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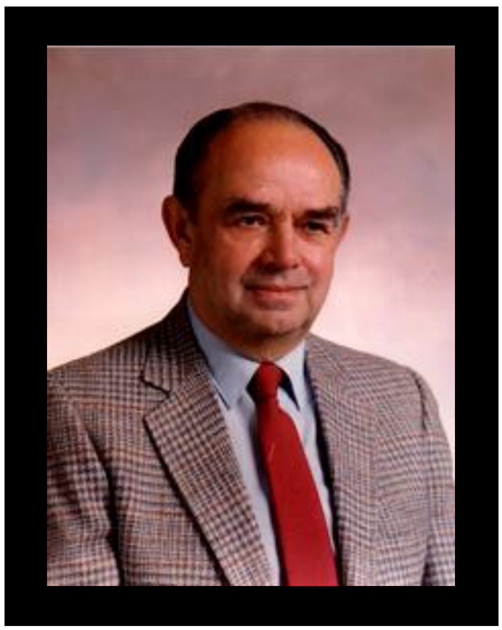
PREFACE

In Honor and Memory of Professor Lajos Takács

August 21, 1924 - December 4, 2015



Professor Emeritus of Mathematics & Statistics
Case Western Reserve University, Cleveland, Ohio, USA
&
An elected member of the Hungarian Academy of Sciences

A World Leading Mathematician, a Pioneer in the field of Queueing Theory,
Author of 225 Original Papers and
three clelbrated volumes in English language:
Introduction to the Theory of Queues
Combinatorial Methods in the Theory of Stochastic Processes
Stochastic Processes, Problems and Solutions

This issue of AAM is dedicated to honoring and remembering Professor Lajos Takács. While wrapping up the manuscript of my book (co-authored by Dr. Dimitar Mishev): Delayed and Network Queues, I went back to celebrate his 1962 book, Introduction to the Theory of Queues, where he gives an example illustrating a waiting time paradox, where the waiting time of a passenger waiting for a bus at a bus stop is infinite, while, in reality, he will wait a finite unit of time before a bus arrive. I sent Professor Takács an e-mail on December 4, 2015, inquiring if he had come up with a solution to resolve this paradox. In response, I received an e-mail from Dalma, his wife, that broke down me to tears by the sad news she broke that he passed away on the day I sent my e-mail.

I am extremely sorry for the loss of such a great human being, teacher, scholar and a world-leading scientist. He was my teacher, mentor and loving professor Lajos Takács. However, we are consoled by the belief that he is alive and well among the world's mathematics and science society forever. I am sure he is at peace. He has my prayers.

I cannot say more about him than what I had prepared to present in his honor as a Preface to the AAM Vol. 10, Issue 2 (December 2015). I am talking about a humble human being, a mentor, a caring teacher, a brilliant Hungarian mathematician, scientist, probabilist, statistician, a pioneer of queueing theory, honored by many organizations, a winner many prizes, including the John von Neumann Theory prize, and doctoral dissertation guide of 23 students, including Paul Burke and myself. The impact of his 225 original papers and his two books, Introduction to the Theory of Queues and Combinatorial Methods in Theory of Stochastic Processes is tremendous. One could not imagine what the impact of his 1600 unpublished materials on the theory of random fluctuation would be on the theory of probability.

While I was writing about his famous Integro-Differential Equation in chapter 5 of my book and reviewing the literature, I noticed that this 1962 formula was applied by Hága et al. (2006) regarding an Efficient and reliable available bandwidth measurement. Another paper I came to notice was by Kim (2014) regarding an available server management in the Internet connected network environments, in which local backup servers are hooked up by LAN and remote backup servers are hooked up by VPN (Virtual Private Network) with high-speed optical network citing Takács (1958), Theory of Telephone Traffic. So, it is not only that his 1962 book has been cited hundreds of times, but even his older papers are also being used these days.

The rest of this Preface was presented at the The 8th International Conference on Lattice Path Combinatorics and Applications, on August 18, 2015, at California Polytechnic State University Pomona, California, United States of America, at his 91st birthday, in honor of Professor Lajos Takács. The presentation was made jointly by Professors Aliakbar Montzer Haghighi and Sri Gopal Mohanty.

Part I

Lajos Takács' Life and Contribution to Combinatorics

Presented by Invited Guest Speaker

Dr. Aliakbar Montzer Haghighi

Professor and Head of Department of Mathematics
Prairie View A&M University, Priarie View, Texas, USA

A graduate student of Lajos Takács at CWRU, who did his dissertation under his guidance and graduated in 1976

The Life of Lajos Takács

► How did I come to know Takács?

At San Francisco State University in northern California, I first heard the name Lajos Takács's while I was enrolled in a graduate class with Dr. Siegfried F. Neustadter (1924–2012).

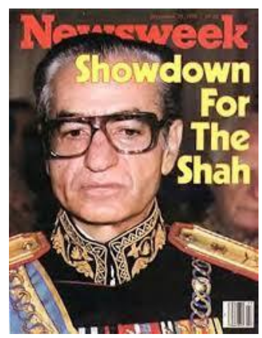

Siegfried F. Neustadter

Professor Neustadter received (known by Fred) his Ph.D. from Berkeley and spent several years at Harvard and MIT before joining SF State in 1958. He was a consultant Mathematician at Sylvania Electronics in Waltham, Massachusetts, at a time when Takács was working at Bell Lab.

So, Neustadter already aware of the great Takács, brought the celebrated Takács's 1962 book, Introduction to the Theory of Queues, to the class's attention and taught the single-server queue, $M/M/1$, from that book. That class left an indelible impression in my mind and was determined to do more graduate work under Professor Takács.

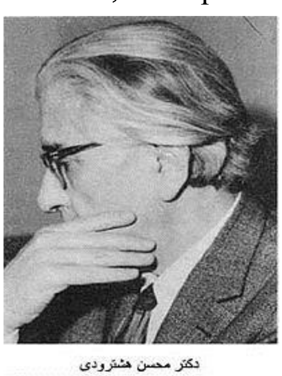
♦ Left the USA, back to Iran

I permanently left the US in 1968 before completing my Master's degree in response to my mother's declining health problem in Iran. However, because of my political activities in 1960's against the late Shah of Iran's regime while I was student at SF State, it was sanctioned by the Shah's Secret Service, called "Savak" (ساواک), that I should not leave the country for 5 years and two other sanctions.



