See Fig. 1 for the graphs of ${}^3\phi_3$ and ${}^5\phi_3$. For ${}^a\phi_5 \in \mathbb{PP}_5$, we have the following.

Theorem 2: The scaling function ${}^a\phi_5 \in \mathbb{PP}_5$ with the smallest support, is determined from the two-scale symbol ${}^aP_5$ of the form

$$ \begin{aligned} {}^aP_5(z) = z^{(1-5a)/2}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^5 \Bigg[& \frac{(a-1)(a+1)(3a-1)(3a+1)}{384} \\ & -\frac{(a-1)(a+1)(9a^2+19)}{96}z+\frac{115+50a^2+27a^4}{192}z^2 \\ & -\frac{(a-1)(a+1)(9a^2+19)}{96}z^3+\frac{(a-1)(a+1)(3a-1)(3a+1)}{384}z^4\Bigg]. \end{aligned} \tag{2} $$

See Fig. 2 for the graphs of ${}^3\phi_5$ and ${}^5\phi_5$. It is also easy to verify that

$$ \operatorname{supp}{}^a\phi_3 = \left[-\frac{3a-1}{2(a-1)}, \frac{3a-1}{2(a-1)}\right], \qquad \operatorname{supp}{}^a\phi_5 = \left[-\frac{5a-1}{2(a-1)}, \frac{5a-1}{2(a-1)}\right]. $$



(a) Graph of ${}^3\phi_5(\cdot)$ (b) Graph of ${}^5\phi_5(\cdot)$

Fig. 2. The interpolatory scaling functions ${}^3\phi_5(\cdot)$ and ${}^5\phi_5(\cdot)$ determined from the two-scale equations in (2) when $a = 3$ and 5, where supp ${}^3\phi_5 = [-7/2, 7/2]$ and supp ${}^5\phi_5 = [-3, 3]$, respectively.

Indeed, if $\operatorname{supp} {}^a\phi_3 = [\ell_3, r_3]$, it follows from (1) that the left-most contribution to ${}^a\phi_3(x)$ is ${}^a\phi_3\left(ax + \dfrac{3a-1}{2}\right)$ while the right-most contribution to ${}^a\phi_3(x)$ is ${}^a\phi_3\left(ax - \dfrac{3a-1}{2}\right)$. Hence, $\ell_3 \le ax + \dfrac{3a-1}{2}$ and $ax - \dfrac{3a-1}{2} \le r_3$, which leads to

$$\frac{1}{a}\left(\ell_3 - \frac{3a-1}{2}\right) = \ell_3, \qquad \frac{1}{a}\left(r_3 + \frac{3a-1}{2}\right) = r_3,$$

so that $\ell_3 = -\dfrac{3a-1}{2(a-1)}$ and $r_3 = \dfrac{3a-1}{2(a-1)}$. Meanwhile, if $\operatorname{supp} {}^a\phi_5 = [\ell_5, r_5]$, completely analogous process leads to $\ell_5 = -r_5 = -\dfrac{5a-1}{2(a-1)}$.

TABLE I

WEIGHTS OF a-ARY 3-POINT SUBDIVISION SCHEME

	$\lambda_{k-1}^{(n)}$	$\lambda_k^{(n)}$	$\lambda_{k+1}^{(n)}$
$\lambda_{ak-(a-1)/2}^{(n+1)}$	${}^a_3p_{(a+1)/2}$	${}^a_3p_{-(a-1)/2}$	${}^a_3p_{-(3a-1)/2}$
$\lambda_{ak-(a-3)/2}^{(n+1)}$	${}^a_3p_{(a+3)/2}$	${}^a_3p_{-(a-3)/2}$	${}^a_3p_{-(3a-3)/2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\lambda_{ak-1}^{(n+1)}$	${}^a_3p_{a-1}$	${}^a_3p_{-1}$	${}^a_3p_{-a-1}$
$\lambda_{ak}^{(n+1)}$		1	
$\lambda_{ak+1}^{(n+1)}$	${}^a_3p_{a+1}$	${}^a_3p_{1}$	${}^a_3p_{-a+1}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\lambda_{ak+(a-3)/2}^{(n+1)}$	${}^a_3p_{(3a-3)/2}$	${}^a_3p_{(a-3)/2}$	${}^a_3p_{-(a+3)/2}$
$\lambda_{ak+(a-1)/2}^{(n+1)}$	${}^a_3p_{(3a-1)/2}$	${}^a_3p_{(a-1)/2}$	${}^a_3p_{-(a+1)/2}$

