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Wavelet Transform of Fractional Integrals for Integrable Boehmians

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Abstract

The present paper deals with the wavelet transform of fractional integral operator (the Riemann-Liouville operators) on Boehmian spaces. By virtue of the existing relation between the wavelet transform and the Fourier transform, we obtained integrable Boehmians defined on the Boehmian space for the wavelet transform of fractional integrals.

Keywords: Wavelet transform, Fourier transform, Riemann-Liouville fractional integral operators, distribution spaces, Boehmian, Integrable Boehmians

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1. Introduction

Wavelet is a new area that stands at the intersection of frontiers of mathematics, scientific computing and signals and image processing. It has been one of the major research directions in science in the last decade and is still undergoing rapid growth. Some group of mathematician view it as a new basis for representing function; some consider it as a technique for time frequency analysis. Wavelet analysis is an abstract branch of mathematics that is originated as a lack in Fourier analysis. In order to eliminate the weakness of finding the frequency spectrum of a signal locally in time, Gabor (1946) first introduced the Windowed – Fourier transform (or short – time Fourier transform) or Gabor transform by using a Gaussian distribution function as the window function Gabor (1946). The concept of *wavelets* or *ondelettes* started to appear in the literature only in early in 1980s.

Morlet et al. (1982) introduced the idea of wavelet transform as a new tool for seismic signal analysis. Grossman broadly defined wavelets in the context of quantum physics. Then by the joint venture of mathematical group in Marseilles, led by Grossman, in collaboration with Daubechies, Paul and others, extended Morlet discrete version of wavelet transform to the continuous version by relating it to the theory of coherent states in quantum physics. See Grossman and Morlet (1984), and Daubechies (1992, 1998a, 1998b).

Meyer learnt about the work of Morlet and Marseilles group and applied the Little-wood Paley theory to the study of wavelet decomposition Meyer (1986), where he also explains the construction of wavelets and the application of wavelet series representation to the analysis for function spaces such as Hölder, Hardy, Besov and studied the notion of holomorphic wavelets. Creditable contributions to the wavelet theory is made by many authors Chui (1992), Daubechies (1992, 1998a), Grossman and Morlet (1984), Janseen (1981a, 1981b), Mikusiński (1983, 1988).

The Gabor transform (i.e., the windowed Fourier transform) of *f* with respect to *g* [cf. Debnath (1998, p.688)] is

$$\mathrm{G}[f](\upsilon,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\overline{g}(\tau - t)\, e^{i\upsilon t} d\tau \tag{1}$$

$$= \frac{1}{\sqrt{2\pi}} (f, \overline{g_{\upsilon,t}}), \tag{2}$$

where $f, g \in L^2(R)$ with the inner product (f, g). For a fixed *t*,

$$\mathrm{G}|f|(\omega,t) = \widetilde{f}_g(\upsilon,t) = \mathrm{F}\{f_t(\tau)\} = \hat{f}_t(\upsilon), \tag{3}$$

where F is the Fourier transform and G is the Gabor transform. *Parseval formula* for Gabor transform is given by

$$\left(\tilde{f},\tilde{g}\right)=\|g\|^{2}\left(f,h\right), \tag{4}$$

whereas the inversion formula is

$$f(\tau)=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\|g\|^{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{g}(\upsilon,t))\overline{g}(\tau-t)\,e^{-i\upsilon t}d\upsilon dt. \tag{5}$$

Basic properties of the Gabor transform are linearity, translation, modulation, and conjugation. The theory of Gabor transform has been generalized by Janssen (1981a, 1981b) for tempered distributions S'. The treatment of wavelet transform with the Schwartz distribution was explained by authors Pandey (1999), Pathak (1998), Walter (1992), Walter (1993), Walter (1994), among others.

In Banerji et al. (2004), authors investigate the Gabor transform for integrable Boehmians. Definition and terminologies, relevant to present work and the convergence for the Boehmian space are explained later.

The fractional calculus (fractional integrals and derivatives, also called fractional differintegrals) have several applications in integral transforms and distribution spaces, which are called fractional transforms, see for instance, Loonker and Banerji (2007).

In the present work, using relation between the Fourier and the Wavelet transform, we have obtained the Gabor transform for fractional integral operator (Riemann-Liouville type) which is further proved for integrable Boehmians.

Definition 1.