Mohammad Reza Pahlavi

After some communication with the mathematics department at SF State, I completed my Master's degree by taking a written comprehensive exam at the US Embassy in Tehran under the auspices of a University of Tehran's professor, the late Dr. Mohsen Hashtroodi.



Mohsen Hashtroodi

During that 5-year period, while writing and translating books, I supervised one of my students to translate a part of Takács's book in partial fulfillment of her master thesis.

► At Case Western Reserve University

Nearing my release date, I decided that I want to meet Takács. I applied for and received acceptance at the Department of Operation Research at Case Western Reserve University (CWRU) in Cleveland, Ohio, United States of America, where I thought Takács was teaching so he guide me on my doctorate degree.

When I was allowed to leave the county, I went to the Netherland to secure a visa to enter the US, got it, and finally left Iran.

I arrived at CWRU in 1973 with my family, and registered for courses in the OR Department. But, I soon found out that Takács was now in the Mathematics and Statistics Department. (Of course, there was no Google to search in those days and everything took time).

So, after a complete semester at the Operations Research Department, Professor Shelemyahu Zacks, the then department chair (now Professor Emiratous at SUNY-Binghamton University), arranged for me to continue my study in the Mathematics and Statistics Department. I enrolled in Stochastic Processes with Takács!


Shelemyahu Zacks

Case Western Reserve University

Even before taking the Graduate Record Exam (GRE), Professor Takács gave me the honor of to start working on my dissertation under his guidance. It seemed he believed that I will pass GRE! And I did, continued working on my dissertation and received Ph.D. degree in January 1976 (but, was confirmed in June of that year).

► Left the USA, back to Iran again!

A few days after my successful defense of the dissertation, I returned home to Iran, only to get plunged into the 1978 Iranian revolution two years later.

► Meeting Professor Mohanty in Canada

After teaching and serving the administration at some of the highest academic levels during the first years of revolution, when the war between Iran and Iraq was going on some unpleasant things began to happen in the universities, I decided that I needed a peaceful atmosphere to live and protect my family since they were taking boys to the war at age 14 and up; and the age of my son was soon reaching 14. So, I contacted Takács, asked for help and a source of financial support.

Iran-Iraq War, 1980 to 1988

Takács was not a fan of obtaining research financial support and grant funds, as he believed the work completed should be awarded rather than a work being promised.

So, he referred me to Professor Sri Gopal Mohanty at McMaster University and asked me to contact him. To him, Mohanty was the BANK, with much grant funds he was receiving, I guess!

After contacting Mohanty, he graciously offered me three months teaching and research during the summer of 1984 at McMaster University.

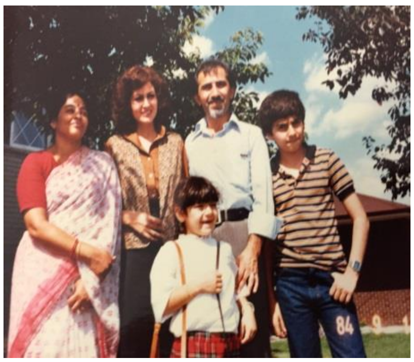
Sri Gopal Mohanty

McMaster University

At that time, two other professors, Professor Jyotiprasad Medhi (from India, at the far right) and Chaći (from Hungary, the second from the right) were there, too. As a result of this acquaintance and friendship, research collaboration among us started.

Mohanty, Haghighi, Chaći, Medhi (from left)

We also developed an enduring family friendship with Professor Mohanty.



His wife, Shanti, was a kind host and very hospitable. She makes good food and excellent chai (tea)! My entire family (Shahin, Ali, Mahyar and Mahroo) enjoyed the relationship.

(In the piture, from left, Shanti Mohanty, Shain, Aliakbar, Mahyar and Mahroo, in front).

► Back to the USA and started a Life-long Living!

Visiting Mohanty was the preparation to leave Iran and so I along with my wife and two children left Iran and arrived in the Los Angeles in February of 1985 to spend my sabatical leave at California State University- Fullerton. After 6 months, we moved to Columbia, South Carolina, stayed there until our both children received their first university degrees and left us. Hence, after 17 years, my wife and I moved to Houston, Texas, where we currently live. Our son followed his eduction and now practices medicine and lives with his wife (Roshni Patel, also a physician) and two children (Maya and Kayvan) in Jackson, Florid. Our dauther received her MBA and working and living with her daghter (Leila) in Charlot, North Carolina.

► Remembrances

Now, I will give some excerpts of my remembrance from the time I was a student of Takács:

1. Television weather forcasters (Meteorologists) used to use the word “chance” of rain or snow, when I was in Cleveland, Ohio. However, during my stay, June 1973 – January 1976, the use of “chance” changed and they started using the word “probability”.

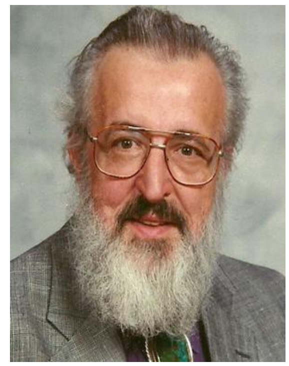
This change of word made Profesor Takács very excited that the word “probability” finally found its position in the media, at least among Television weather forcasters (meteorologists) in Cleveland!

2. Takács was criticized because he did not present real-life examples for his deep theoretical work to make the theory easier to understand.

His response, as he used to tell me, was that he prepares medicines and leaves them in the phamacy’s shelf. It is up to the doctors to recognize these medicines’ availability and values and then prescribe them for their pacients.

This response made Professors Joe Gani and the late Marcel Nuets laugh and enjoy it when I told them at a conference in Honor of Gani in Athen, Greece, 1995.

Joe Gani
UCSB- Statistics



Marcel Neuts
1935- 2014

3. As Takács entered our classroom at Case, he would go to the blackbord (there was no whiteboard those days) and start writing from left to write and from top to bottom of the board, erase and continue writing all over the board, without consulting notes or working at us, the students. It was up to us, the studnedts, to figure out later what he was talking about.

I later discovered that his lectures were from published research papers and because he understood then, he ofered no detail and explanation to us!

Once in one of my queueing theory classes, Professor Otomar Hájek from the department of mathematics and Statistics was sitting to audit the course, too. (Emeritus Professor Otomar Hájek is originally from Czechoslovakia. He is a recipient of *von Humboldt* award at the TH Darmstadt, Fachbereich Mathematik and also known for his contributions to dynamical systems, game and control theories.)