If we write ${}^{a}P_3$ in (1) and ${}^{a}P_5$ in (2) by

$$
{}^{a}P_3(z) = \frac{1}{a} \sum_{k=-3a+1}^{3a-1} {}^{a}_{3}p_k z^k, \qquad {}^{a}P_5(z) = \frac{1}{a} \sum_{k=-5a+1}^{5a-1} {}^{a}_{5}p_k z^k,
$$

the a-ary 3- and 5-point interpolatory subdivision schemes for curve design can be given by Table I and Table II, i.e., the 3-point scheme is given by

$$
\lambda^{(n+1)}_{ak+\ell} = \sum_{j=-1}^{1} {}^{a}_{3}p_{-aj+\ell} \lambda^{(n)}_{k+j}, \quad \ell = -(a-1)/2, \ldots, (a-1)/2; \quad n \in \mathbb{Z}_+, \tag{3}
$$

while the 5-point a-ary scheme is given by

$$
\lambda^{(n+1)}_{ak+\ell} = \sum_{j=-2}^{2} {}^{a}_{5}p_{-aj+\ell} \lambda^{(n)}_{k+j}, \quad \ell = -(a-1)/2, \ldots, (a-1)/2, \quad n \in \mathbb{Z}_+. \tag{4}
$$

The two-scale sequences $\{{}^{a}_{3}p_k\}_{k\in\mathbb{Z}}$ and $\{{}^{a}_{5}p_k\}_{k\in\mathbb{Z}}$ are listed explicitly in the following,

$$
{}^{a}_{3}p_{-k} = {}^{a}_{3}p_k = \frac{1}{a^2}(a+k)(a-k), \quad k = 0, \ldots, (a-1)/2; \tag{5}
$$

$$
{}^{a}_{3}p_{-k} = {}^{a}_{3}p_k = \frac{1}{2a^2}(a-k)(2a-k), \quad k = (a+1)/2, \ldots, (3a-1)/2; \tag{6}
$$

$$
{}^{a}_{3}p_k = 0, \quad |k| \geq (3a-1)/2, \tag{7}
$$

and

$$
{}^{a}_{5}p_{-k} = {}^{a}_{5}p_k = \frac{1}{4a^4}(a-k)(a+k)(2a-k)(2a+k), \quad k = 0, \ldots, (a-1)/2; \tag{8}
$$

$$
{}^{a}_{5}p_{-k} = {}^{a}_{5}p_k = \frac{1}{6a^4}(a-k)(a+k)(2a-k)(3a-k), \quad k = (a+1)/2, \ldots, (3a-1)/2; \tag{9}
$$

$$
{}^{a}_{5}p_{-k} = {}^{a}_{5}p_k = \frac{1}{24a^4}(a-k)(2a-k)(3a-k)(4a-k), \quad k = (3a+1)/2, \ldots, (5a-1)/2; \tag{10}
$$

$$
{}^{a}_{5}p_k = 0, \quad |k| \geq (5a-1)/2. \tag{11}
$$

The interpolatory property of both schemes in (3) and (4) follows from (5)–(7) and (8)–(11).

More explicitly, it follows from (3) and (5)–(7) that the 3-point a-ary interpolatory subdivision scheme is given by

$$
\lambda^{(n+1)}_{ak-(a+1)/2+\ell} = \frac{(a+1-2\ell)(3a+1-2\ell)}{8a^2} \lambda^{(n)}_{k-1} + \frac{(a-1+2\ell)(3a+1-2\ell)}{4a^2} \lambda^{(n)}_{k} - \frac{(a+1-2\ell)(a-1+2\ell)}{8a^2} \lambda^{(n)}_{k+1}, \qquad \ell = 1, \ldots, (a-1)/2; \tag{12}
$$

$$
\lambda^{(n+1)}_{ak} = \lambda^{(n)}_{k}, \tag{13}
$$

$$
\lambda^{(n+1)}_{ak+\ell} = -\frac{\ell(a-\ell)}{2a^2} \lambda^{(n)}_{k-1} + \frac{(a-\ell)(a+\ell)}{a^2} \lambda^{(n)}_{k} + \frac{\ell(a+\ell)}{2a^2} \lambda^{(n)}_{k+1}, \qquad \ell = 1, \ldots, (a-1)/2. \tag{14}
$$