Samko et al. (1993, p. 33)**:** Let $\varphi(x)\in L_{1}(a,b)$. Then the integrals

$$(I_{a+}^{\alpha}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\varphi(t)dt,\ x>a, \tag{6}$$

$$(I_{b-}^{\alpha}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\varphi(t)dt,\ x<b, \tag{7}$$

where $\alpha>0$, are Riemann-Liouville fractional integrals of order α. They are also known as *left-sided* and *right-sided fractional integrals*, respectively. Indeed, these integrals are extensions from the case of a finite interval $[a,b]$ to the case of a half-axis, given by

$$(I_{0+}^{\alpha}\phi)(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}\phi(t)dt, \quad 0<x<\infty, \tag{8}$$

while for the whole axis, it is given, respectively, by Samko et al. (1993, p. 94) as

$$(I_{+}^{\alpha}\phi)(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}\phi(t)dt, \quad -\infty<x<\infty, \tag{9}$$

and

$$(I_{-}^{\alpha}\phi)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{-\infty}(t-x)^{\alpha-1}\phi(t)dt, \quad -\infty<x<\infty. \tag{10}$$

The *convolution* of formulae (9) and (10) is

$$(I_{\pm}^{\alpha}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\infty}t_{\pm}^{\alpha-1}\varphi(x-t)dt,$$

$$=\frac{1}{\Gamma(\alpha)}\int_{0}^{\infty}t^{\alpha-1}\varphi(x-t)dt, \tag{11}$$

where

$$t_{+}^{\alpha-1}=\begin{cases} t^{\alpha-1}, & t>0, \\ 0, & t<0, \end{cases} \tag{12}$$

and

$$t_{-}^{\alpha-1}=\begin{cases} 0, & t>0, \\ |t|^{\alpha-1}, & t<0. \end{cases} \tag{13}$$

Fractional integral for function $\varphi\in L_p(-\infty,\infty)$, $0<\alpha<1$ and $1\le p\le 1/\alpha$, is given by

$$(I_{+}^{\alpha}\varphi)(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{1}t^{\alpha-1}\varphi(x-t)dt+\frac{1}{\Gamma(\alpha)}\int_{1}^{\infty}t^{\alpha-1}\varphi(x-t)dt. \tag{14}$$

There are two ways to define the fractional integrals and derivatives of generalized functions Samko et al. (1993, p. 146). The *first* is based on the definition of a fractional integral operator as a convolution

$$\frac{1}{\Gamma(\alpha)}(x_{\pm})^{\alpha-1} * f , \tag{15}$$

of the function $\frac{1}{\Gamma(\alpha)} x_{\pm}^{\alpha-1}$, with the generalized function *f*. The *second* is by virtue of the use of adjoint operators. By employing fractional integration by parts, formulae (6) and (7) assume the form

$$(I_{a+}^{\alpha} f)(\varphi) = (f , I_{b-}^{\alpha})(\varphi) \ . \tag{16}$$

The function *f*, in (16) may, indeed, be defined as the generalized function if I_{b-}^{α} maps continuously the space of test functions X into itself. When f and $I_{a+}^{\alpha}(f)$ are considered to be generalized functions on different spaces of test function X and Y such that $f \in X'$ (the dual of the test function space *X)* and $I_{a+}^{\alpha}(f) \in Y'$ (the dual of the test function space *Y)*, I_{b-}^{α} maps Y into X continuously.

The fractional integration $I_{\pm}^{\alpha}$ of a generalized function $f \in \Phi'$ (the dual of Φ) is given by

$$(I_{\pm}^{\alpha} f , \varphi) = (f , I_{\mp}^{\alpha} , \varphi), \quad \varphi \in \Phi \ . \tag{17}$$

Indeed, using (17), the Fourier transform is given by

$$(f , I_{\pm}^{\alpha} \varphi) = (\widetilde{f} , \widehat{I_{\mp}^{\alpha} \varphi}) = (\widetilde{f} , (\pm ix)^{-\alpha} \, \hat{\varphi}(x)), \tag{18}$$

which is derived by virtue of the notions of convolution prescribed in (16). The Fourier transform of the fractional integrals $I_{\pm}^{\alpha} \varphi$ are [Samko et al. (1993, p. 147)]

$$\mathsf{F}(I_{\pm}^{\alpha} \varphi) = (\mp ix)^{-\alpha} \, \hat{\varphi}(x), \quad \varphi \in L_1(a,b). \tag{19}$$

Study of *regular operators* of Mikusiński by Boehme (1973) resulted into the theory of Boehmians, the generalization of Schwartz distribution theory. These regular operators form subalgebra of Mikusiński operators such that they include only such functions whose support is bounded from the left, and at the same time do not have any restriction on the support. The general construction of Boehmians gives rise to various function spaces, which are known as **Boehmian spaces** [cf. Mikusiński and Mikusiński (1981) and Mikusiński (1983, 1988)]. It is observed that these spaces contain all Schwartz distributions, Roumieu ultradistributions and tempered distributions.