As Takács continued to write, seemingly endlessly, he suddenly stopped. Professor Hájek asked Takács what was it that he forgot. Takács turnd his face to him,

Otomar Hájek

just stirred at him for a few seconds, went back to the board and continued writing (he remmebered what he had forgotten)! That was the only time I noticed him forget somethin during teaching without his notes.

4. As mentioned, I attended CWRU in June 1973. I was with my, then, 2-years old son, Mahyar (now called Micheal), and my wife, Shahin. We were staying in a graduate residence hall, right in the heart of downtown Cleveland, Ohio, where crimes and particularly gun shooting was high.


Happy turning three, my son, Mhyar!

After two years of frustration, my wife decided to return home, to Tehran, with my son, after his 4th birthday, so that I can rush to graduate and they can be safe. After they left, I practically lived in my office, mostly with very brief naps at my desk and trying to finish my dissertation.

During the Christmas Holidays of 1975, when I was preparing the defense of my dissertation, I needed help. I called Professor Takács and ask him for help. He responded without hesitation, went to his office and helped me out. This is one of many caring experiences I remember from this great teacher of mine at the time I was a student at Case Western Reserve University.

I am extremely thankful of him, among other things, for him being a very caring professor. With all his clout that could have made him arrogant (as some of us professors are!) he was so much down to earth, caring and humble.

► A Biographical Sketch of Lajos Takács

♦ Takács's Childhood

- Lajos was born on August 21, 1924 in Maglód (a little town 16 miles from Budapest).

- At age 4, Lajos could multiply. This is when her mother was helping his dad at a general store he owned. A neighbor told her mother that she sold her pig for 40 pengös (Hungarian currency during 1927-1946). Lajos, asked her mother how many fillérs was in a pengö?

pengö

After her mother responded 100, he thought a moment and told her that the neighbor has 4000 fillérs! So, he became a locally known mathematician at age 4 in his small town, Maglód!

♦ **Takács in Elementary School**

- While in elementary school, Lajos took note of all of his learning and carried the notebook with him. He was interested in technology, arithmetic, electronic and radio.

- A mathematical problem Lajos solved while in elementary school was the magic property of π in calculating the area of a circle and consequently the exact volume of a cylindrical barrel in his yard.

- Takács calculated the chess game's grain sum reward while he was in elementary school as one of his mathematical problems at the young age. The problem is following:

The chess game inventor asked that as a reward he be given one grain of wheat for the first square of the chessboard, two for the second, and double the previous one for the subsequent square.

At that age he was not aware of how to calculate the sum of the geometric series, to find out the total number of grains to be awarded. So, he just added the numbers for all 64 squares and found the exact sum in kilograms and sacks.

The question is, how Takács calculated the number of sacks needs? So, I asked Takács the following question in an e-mail:

In elementary school, how did you know the weight of a wheat grain and a sack weight capacity to figure out how many sacks of grain the reward would be?

Here is Takács's response and his wish for this conference in his e-mail on July 23, 2015:

"Dear Ali,

To answer your question, I had no way to measure the weight of one grain. My mother had a general store, not a pharmacy! But she did have an old-fashioned scale with a collection of weights of various sizes. I put the 100 gram weight in one tray of the scale, and enough wheat grains in the other to tip the scale. Then I counted the number of grains in 100 grams, and divided 100 by the number of grains to find the weight of one grain of wheat. I already knew that the standard weight capacity of a sack of grain was 80 kilograms.

I checked the accuracy of my calculations by taking into consideration that 2^{10} =1024.

Best wishes for a very successful conference,
Lajos"

♦ **Takács in Secondary School**

- By the time Takács attended the secondary school, his mentioned notebook was full of everything he had learned about technology and arithmetic.

- To attend his secondary school, Takács had to commute with train to Budapest.

  Riding train was an opportunity for him to borrow books from older students and study them.

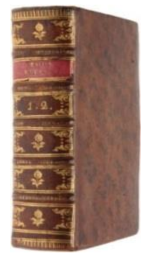
  He studies many books on his own, including the Differential and Integral Calculus by Manó (Emanuel) Beke that he bought.

Manó Beke
1862-1946

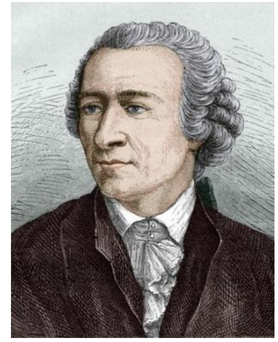
- At the age 14, Takács lost he father and had to help his mother at the general store in Maglód. Hence, he had to take care of errands of the business in Budapest. Thus, during the high school period, he was the manager and buyer for his mother's business.

- At age 15, Takács approached his mathematics teacher, Dezső Vörös, asking a mathematics book to read and the teacher suggested Euler’s Algebra.

  Euler’s Algebra (1770) is an extremely complicated and difficult German language book with old-fashion Gothic letters (relating to the Goths or their extinct East Germanic language).



Euler’s Algebra



Leonhard Euler (1707-1783)

  Nonetheless, under insistence of the teacher, Takács purchased the book and spent Christmas vacation of 1939 to study it. As soon as he mastered the Euler’s syntax, he was amazed by the contents of the book of this Swiss Mathematician such as solution of cubic and quadratic equations, Diophantine equation, and solution of Fermat’s problem for $n = 3$ and $n = 4$.

- Throughout high school, Takács spent his free times and vacations reading and studying many books with different topics, including Diophantine equations in Euler and notes of Vörös on Gusztav Rados’s lectures on number theory and some other books on number theory, including, Dirichlet (1871) and Dickson (1929).

- Takács became, particularly, interested in Number Theory and out of his readings; he made some discoveries at the time. For instance, in a book Takács read that the number of branches on a tree increases annually according to the sequence 1, 2, 3, 5, 8, 13, …. Assuming this statement is true, while he had no knowledge of “Calculus of Finite Differences”, he wanted to find a general formula for the number of branches of a tree in the n^{th} year.

  Takács solved the problem posed in a *roundabout* way by observing that the numbers in the sequence 1, 2, 3, 5, 8, … are positive integers in y of the Diophantine equation $x^2 - 5y^2 = \pm 4$ that he could solve. However, he was not aware that solution of the problem already existed in the literature of Fibonacci numbers.