TABLE II

WEIGHTS OF a-ARY 5-POINT SUBDIVISION SCHEME

	$\lambda_{k-2}^{(n)}$	$\lambda_{k-1}^{(n)}$	$\lambda_{k}^{(n)}$	$\lambda_{k+1}^{(n)}$	$\lambda_{k+2}^{(n)}$
$\lambda_{ak-(a-1)/2}^{(n+1)}$	${}_5^a p_{(3a+1)/2}$	${}_5^a p_{(a+1)/2}$	${}_5^a p_{-(a-1)/2}$	${}_5^a p_{-(3a-1)/2}$	${}_5^a p_{-(5a-1)/2}$
$\lambda_{ak-(a-3)/2}^{(n+1)}$	${}_5^a p_{(3a+3)/2}$	${}_5^a p_{(a+3)/2}$	${}_5^a p_{-(a-3)/2}$	${}_5^a p_{-(3a-3)/2}$	${}_5^a p_{-(5a-3)/2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\lambda_{ak-1}^{(n+1)}$	${}_5^a p_{2a-1}$	${}_5^a p_{a-1}$	${}_5^a p_{-1}$	${}_5^a p_{-a-1}$	${}_5^a p_{-2a-1}$
$\lambda_{ak}^{(n+1)}$			1		
$\lambda_{ak+1}^{(n+1)}$	${}_5^a p_{2a+1}$	${}_5^a p_{a+1}$	${}_5^a p_{1}$	${}_5^a p_{-a+1}$	${}_5^a p_{-2a+1}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\lambda_{ak+(a-3)/2}^{(n+1)}$	${}_5^a p_{(5a-3)/2}$	${}_5^a p_{(3a-3)/2}$	${}_5^a p_{(a-3)/2}$	${}_5^a p_{-(a+3)/2}$	${}_5^a p_{-(3a+3)/2}$
$\lambda_{ak+(a-1)/2}^{(n+1)}$	${}_5^a p_{(5a-1)/2}$	${}_5^a p_{(3a-1)/2}$	${}_5^a p_{(a-1)/2}$	${}_5^a p_{-(a+1)/2}$	${}_5^a p_{-(3a+1)/2}$

Similarly, it is clear from (4) and (8)–(11) that the 5-point a-ary interpolatory subdivision scheme is explicitly given by

$$\begin{aligned}\lambda_{ak-(a+1)/2+\ell}^{(n+1)} = &-\frac{(a-1+2\ell)(a+1-2\ell)(3a+1-2\ell)(5a+1-2\ell)}{384a^4}\lambda_{k-2}^{(n)}\\ &+\frac{(a+1-2\ell)(3a-1+2\ell)(3a+1-2\ell)(5a+1-2\ell)}{96a^4}\lambda_{k-1}^{(n)}\\ &+\frac{(a-1+2\ell)(3a-1+2\ell)(3a+1-2\ell)(5a+1-2\ell)}{64a^4}\lambda_{k}^{(n)}\\ &-\frac{(a+1-2\ell)(a-1+2\ell)(3a-1+2\ell)(5a+1-2\ell)}{96a^4}\lambda_{k+1}^{(n)}\\ &+\frac{(a+1-2\ell)(a-1+2\ell)(3a+1-2\ell)(3a-1+2\ell)}{384a^4}\lambda_{k+2}^{(n)},\\ &\qquad \ell = 1,\ldots,(a-1)/2;\end{aligned} \tag{15}$$

$$\lambda_{ak}^{(n+1)} = \lambda_{k}^{(n)}; \tag{16}$$

$$\begin{aligned}\lambda_{ak+\ell}^{(n+1)} = &\frac{\ell(a-\ell)(a+\ell)(2a-\ell)}{24a^4}\lambda_{k-2}^{(n)} - \frac{\ell(a-\ell)(2a-\ell)(2a+\ell)}{6a^4}\lambda_{k-1}^{(n)}\\ &+\frac{(a-\ell)(a+\ell)(2a-\ell)(2a+\ell)}{4a^4}\lambda_{k}^{(n)} + \frac{\ell(a+\ell)(2a-\ell)(2a+\ell)}{6a^4}\lambda_{k+1}^{(n)}\\ &-\frac{\ell(a-\ell)(a+\ell)(2a+\ell)}{24a^4}\lambda_{k+2}^{(n)}, \qquad \ell = 1,\ldots,(a-1)/2.\end{aligned} \tag{17}$$

We end this section by pointing out that the graphs of ${}^3\phi_3$ and ${}^5\phi_3$ in Fig. 1 and the graphs of ${}^3\phi_5$ and ${}^5\phi_5$ in Fig. 2 can also be obtained by the two subdivision schemes (12)–(14) and (15)–(17) with the initial sequence $\lambda_k^{(0)} = \delta_{k,0}, k \in \mathbb{Z}$.