The name *Boehmian* is used for all objects by an algebraic construction, which is similar to the construction of the field of quotients. Suppose G is an additive commutative semigroup, S be a subset of group G such that $S \subseteq G$ is a sub semigroup, for which we define a mapping $*$ from $G \times S$ to G such that following conditions are satisfied (these condition are for the mapping $*$):

(i) if $\delta, \eta \in S$ then $(\delta * \eta) \in S$ and $\delta * \eta = \eta * \delta$

(ii) if $\alpha \in G$, $\delta, \eta \in S$ then $(\alpha * \delta) * \eta = \alpha * (\delta * \eta)$

(iii) if $\alpha, \beta \in G, \delta \in S$ then $(\alpha + \beta) * \delta = (\alpha * \delta) + (\beta * \delta)$.

The delta sequence, denoted by Δ, is defined as members of class delta which are the sequences of subset *S*, and satisfies the conditions

(i) if $\alpha, \beta \in G, (\delta_n) \in \Delta$ and $\alpha * \delta_n = \beta * \delta_n, \forall\ n$ then $\alpha = \beta$ in *G*.

(ii) if $(\delta_n), (\phi_n) \in \Delta$ then $(\delta_n * \phi_n) \in \Delta$.

Then the *quotient of sequences* is defined as the element of certain class *A* of pair of sequences defined by

$$A = \{(f_n), (\varphi_n) : (f_n) \subseteq G^N, (\varphi_n) \in \Delta\} .$$

This is denoted by f_n / φ_n such that

$$f_m * \varphi_n = f_n * \varphi_m \quad , \quad \forall m, n \in N .$$

Further, the quotients of sequences f_n / φ_n and g_n / ψ_n are called *equivalent* if

$$f_n * \psi_n = g_n * \varphi_n \quad , \quad \forall n \in N .$$

The equivalence relation defined on *A* and the equivalence classes of quotient of sequence are called *Boehmians***.**

The *space of all Boehmians***,** denoted by *B*, has the properties addition, multiplication and differentiation. The Boehmian space B_{L_1} will be called the space of *locally integrable Boehmians* if the group *G* be the set of all locally integrable function on *R* and possibly two such functions are identified with respect to Lebesgue measure (these functions are equal almost everywhere) and the topology of this space is taken to be the semi-norm topology generated by

$$p_n(f) = \int_{-n}^{n} | f | d\lambda, \quad n = 1, 2, 3, \cdots,$$

where λ is the usual Lebesgue measure on R and $D(R)$. In other words, if $f \in L_1$ and (δ_n) is the delta sequence, then $\|(f * \delta_n) - f\| \to 0$, as $n \to \infty$. A pair of sequences (f_n, φ_n) is called a quotient of sequences, and is denoted by f_n / φ_n, if $f_n \in L_1 (n = 1,2,\ldots)$, where (φ_n) is a delta sequence and $f_m * \varphi_n = f_n * \varphi_m, \forall m,n \in N$. Two quotients of sequences f_n / φ_n and g_n / ψ_n are equivalent if $f_n * \psi_n = g_n * \varphi_n, \forall n \in N$. The equivalence class of quotient of sequences will be called an *integrable Boehmain*, the space of all integrable Boehmian will be denoted by B_{L_1}. Convergence of Boehmians is defined in Mikusiński (1983). The terminologies regarding Boehmians and Boehmian spaces can be referred to in Mikusiński and Mikusiński (1981), Mikusiński (1983, 1988). We remark that present investigations are independent of the results given in Bargmann (1961, 1967).