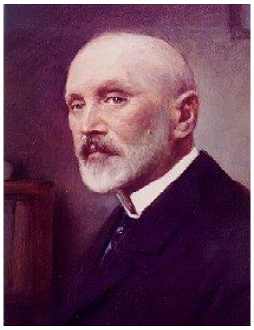
- During high school years, Takács was familiar with ***combinatorics*** and solved many probability problems. However, he considered probability as a branch of combinatorics. Much later, he realized the importance of the probability theory and its role in describing the physical world.

- Takács had particular interest in radio technology. Hence, he bought the monthly journal *Rádió Tecknika* and constructed different kinds of electronic equipment. He learned logarithm, trigonometry and complex numbers from a mathematical column of this journal.

- Although during high school years, Takács was spending most of his times studying mathematics books, he also studied classical and modern physics. However, he did not ignore physical activities. He was proud of his accomplishment in high jump, long jump, running, swimming, and skating. He also was an entertainer for his friends by mathematical puzzles and tricks.

- **Takács's Eötvös Prize**

In 1943 when graduated from high school, Takács won the second prize in the 47th Loránd Eötvös mathematical and physical society competition for high school graduates in Hungary. The award was established in 1894, given to two high school students annually. Including in the list of winners are great mathematicians like the following:



Loránd Eötvös

First Name	Last Name	Year Won	Year Born	Year Died
Lipót	Fejér	1897	1880	1959
Theodor	Kármán	1898	1881	1963
Dénes	König	1902	1884	1944
Alfréd	Haar	1903	1885	1933
Maecel	Riesz	1904	1886	1969
Gábor	Szegö	1912	1895	1985
Tibor	Radó	1913	1895	1965
László	Rédei	1918	1900	1967
László	Kalmár	1922	1905	1976
Edward	Teller	1925	1908	2003
Lajos	Takács	1943	1924	

Problems and their solutions for the 47th competition may be found online at http://www.math-olympiad.com/47th-eotvos-competition-1943-problems-solutions.htm

♦ **Takács in University**

- Takács studied at the Technical University of Budapest from 1943 to 1948. While studying at the Technical University of Budapest, he was also attending classes at the Pázmány University.

- At the end of 1943 school year, the Second World War reached the door step of the university, and as a result, academic activities were interrupted until the fall of 1945. Thus, Takács was spending his time on studying by himself.

Back to the university that resumed in 1945, Takács took courses in probability theory and mathematical statistics with Charles Jordán; courses that made Takács's mind to take the area as his career.


Charles Jordán
1871–1959

Jordán became Takács's mentor during the years he was a student. Takács not only learned from Jordán, but contributed to his class too. Here is an example: On one occasion, when Jordán was discussing "matching" he stated:

Let $Q(n)$ denote the number of permutation of $1, 2, \cdots, n$, in which no match occurs. Then,

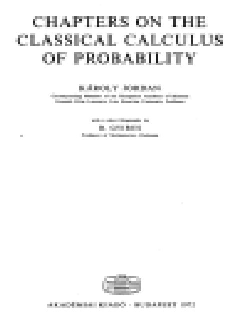
$$Q(n) = \begin{cases} 1, & n = 0, \\ 0, & n = 1, \\ \dfrac{n-1}{Q(n-1) + Q(n-2)}, & n \geq 2. \end{cases}$$

This result was found by Pierre Raymond de Montmort in 1708 and Jordán did not offer the proof. Takács, then, gave the proof based on the observation that $Q(n)$ is also the number of terms that contain no diagonal element in the expansion of a determinant of order of n. Jordán added this short note from Takács in already finished manuscript of his book of probability theory.


Montmort
1678-1719

Charles (Károly) Jordán (1871 – 1959) best known for his work on probability theory and finite differences, was a founder of the Hungarian School of Probability. He published 5 books and 83 papers.

Later, Takács (1961) wrote an article in his memory, while at Columbia University in New York.



- From 1945 to 1948, Takács was a student assistant to Professor Zoltán Bay, a professor of atomic physics, and participated in his famous experiment of receiving microwave echoes from the Moon, 1946.

  Zoltán Bay (1900-1992) was a Hungarian physicist, and engineer who developed microwave technology, including tungsten lamps. Bay was the second person to observe radar echoes from the Moon.

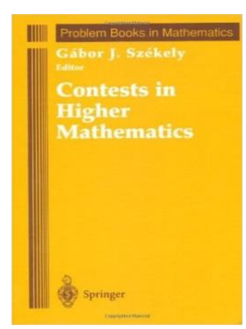

Zoltán Bay

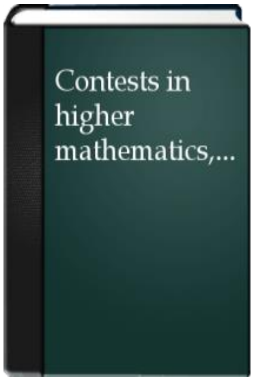
As a student assistant, he was appointed as the consultant to Bay's research laboratory consisting of mostly university researchers. Takács's job here was to calculate the position of the moon at 15-minutes interval and participate in the nightly experiment. The success arrived in the evening of February 6, 1946, also a triumph of probability theory, Takács calls it. The experiment showed that the signal/noise ratio was very small, but the effect of signal could be detected by averaging out the noise.

- Takács received a Doctorate degree (Ph. D.) in 1948 with his thesis title as "*Probability Theoretical Investigation of Brownian Motion*", that was refereed by Charles Jordan. However, he continued his registration at the university to become a teacher.

- **Takács's Miklós Schweitzer Prize.** Takács won the first prize in the Schweitzer mathematical competition for university graduates in Hungary in 1949.

  The Miklós Schweitzer Competition (*Schweitzer Miklós Matematikai Emlékverseny*) is an annual Hungarian mathematics competition for university recent graduates, established in 1949.





Hungary,
1949-1961

Schweitzer Competition is uniquely high-level among mathematics competitions. The problems on the competition can be classified roughly in the following categories: Algebra, Combinatorics, Theory of Functions, Geometry, Measure Theory, Number Theory, Operators, Probability Theory, Sequences and Series, Topology, and Set Theory. For sample questions, see: http://www.artofproblemsolving.com/community/c3253_mikls_schweitzer**.**

- **Takács's Prize for his Paper**

Takács won the second Gründwald Prize awarded by the János Bolyai Mathematical Society in 1952 for his paper "*Investigation of waiting time problems by reduction to Markov processes*". See Takács (1955).