3. Proofs of Main Results

Proof of Theorem 1.

First, an a-ary 3-point scheme needs at most $3a$ weights, i.e., the two-scale sequence $\{{}_3^a p_k\}_{k\in\mathbb{Z}}$ of ${}^a\phi_3$ has at most $3a$ consecutive nontrivial entries. Secondly, for ${}^a\phi_3$ to have the highest possible m of $\mathbb{PP}_m$, its two-scale symbol ${}^aP_3$ has to have the highest possible order of factor of $(1+z+\cdots+z^{a-1})$. This leads to both $m=3$ and ${}^aP_3$ must have the form

$$ {}^aP_3(z) = z^{(1-3a)/2}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^3\left(s_0+s_1z+s_2z^2\right) $$

for some constant s_0, s_1, and s_2 satisfying $s_2=s_0$ and $s_0+s_1+s_2=1$. By using $(1-z)^{-3} = \sum_{\ell=0}^{\infty}\binom{2+\ell}{2}z^\ell$ we have

$$ \begin{aligned} &\left(s_0+s_1z+s_2z^2\right)(1-z)^{-3} = \sum_{\ell=0}^{\infty}\mu_\ell z^\ell, \quad \text{where} \\ &\mu_\ell = \binom{\ell+2}{2}s_0+\binom{\ell+1}{2}s_1+\binom{\ell}{2}s_2, \quad \ell\in\mathbb{Z}_+. \end{aligned} \tag{18} $$

Hence, by defining $\mu_\ell=0$ for all $\ell<0$ and multiplying by the expansion of $(1-z^a)^3$ we obtain the explicit expressions for $\{{}_3^a p_k\}_{k\in\mathbb{Z}}$ in terms of $\{\mu_\ell\}$, namely,

$$ \begin{aligned} {}_3^a p_k &= \frac{1}{a^2}(\mu_{(3a-1)/2+k}-3\mu_{(a-1)/2+k}+3\mu_{-(a+1)/2+k}-\mu_{-(3a+1)/2+k}), \\ &k=-(3a-1)/2,\ldots,(3a-1)/2. \end{aligned} \tag{19} $$

Next, the three identities ${}_3^a p_{-a}=0, {}_3^a p_0=1$, and ${}_3^a p_a=0$, lead to

$$ \begin{aligned} &\frac{1}{a^2}\mu_{(a-1)/2}=0, \\ &\frac{1}{a^2}(\mu_{(3a-1)/2}-3\mu_{(a-1)/2})=1, \\ &\frac{1}{a^2}(\mu_{(5a-1)/2}-3\mu_{(3a-1)/2}+3\mu_{(a-1)/2})=0, \end{aligned} $$

or simply $\mu_{(a-1)/2}=0, \mu_{(3a-1)/2}=a^2, \mu_{(5a-1)/2}=3a^2$, or, equivalently,

$$ \begin{aligned} &\binom{\frac{a+3}{2}}{2}s_0+\binom{\frac{a+1}{2}}{2}s_1+\binom{\frac{a-1}{2}}{2}s_2=0, \\ &\binom{\frac{3a+3}{2}}{2}s_0+\binom{\frac{3a+1}{2}}{2}s_1+\binom{\frac{3a-1}{2}}{2}s_2=a^2, \\ &\binom{\frac{5a+3}{2}}{2}s_0+\binom{\frac{5a+1}{2}}{2}s_1+\binom{\frac{5a-1}{2}}{2}s_2=3a^2. \end{aligned} $$

By solving this linear system, s_0, s_1, and s_2 are given by

$$ s_0=s_2=\frac{1-a^2}{8}, \quad s_1=\frac{a^2+3}{3}, $$

as they were in (1). Substituting s_0, s_1, and s_2 into (18) leads to

$$\mu_\ell = -\frac{1}{8}(a^2 - (2\ell+1)^2), \qquad \ell \in \mathbb{Z}_+.$$

Finally, by substituting μ_ℓ's into (19) we arrive at the explicit expressions for ${}^a_3p_k$'s in (5)–(7). This completes the proof of Theorem 1. $\square$

Proof of Theorem 2.