2. Wavelet Transform of Fractional Integrals for Integrable Boehmians

Using the relation between the Gabor and the Fourier transform, relations (3) and (19), respectively, the fractional integrals for the Gabor transform, can be written in the form

$$\mathsf{F}(I_{\pm}^{\alpha} f_t(\tau)) = (\mp i\upsilon)^{-\alpha} \hat{f}_t(\upsilon), \quad f \in L_1(a,b). \tag{20}$$

In other words, (20) can be written as

$$\mathsf{G}(I_{\pm}^{\alpha} f) = (\mp i\upsilon)^{-\alpha} \hat{f}_t(\upsilon), \tag{21}$$

i.e.,

$$\begin{aligned} \mathsf{G}(I_{\pm}^{\alpha} f_n) &= (\mp i\upsilon)^{-\alpha} (\hat{f}_t(\upsilon))_n \\ &= (\mp i\upsilon)^{-\alpha} (\hat{f}_t)_n(\upsilon). \end{aligned} \tag{22}$$

Theorem 1:

If $[f_n / \delta_n] \in B_{L_1}$, then the sequence

$$\mathsf{G}(I_{\pm}^{\alpha} f_n) = (\mp i\upsilon)^{-\alpha} (\hat{f}_t)_n(\upsilon) \tag{23}$$

converges uniformly on each compact set in R.

Proof:

If (δ_n) is a delta sequence, then $(\hat{\delta}_t)_n$ converges uniformly on each compact set to the constant function unity. Therefore, $(\hat{\delta}_k) > 0$ on *K* (the compact set) and, thus, the left hand side of (23) gives

$$\mathsf{G}(I_{\pm}^{\alpha} f_n) = \frac{(I_{\pm}^{\alpha} \hat{f}_n)(\hat{\delta}_k)}{(\hat{\delta}_k)} = \frac{(I_{\pm}^{\alpha} f_n * \delta_k)^{\wedge}}{(\hat{\delta}_k)} = \frac{(I_{\pm}^{\alpha} \hat{f}_k)(\hat{\delta}_n)}{(\hat{\delta}_k)} \quad \text{on } K$$

$$= \frac{(\mp i\upsilon)^{-\alpha}(\hat{f}_t)_n(\hat{\delta}_n),}{(\hat{\delta}_k)} \qquad \text{[cf. Equation (22)].}$$

This shows that the Gabor transform of fractional integrals for an integrable Boehmian $F = [f_n / \delta_n]$ can be expressed as the limit of the sequence $\mathsf{G}(I_{\pm}^{\alpha} f_n)$, which, in fact, is the space of all continuous functions on *R*. This proves the theorem completely.

Property 1:

Let $[f_n / \delta_n] \in B_{L_1}$. Then, $\Delta - \lim_{n\to\infty} F_n = F$, $\mathsf{G}(I_{\pm}^{\alpha} F_n) \to \mathsf{G}(I_{\pm}^{\alpha} F)$ uniformly on each compact set.

Proof:

We have $\delta - \lim_{n\to\infty} F_n - F \Rightarrow \mathsf{G}\{F_n\} \to \mathsf{G}\{F\}$, uniformly on each compact set. The sequence can be expressed as $F_n * \delta_k, F * \delta_k \in L_1$, $\forall n, k \in N$, which has a norm

$$\left\| (F_n - F) * \delta_k \right\| \to 0, \text{ as } n \to \infty, \forall k \in N,$$

where *K* is well defined. Since $\mathsf{G}\{\delta_k\}$ is a continuous function, we have $\mathsf{G}\{\delta_k\} > 0$ on *K* for $k \in N$. It is, therefore, enough to prove that

$$\mathsf{G}\{F_n\} \cdot \mathsf{G}\{\delta_k\} \to \mathsf{G}\{F\} \cdot \mathsf{G}\{\delta_k\},$$

uniformly on *K*. We have,

$$\mathsf{G}\{F_n\} \cdot \mathsf{G}\{\delta_k\} - \mathsf{G}\{F\} \cdot \mathsf{G}\{\delta_k\} = \mathsf{G}\{(F_n - F) * \delta_k\},$$

such that $\|(F_n - F) * \delta_k\| \to 0$, as $n \to \infty$.

This justifies the existence and validity of the property.

3. Conclusions

The present paper focuses on the application of the Riemann Liouville type fractional integral operator to the Gabor transform and the integrable Boehmians. The fractional integral formula for the Gabor transform is given by using the relation between the Gabor and the Fourier transforms. The formula and the property established in this paper are suitable for certain Boehmian space for an integrable Boehmian. The compact set and the continuity of the function used, approves the existence of the results given in this paper.

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