János Bolyai
(1802-1860)

The János Bolyai Mathematical Society (Bolyai János Matematikai Társulat, BJMT), founded in 1947, is the Hungarian mathematical society, named after János Bolyai, a 19th-century Hungarian mathematician, a co-discoverer of non-Euclidean geometry. (Portrait by Ferenc Márkos -2012).


Alfréd Rényi
(1921-1970)

The paper was presented by Alfréd Rényi in Budapest.

Rényi is a Hungarian mathematician who made contributions in combinatorics, graph theory, number theory and in probability theory.

With the win of this paper, Takács posed himself to be considered among the frontiers in queueing theory.

- **Now, Dr. Dr. Lajos Takács**

In 1957 Takács received the Academic Doctor's Degree (D. Sc.) in Mathematics with his dissertation entitled:

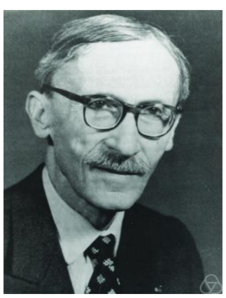
"Stochastic processes arising in the theory of particle counters",

at the Department of Mathematics of the L. Eötvös University.

► **Takács's Employments and Further Achievements**

♦ Takács is a Hungarian born mathematician, known for his contributions to probability theory; stochastic processes; in particular, queueing theory; and combinatorics. He was one of the first to introduce semi-Markov processes in queueing theory in 1952.

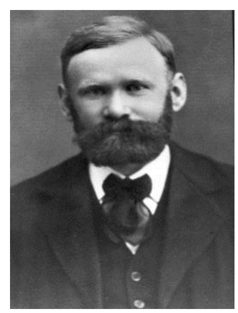
♦ Takács worked as a mathematician at the Tungsram Research Laboratory (1948-55), While at the Tungsram Research Laboratory, Takács also accepted a staff position at the newly created Research Institute for Mathematics of the Hungarian Academy of Sciences (1950-58), during which he published several papers on queueing theory, involving applications to telephone traffic, inventories, dams, and insurance risk.



Paul Lévy
1886-1971

During this period, Takács developed the theory of point processes and introduced the process later introduced as *semi-Markov processes* by Paul Lévy.

Takács also gave the generalization of Agner Krarup Erlang's telephone traffic congestion formula that later was discussed in his celebrated (1962) book, Introduction to the Theory of Queues.



A. K. Erlang
1878-1929

♦ Takács was an associate professor in the department of mathematics of the L. Eötvös University (1953-58), formerly called Pázmány University.

Eötvös Loránd
Tudományegyetem

♦ Takács took a Lecturer appointment at Imperial College in London and London School of Economics (1958), where he lectured on the theory of stochastic processes and queueing theory.

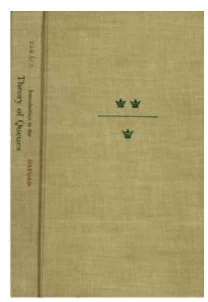
Imperial College London

♦ Between 1954 and 1958, Professor Takács published no fewer than 55 research papers on various topics in stochastic processes and the foundations of modern queueing theory.

♦ Takács's early research in queueing theory was summarized in one of his finest works, "Some Probability Questions in the Theory of Telephone Traffic," Takács (1958). This paper just recently has been cited by Kim (2014). Interestingly, Kim's paper recent is focused on available server management in the Internet connected network environments, in which local backup servers are hooked up by LAN and remote backup servers are hooked up by VPN (Virtual Private Network) with high-speed optical network.

♦ The year 1958, while at the Imperial College of London, was a turning point in Takács's life. He decided to leave Hungary forever and move to the United States. As his first job in the United States, in 1959 Takács received an offer from Columbia University in New York as Assistant Professor that he accepted and after a year, in 1960, he was promoted to the rank of Associate Professor. Takács stayed at Columbia University for the next 7 years teaching probability theory and stochastic processes. While at Columbia University, Takács had a consulting job at Bell Laboratories and at IBM.

♦ In summer of 1961, Takács had a visiting position at Boeing Research Laboratories in Seattle, Washington. Takács also spent a summer sabbatical at Stanford University working on his first celebrated bestseller book, *Introduction to the theory of queues,* Takács (1962).



Takács first published in Hungarian and later on translated into various languages. He developed a large variety of multichannel queueing systems, applying his extraordinary fluency in combinatorial and continuous mathematics, including primarily results for embedded queueing processes. Their extensions to continuous-time-parameter queueing processes were later included in his celebrated monograph, Introduction to the Theory of Queues, that appeared in 1962.

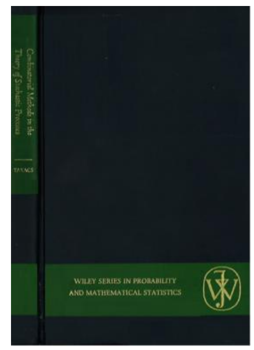
♦ In early 1960s, Takács developed the time-dependent behavior of various queuing processes, specifically, the *virtual waiting-time process*, that now is referred to as the *Takács Process*.

During his tenure at Columbia University, Takács attended many conferences in Europe and the US focusing on his research on queueing problems, Ballot theorem, and order statistics. He found *a generalization of the classical ballot theorem* of Bertrand, which made it possible to solve many problems in queueing theory, in the theory of dams, and in order statistics. More on this topic will be discussed in part II of this presentation.

While at Columbia University, Takács also developed queues with feedback, balking, various orders of service, and priority queues. Multi-server with Feedback was, indeed, the title of my dissertation.

During his stay at Columbia University, Takács also explored the theory of fluctuations, that he has now written 1600 pages on the subject and is waiting to be published on his own terms!

In 1966 Takács spent part of his sabbatical at Stanford University working on his second bestseller book, *combinatorial methods in theory of stochastic processes*, Takács (1967).



♦ At Columbia University, Takács advised 9 Ph.D. students with their dissertations. They are as follows:

1. **Paul J. Burke,**
2. Ora Engelberg,
3. Joseph Gastwirth,
4. Peter Linhart,
5. Clifford Marshall,
6. Lloyd Rosenberg,
7. Saul Shapiro,
8. Lakshmi Venkataraman, and
9. Peter Welch.

♦ In 1966, Takács accepted the appointment as Professor of Mathematics at Case Western Reserve University in Cleveland, Ohio, where he held this position until 1987, when he retired as a Professor Emeritus.

During his tenure at Case Western Reserve University Takács wrote over 100 monographs and research papers. By this time, Takács's major research areas has become sojourn time, fluctuation theory and random trees.