Similar to the proof of Theorem 1, the two-scale symbol ${}^aP_5$ of ${}^a\phi_5$ must have the form

$${}^aP_5(z) = z^{(1-5a)/2}\left(\frac{1}{a}\frac{1-z^a}{1-z}\right)^5 \left(s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4\right)$$

for some constants $s_0, \ldots, s_4$ satisfying $s_4 = s_0, s_3 = s_1$, and $s_0 + \cdots + s_4 = 1$. First, multiply $s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4$ and $(1-z)^{-5} = \sum_{\ell=0}^{\infty} \binom{4+\ell}{4} z^\ell$ to get

$$\begin{aligned} &\left(s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4\right)(1-z)^{-5} = \sum_{\ell=0}^{\infty} \nu_\ell z^\ell, \quad \text{where} \\ &\nu_\ell = \binom{\ell+4}{4}s_0 + \binom{\ell+3}{4}s_1 + \binom{\ell+2}{4}s_2 + \binom{\ell+1}{4}s_3 + \binom{\ell}{4}s_4, \quad \ell \in \mathbb{Z}_+. \end{aligned} \tag{20}$$

Secondly, multiply by the expansion of $(1-z^a)^5$, $\{{}^a_5p_k\}_{k\in\mathbb{Z}}$ can be expressed in terms of $\{\nu_\ell\}$ in (20). Then, with $\nu_\ell = 0$ for all $\ell < 0$, all coefficients of ${}^aP_5(z)$ are now in terms of $s_0, \ldots, s_4$, namely,

$$\begin{aligned} {}^a_5p_k = \frac{1}{a^4}(&\nu_{(5a-1)/2+k} - 5\nu_{(3a-1)/2+k} + 10\nu_{(a-1)/2+k} - 10\nu_{-(a+1)/2+k} \\ &+5\nu_{-(3a+1)/2+k} - \nu_{-(5+2)/2+k}), \qquad |k| \le (5a-1)/2. \end{aligned} \tag{21}$$

The five requirements

$${}^a_5p_{-2a} = {}^a_5p_{-a} = 0, \qquad {}^a_5p_0 = 1, \qquad {}^a_5p_a = {}^a_5p_{2a} = 0$$

yield

$$\begin{aligned} &\nu_{(a-1)/2} = 0, \\ &\nu_{(3a-1)/2} - 5\nu_{(a-1)/2} = 0, \\ &\nu_{(5a-1)/2} - 5\nu_{(3a-1)/2} + 10\nu_{(a-1)/2} = a^5, \\ &\nu_{(7a-1)/2} - 5\nu_{(5a-1)/2} + 10\nu_{(3a-1)/2} - 10\nu_{(a-1)/2} = 0, \\ &\nu_{(9a-1)/2} - 5\nu_{(7a-1)/2} + 10\nu_{(5a-1)/2} - 10\nu_{(3a-1)/2} + 5\nu_{(a-1)/2} = 0, \end{aligned}$$

which is equivalent to

$$
\begin{aligned}
&\binom{\frac{a+7}{2}}{4}s_0+\binom{\frac{a+5}{2}}{4}s_1+\binom{\frac{a+3}{2}}{4}s_2+\binom{\frac{a+1}{2}}{4}s_3+\binom{\frac{a-1}{2}}{4}s_4=0,\\
&\binom{\frac{3a+7}{2}}{4}s_0+\binom{\frac{3a+5}{2}}{4}s_1+\binom{\frac{3a+3}{2}}{4}s_2+\binom{\frac{3a+1}{2}}{4}s_3+\binom{\frac{3a-1}{2}}{4}s_4=0,\\
&\binom{\frac{5a+7}{2}}{4}s_0+\binom{\frac{5a+5}{2}}{4}s_1+\binom{\frac{5a+3}{2}}{4}s_2+\binom{\frac{5a+1}{2}}{4}s_3+\binom{\frac{5a-1}{2}}{4}s_4=a^4,\\
&\binom{\frac{7a+7}{2}}{4}s_0+\binom{\frac{7a+5}{2}}{4}s_1+\binom{\frac{7a+3}{2}}{4}s_2+\binom{\frac{7a+1}{2}}{4}s_3+\binom{\frac{7a-1}{2}}{4}s_4=5a^4,\\
&\binom{\frac{9a+7}{2}}{4}s_0+\binom{\frac{9a+5}{2}}{4}s_1+\binom{\frac{9a+3}{2}}{4}s_2+\binom{\frac{9a+1}{2}}{4}s_3+\binom{\frac{9a-1}{2}}{4}s_4=15a^4.
\end{aligned}
$$

Solving this linear system we have $s_0,\dots,s_4$ in (2), i.e.,

$$
\begin{aligned}
s_0&=s_4=\frac{1}{384}(a^2-1)(9a^2-1),\\
s_1&=s_3=-\frac{1}{96}(a^2-1)(9a^2+19),\\
s_2&=\frac{1}{192}(115+50a^2+27a^4).
\end{aligned}
$$

Substitute $s_0,\dots,s_4$ into (20) to get

$$
\nu_\ell=\frac{1}{384}(a^2-(2\ell+1)^2)(9a^2-(2\ell+1)^2),\qquad \ell\in\mathbb{Z}_+.
$$

Then ${}_5^a p_k$'s in (8)–(11) subsequently follow. This completes the proof of Theorem 2. $\square$

Fig. 3. The geometric illustration of the 3-point ternary subdivision scheme in (22).

4. Applications to Curve Design

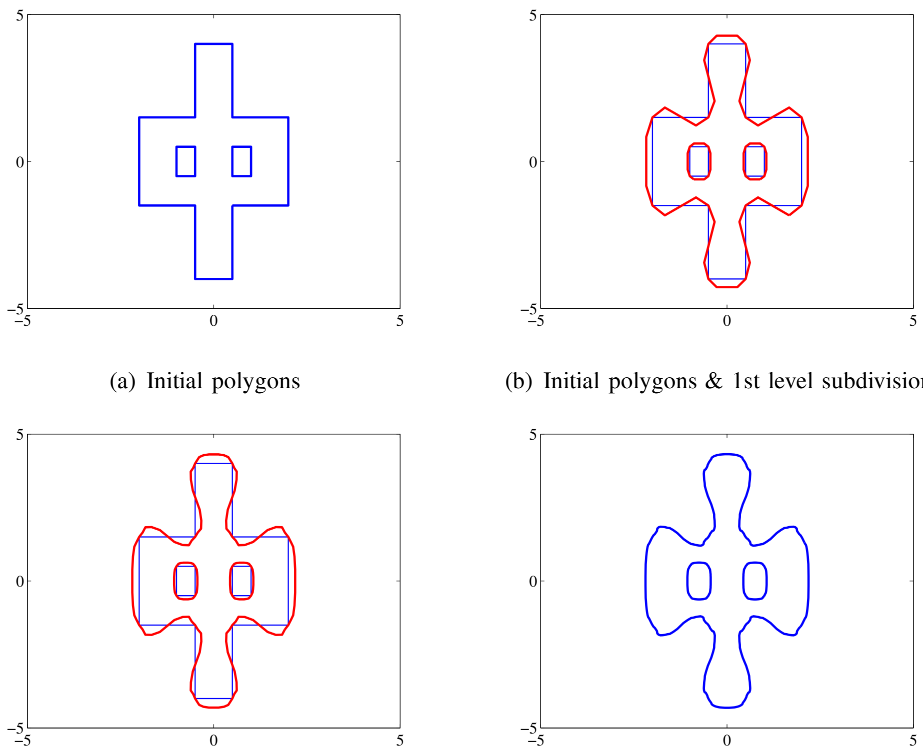
With $a=3$, it follows either from Table I and (5)–(7) or directly from (12)–(14) that the 3-point ternary interpolatory subdivision scheme is

$$\begin{aligned} \lambda_{3k-1}^{(n+1)} &= \frac{2}{9}\lambda_{k-1}^{(n)} + \frac{8}{9}\lambda_k^{(n)} - \frac{1}{9}\lambda_{k+1}^{(n)}, \\ \lambda_{3k}^{(n+1)} &= \lambda_k^{(n)}, \\ \lambda_{3k+1}^{(n+1)} &= -\frac{1}{9}\lambda_{k-1}^{(n)} + \frac{8}{9}\lambda_k^{(n)} + \frac{2}{9}\lambda_{k+1}^{(n)}, \qquad k \in \mathbb{Z}_+. \end{aligned} \tag{22}$$

By observing from (22) that

$$\begin{aligned} \lambda_{3k-1}^{(n+1)} &= \frac{2}{9}\lambda_{k-1}^{(n)} + \frac{7}{9}\lambda_k^{(n)} + \frac{1}{9}\left(\lambda_k^{(n)} - \lambda_{k+1}^{(n)}\right), \\ \lambda_{3k+1}^{(n+1)} &= \frac{7}{9}\lambda_k^{(n)} + \frac{2}{9}\lambda_{k+1}^{(n)} + \frac{1}{9}\left(\lambda_k^{(n)} - \lambda_{k-1}^{(n)}\right), \qquad k \in \mathbb{Z}_+. \end{aligned}$$

the 3-point ternary scheme has a clear geometric interpretation as illustrated by Fig. 3. We also point out that the ternary scheme (22) was also studied in (Hassan & Dodgson [3]) by using the method of "generating function formalism."