During his tenure at Case Western Reserve University, Takács guided the following additional 14 Ph.D. students, including me:

1. Roberto Altschul, 1973
2. John Bushnell,
3. Daniel Michael Cap, 1985,
4. Jin Yuh Chang, 1976,
5. Sara Debanne, 1977,
6. Nancy (Mailyn) Geller,
7. **Aliakbar Montazer-Haghighi,**
8. Andreas Papanicolau,
9. Pauline Ramig,
10. Josefina De los Reyes,
11. Douglas Rowland,
12. Elizabeth Van Vought,
13. Enio E. Velazco, and
14. Fabio Vincentini.

► Takács's Number of Publications

♦ Papers

In summary, by his own count, as of May 23, 2015, Takács has published 225 papers, many of which have had a huge impact on the contemporary theory of probability and stochastic processes. His last paper appeared in 1999. His work may be summarized in the following topics:

1. Combinatorial Problems
2. Ballot Theorems
3. Random Walks
4. Random Graphs
5. Point Processes
6. Queueing Processes
7. Binomial Moments
8. Sojourn Time Problems
9. Branching Processes
10. Fluctuation Theory
11. Order Statistics

- Here is a small sample of his publications up to 1994:

1954. Some Investigations Concerning Recurrent Stochastic Processes of a Certain Kind, Magyar Tud. Akad. Alk. Mat.Int. Kozl. vol. 3, pp. 115-128.
1955. Investigations of Waiting Time Problems by Reduction to Markov Processes, Acta Math. Acad. Sci. Hung. vol. 6, pp. 101-129.
1970. On the distribution of the supremum for stochastic processes, Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques, 6(3):237-247.
1977. An Identity for Ordered Partial Sums, J. Comb. Theory, Ser. A 23(3): 364-365.
1981. On the "problème des ménages", Discrete Mathematics 36(3): 289-297.
1981. On a Combinatorial Theorem Related to a Theorem of G. Szegö, J. Comb. Theory, Ser. A 30(3): 345-348.
1988. Queues, random graphs and branching processes, Journal of Applied Mathematics and Stochastic Analysis, vol. 1, no. 3, pp. 223–243.
1990. Counting forests, Discrete Mathematics 84(3): 323-326.
1990. On Cayley's formula for counting forests, J. Comb. Theory, Ser. A 53(2): 321-323.
1990. A generalization of an inequality of Stepanov, J. Comb. Theory, Ser. B 48(2): 289-293.
1990. On the Number of Distinct Forests, SIAM Journal on Discrete Mathematics 3(4): 574-581.
1991. On a probability problem connected with railway traffic, Journal of Applied Mathematics and Stochastic Analysis, vol. 4, no. 1, pp. 1–27.
1991. Conditional limit theorems for branching processes, Journal of Applied Mathematics and Stochastic Analysis, vol. 4, no. 4, pp. 263 – 292.
1991. On the distribution of the number of vertices in layers of random trees, Journal of Applied Mathematics and Stochastic Analysis, vol. 4, no. 3, pp. 175–186.
1993. Limit distributions for queues and random rooted trees, Journal of Applied Mathematics and Stochastic Analysis, vol. 6, no. 3, pp. 189–216.
1994. On the Total Heights of Random Rooted Binary Trees, J. Comb. Theory, Ser. B 61(2): 155-166.

- **And, here is the list of Takács papers to the last:**

[1] - [211] *Journal of Applied Mathematics and Stochastic Analysis,* Volume *7* Fall 1994 Number III pp. 229-237.

[212] (1995). On the local time of the Brownian motion. *The Annals of Applied Probability* 5, 741-756.

[213] (1995). Brownian local times. *Journal of Applied Mathematics and Stochastic Analysis* 8. 209-232.

[214] (1996). On a test for uniformity of a circular distribution. *Mathematical Methods of Statistics* 5, 77-78.

[215] (1996). On a three-sample test. *Athens Conference on Applied Probability and Time Series.* Volume 1: Applied Probability. In Honor of J.M. Gani. Edited by C.C. Heyde, Yu. V. Prohorov, R. Pyke and S.T. Rachev. Springer-Verlag, N.Y., pp. 433-448.

[216] (1996). On a generalization of the arc-sine law. *The Annals of Applied Probability*. 6, 1035-1040.

[217] (1996). Sojourn times. *Journal of Applied Mathematics and Stochastic Analysis* 9, 415-426.

[218] (1996). In memoriam. Pál Erdös March 26, 1913 – September 20, 1996). *Journal of Applied Mathematics and Stochastic Analysis* 9, 563-564.

[219] (1997). Ballot problems. *Advances in Combinatorial Methods and Applications to Probability and Statistics.* Edited by N. Balakrishnan, Birkhäuser, Boston, pp. 97-114.

[220] (1997). Holdvisszhang 1946 február 6-án. *Fizikai Szemle*, 47:1, 20-21.

[221] (1998). On the comparison of theoretical and empirical distribution functions. *Asymptotic Methods in Probability and Statistics. A Volume in Honor of Miklós Csörgö*. Edited by Barbara Szyszkowicz. Elsevier Science B.V., Amsterdam, The Netherlands, pp. 213-231.

[222] (1998). Sojourn times for the Brownian motion. *Journal of Applied Mathematics and Stochastic Analysis*. 11, 231-246.

[223] (1998). On cyclic permutations. *The Mathematical Scientist*. 23 91-94.

[224] (1999). On the local time of the Brownian bridge. *Applied Probability and Stochastic Processes.* Edited by J George Shanthikumar and Ushio Sumita. Kluwer Academic Publishers, Boston, M.A., pp.45-62.

[225] (1999). The distribution of the sojourn time of the Brownian excursion. *Methodology and Computing in Applied Probability*, 7-28.

- **Takács has also published six books, three in Hungarian and three in English. They are as follows:**

(1) Az Elektroncsö (The Vacuum Tube) with A. Dallos, Tankönyv Kiadó, Budapest, 1950.

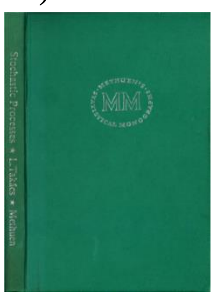
(2) Valószinüségszámitás (Theory of Probability), With M. Ziermann, Tankönyv Kiadó, Budapest, 1955 (original publishing), 1967, 1972.