(a) Initial polygons

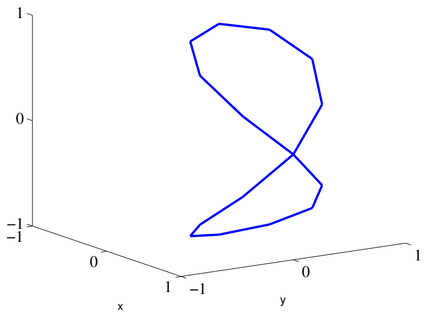
(b) Initial polygons & 1st level subdivision

(c) Initial polygons & 2nd level subdivision

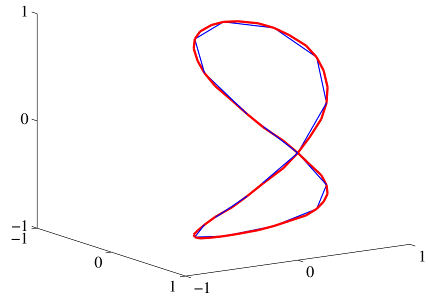
(d) Result after 4th subdivision

Fig. 4. Three planar polygons with 12, 4, and 4 initial control points.

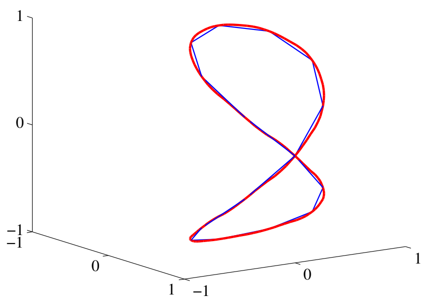
While when $a = 3$, it follows either from Table II together with (8)–(11) or directly from (15)–



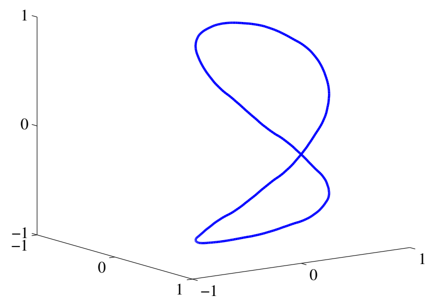
(a) Initial 3D polygon



(b) Initial polygon & 1st level subdivision



(c) Initial polygon & 2nd level subdivision



(d) Result after 4th subdivision

Fig. 5. A space curve with 16 initial control points selected from the Viviani's curve in (24).

(17) that the 5-point ternary interpolatory subdivision scheme is

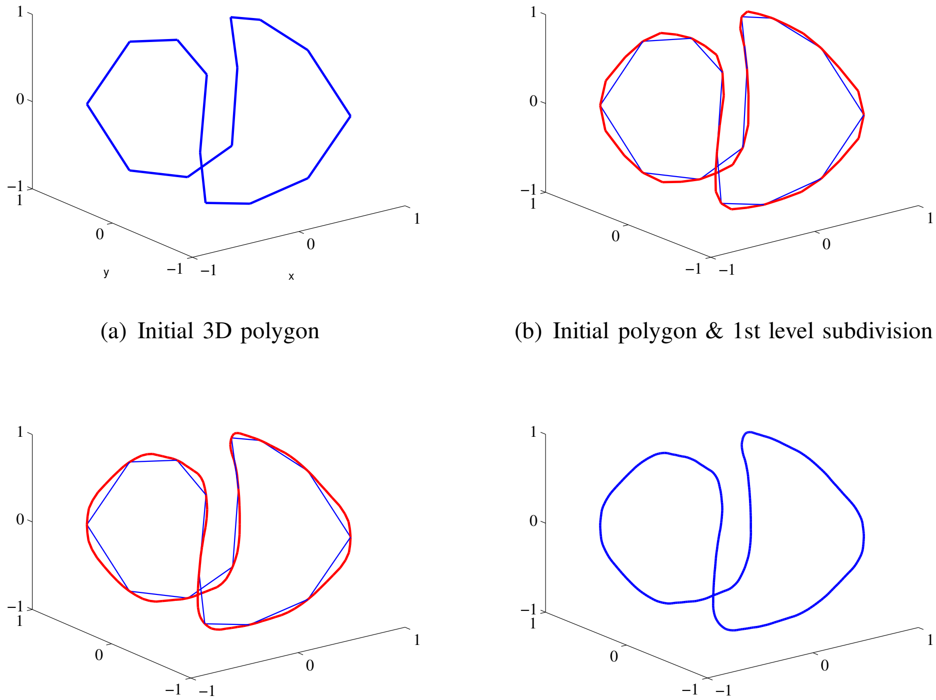
$$\begin{aligned}
\lambda_{3k-1}^{(n+1)} &= -\frac{7}{243}\lambda_{k-2}^{(n)} + \frac{70}{243}\lambda_{k-1}^{(n)} + \frac{70}{81}\lambda_{k}^{(n)} - \frac{35}{243}\lambda_{k+1}^{(n)} + \frac{5}{243}\lambda_{k+2}^{(n)}, \\
\lambda_{3k}^{(n+1)} &= \lambda_{k}^{(n)}, \\
\lambda_{3k+1}^{(n+1)} &= \frac{5}{243}\lambda_{k-2}^{(n)} - \frac{35}{243}\lambda_{k-1}^{(n)} + \frac{70}{81}\lambda_{k}^{(n)} + \frac{70}{243}\lambda_{k+1}^{(n)} - \frac{7}{243}\lambda_{k+2}^{(n)}, \qquad k \in \mathbb{Z}_+.
\end{aligned} \tag{23}$$

To demonstrate the elegance of all these schemes, we apply the 3-point ternary scheme (22) to the 3 closed 2D polygons in Fig. 4(a).

The space polygon in Fig. 5(a) was formed by eight initial control points, selected from the Viviani's curve (Gray [2], p. 201), which is the intersection between a sphere and a right circular cylinder passing through the center of the sphere whose diameter is half of the sphere. Its parametric equation is given by

$$x(t) = \frac{r}{2}(1+\cos 2t), \quad y(t) = \frac{r}{2}\sin 2t, \quad z(t) = -r\sin t, \qquad t \in [0, 2\pi], \tag{24}$$

with r the radius of the sphere. By applying the 3-point ternary scheme (22), the resulting "polygon" after 4^{th} subdivision is shown in Fig. 5(d) .



(a) Initial 3D polygon

(b) Initial polygon & 1st level subdivision

(c) Initial polygon & 2nd level subdivision

(d) Result after 4th subdivision

Fig. 6. A closed space curve with 16 initial control points selected from the baseball's seam curve in (25).

A family of curves of a baseball's seam can be given by the following parametric equation

$$\begin{aligned} x(t) &= r\cos\left(\left(\frac{\pi}{2}-b\right)\cos 2t\right)\cos\left(t+b\sin 4t\right),\\ y(t) &= r\cos\left(\left(\frac{\pi}{2}-b\right)\cos 2t\right)\sin\left(t+b\sin 4t\right),\\ z(t) &= r\sin\left(\left(\frac{\pi}{2}-b\right)\cos 2t\right), \qquad t\in[0,2\pi], \end{aligned} \tag{25}$$

where r is the radius of the baseball, and b is a constant. With the choice of $b = 0.4$, we select 16 points on this curve as shown in Fig. 6(a). We apply the 5-point ternary scheme (23) to get the 3D "curve" in Fig. 6(d) after 4^{th} subdivision.

5. Conclusion

The 3- and 5-point a-ary interpolatery subdivision schemes for curve design were established for any odd integer $a \geq 3$. The polynomial preservation orders of the scaling functions corresponding to these schemes are fixed, namely, either 3 or 5, which is independent of a. The smoothness of the corresponding scaling functions for various values of $a \geq 3$ are needed to and will be studied in detail in the forthcoming paper.

Acknowledgment

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References

[1] Dyn, N., D. Levin, and J. Gregory (1987). A 4-point interpolatory subdivision scheme for curve design, *Computer Aided Geometric Design* **4**, 257-268.

[2] Gray, A. (1997). *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed., CRC Press, Boca Raton, FL.

[3] Hassan, M. F. and N. A. Dodgson (2002). Ternary and three-point univariate subdivision schemes, in: *Curve and Surface Fitting: Saint-Malo 2002*, A. Cohen, J.-L. Merrien, and L. L. Schumaker (eds.), Nashboro Press, Brentwood, TN, 199–208.

[4] Lian, J.-A. (2008). On a-ary subdivision for curve design: I. 4-point and 6-point interpolatory schemes, *Application & Applied Math.* **3**, 18–29.

[5] Weissman, A. (1989). A 6-point interpolatory subdivision scheme for curve design, Masters Thesis, Tel-Aviv University.