(3) Valószinüségszámitás, (Theory of Probability), with P. Medgyessy (Part A: Probability
Theory) and L. Takács (Part B: Stochastic Processes), Tankönyvkiadó, Budapest 1957 (original publishing), 1966, 1973.

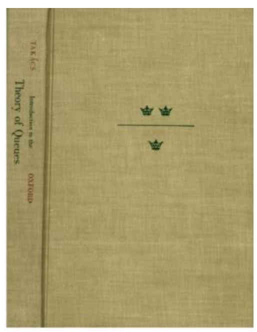
(4) *Stochastic Processes. Problems and Solutions,* Methuen & CO LTD, 1960, Translated by P. Zádor (1962).

(5) *Introduction to the Theory of Queues*, Oxford University Press, 1962.

(6) *Combinatorial Methods in the Theory of Stochastic Processes*, John Wiley, (1967).



(4) Stochastic Processes: Problems and Solutions, 1960



(5) Introduction to the Theory of Queues, 1962, Wiley

(6) Combinatorial Methods in the Theory of Stochastic Processes, 1967, Wiley

► Awards

- 1993, Foreign Membership Magyar Tudomanyos Akademia, Matematikai es Fizikai Tudomanyok Osztalyanak Közlemenyei, Hungarian

- 1994, John von Neumann Theory Prize

John von Neumann (1903 – 1957)

A Hungarian mathematician who made major contributions to a number of fields, including mathematics (foundations of mathematics, functional analysis, ergodic theory, geometry, topology, and numerical analysis), physics (quantum mechanics, hydrodynamics, and fluid dynamics), economics (game theory), computing (Von Neumann architecture, linear programming, self-replicating machines, stochastic computing), and statistics.

- 2002, Fellows Award. Institute for Operations Research and Management Sciences

Honors

- Professor Lajos Takács is noted as "the most well-known, reputed and celebrated Hungarian" in the field of probability and stochastic processes.

- 1994, Many scientific institutions celebrated his 70th birthday, including:

  - The Institute of Mathematical Statistics
  - Operations Research Society of America
  - The Institute of Management Sciences
  - Hungarian Academy of Sciences
  - Jewgeni H. Dshalalow and Ryszard Syski, Lajos Takács and his work, in J. of Applied Math. and Stochastic Analysis, 7(3): (1994) 215-237.
  - A special volume, Studies in Applied Probability, 31A of Journal of Applied Probability, edited by J. Galambos and J. Gani, honoring Professor Takács (1994).

- 1995, Studies in Applied Probability, Papers in Honor of Lajos Takács, by J. Galambos, J. Gani, The Journal of the Operational Research Society, 46(11):1397-98.

- 2015, In Honor of Lajos Takács and his works up to August 21, 2015, presented by Aliakbar Montazer Haghighi and Sri Gopal Mohanty, at the 8th International Conference on Lattice Path Combinatorics and Applications, August 17–20, California Polytechnic State University, Pomona, CA.

Takács's Recent Life by December 4, 2015

- At the present time, Takács is Professor Emeritus at Case Western Reserve University. At age 91, Takács is "confined at home", as he stated, with his wife Dalma. In an e-mail to me just recently, 5/23/2015, Professor Lajos Takács wrote to me:

  "As for me, I am doing reasonably well, but I am more or less confined to my home. Since travel is a serious problem for me, I am unfortunately unable to participate in meetings."

- A great part of his work is waiting for a right publisher who would agree to his terms and conditions. Here is what he described his unpublished work to me in his e-mail dated May 2015 (original in the standard text format):

  "An additional note: In July 1973 I completed my book Theory of Random Fluctuations, which unfortunately is still in manuscript form. While I was working on

the book, John Wiley was so interested that they offered me a contract before the book was finished. I did not like to sign a contract until my work was ready for the press. The finished manuscript turned out to be 1600 pages which Wiley considered too big for a book. They offered to publish it in lecture note form, which was unacceptable to me. I was also unwilling to shorten the MS, so it is still unpublished."

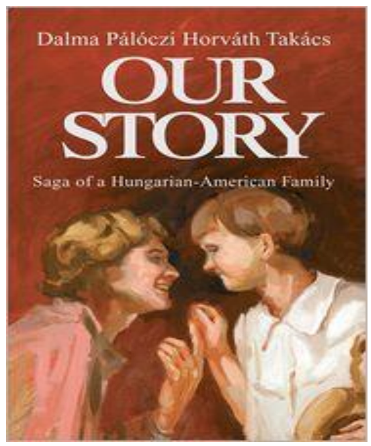
►Takács's Family

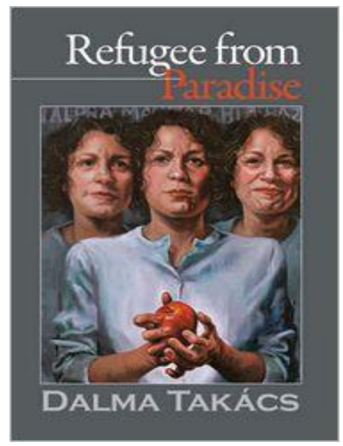
Takács greatest achievement was on April 9, 1959, when Lajos Takács married Dalma Horváth in London with Ityszard Syski as Takács's best man at his wedding. Dalma is now 82.

From their marriage, they have two daughters, Judith, an artist, and Susan, a legal assistant.

Ryszard Syski
(1924-2007)

Dalma is was Professor of English Literature and chair of the English Department at Notre Dame College of Ohio and an author of historical family memoirs, plays and several fiction books as well.





Dalma, painting by Judy on Mother's day of 2013, Mother having a ball, while diagnosed with cancer!

Judy is a commercial artist and illustrator. She is a seven time Best of Show winner, her work has been exhibited in many places. She does Pastel Portraiture, Paintings and Drawings.

Judy Takács,
Self-Portrait

The Applied Mathematical and Statistical world has just lost, by Takács' death, a great leader and compatriot; there is no doubt, however, that his work (some of which still to be published) will endure the ages.

Part II

Contribution to Combinatorics

Presented by the Conference Coordinator
Emeritus Professor
Dr. Sri Gopal Mohanty

Department of Mathematics & Statistics
McMaster University, Hamilton, Ontario, Canada

In this part, a list of theorems regarding ballot Problem that relates to Takács is presented.

Ballot Problems

The combinatorial approach of Takács is based on a generalization of the classical ballot theorem stated as follows:

Theorem 1: The Classical Ballot Theorem (*Theorem 7.2.1 from Takács, L. (1997)*)

If in a ballot, candidate A scores *a* votes and candidate B scores *b* votes, with $a > b\mu$, where μ is a positive integer, then, the probability that throughout the number of votes registered for A is always greater than μ times the number of votes registered for B is given by

$$P(a, b, \mu) = \frac{a - b\mu}{a + b},$$

provided that all the possible voting records are equally probable.

Theorem 2: Takács' generalization (*Theorem 7.5.1 from Takács, L. (1997)*)

Let us suppose that a box contains *n* cards marked with nonnegative integers $k_1, k_2, \cdots, k_n$, such that $k_1 + k_2 + \cdots + k_n = k \leq n$. All the *n* cards are drawn without replacement from the box. Denote by ν_r the number obtained at the r^{th} drawing $(r = 1, 2, \cdots, n)$. Then,

$$P\{\nu_1 + \nu_2 + \cdots + \nu_r < r,\ r = 1, 2, \cdots, n\} = \frac{n-k}{n},$$

provided that all the possible results are equally probable.

Let there be *a* cards marked 0 and *b* cards marked $\mu+1$. Suppose a card marked with 0 corresponds to a vote for A and a card marked with $\mu+1$ corresponds to a vote for B. Then it can be shown that the classical ballot theorem follows as a special case.

Note that letting *x* and *y* be the number of votes for A and B respectively at the r^{th} count, then,

$$x + y = r,\ a + b = n,\ b(\mu + 1) = k.$$

Hence, the event in the Theorem becomes

$$x(0) + y(\mu + 1) < x + y \Leftrightarrow y\mu < x$$

and the probability becomes $(n - k)/n = (a - \mu b)/(a + b)$ that checks the *Classical Ballot Theorem*.

Drawing cards without replacement implies consideration of *n!* permutations of *n* cards. The Theorem is true if permutations are replaced by cyclic permutations and is stated as follows as a counting result:

Theorem 3: (*Theorem 7.5.3 from Takács, L. (1997)*)

Let us suppose that *n* cards are marked with non-negative integers $k_1, k_2, \cdots, k_n$, such that $k_1 + k_2 + \cdots + k_n = k \leq n$. Among the *n* cyclic permutations of the *n* cards, there are exactly $n - k$ in which the sum of the numbers on the first *r* cards is less than *r* for every $r = 1, 2, \cdots, n$.

Its continuous version:

Theorem 4: (*Theorem 1 on page 1 of Takács, L. (1967)*)

Let $\phi(u)$, $0 \leq u \leq t$, be a non-decreasing function for which $\phi'(u) = 0$ almost everywhere and $\phi(0) = 0$. Let $\phi(t + u) = \phi(t) + \phi(u)$ for $0 \leq u \leq t$. For $0 \leq u \leq t$ define

$$\delta(u) = \begin{cases} 1, & \text{if } \phi(v) - \phi(u) \leq v - u,\ u \leq v \leq u + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$\int_0^t \delta(u) = t - \phi(t), \text{for } \phi(t) \le t.$$

Theorem 3 can be formulated in the following more general way:

Theorem 5: (*Theorem 7.5.4 from Takács, L. (1997))*

Let $\nu_1, \nu_2, \cdots, \nu_n$, be interchangeable or cyclically interchangeable discrete random variables which take on non-negative integers only. Write

$$N_r = \nu_1 + \nu_2 + \cdots + \nu_r, \ \ r = 1, 2, \cdots, n, \ \ N_0 = 0.$$

Then, we have

$$\begin{aligned} &P\{N_r < r, 1 \le r \le n, \text{and } N_n = n - i\} \\ &\qquad = \frac{i}{n} P\{N_n = n - i\}, \ 0 \le i \le n, \ n = 1, 2, \cdots. \end{aligned}$$

Theorem 6: (*Theorem 7.5.5 from Takács, L. (1997))*

Let $\nu_1, \nu_2, \cdots, \nu_n$, be interchangeable or cyclically interchangeable discrete random variables which take non-negative integers only. Write

$$N_r = \nu_1 + \nu_2 + \cdots + \nu_r, \ \ r = 1, 2, \cdots, n, \ \ N_0 = 0.$$

We have

$$\begin{aligned} &P\{N_r < r \text{ for at leasr one } r = 1, 2, \cdots, n\} \\ &\qquad = \sum_{i=1}^{n} \frac{1}{i} P\{N_i = i - 1\}, \ n = 1, 2, \cdots. \end{aligned}$$

Another extension of the Ballot Theorem:

While in the ballot problem we consider the number of votes for A is always greater than μ times the number of votes for B, we consider now the number to be greater exactly in j cases (j = 1, 2, …, $a + b$), not always.

This has been studied by Chao and Severo (1991).

Denote by α_r and β_r the number of votes registered for A and B, respectively, among the first r votes counted. Let μ be a positive real number and define

$$P_j(a, b, \mu) = P\{\alpha_r > \beta_r \mu, \text{ for } j \text{ subscripts } r = 1, 2, \cdots, a + b\}$$

for $j = 0, 1, 2, \cdots, a+b$. We can write that

$$P_j(a,b,\mu) = \sum_{0 \le s \le j} P\{\beta_j = s\} P_j(j-s, s, \mu) P_0(a+s-j, b-s, \mu),$$

where

$$P\{\beta_j = s\} = \frac{\binom{j}{s}\binom{a+b-j}{b-s}}{\binom{a+b}{b}} = \frac{\binom{a}{j-s}\binom{b}{s}}{\binom{a+b}{j}},$$

whenever $0 \le s \le j$ and $j - a \le s \le b$. (*Section 7.7 of Takács, L. (1997)).* It follows from the following auxiliary theorem:

Auxiliary Theorem: (*Theorem 7.7.1 from Takács, L. (1997))*

Let $\xi_1, \xi_2, \cdots, \xi_n$ be interchangeable real random variables. Define $\zeta_r = \xi_1 + \xi_2 + \cdots + \xi_r$ for $r = 1, 2, \cdots, n$ and $\zeta_0 = 0$. Denote by Δ_n the number of subscripts $r = 1, 2, \cdots, n$ for which $\zeta_r > 0$. Then,

$$P\{\Delta_n = j\} = P\{\zeta_r < \zeta_j,\ 0 \le r < j, \text{ and } \zeta_r \le \zeta_j,\ j \le r \le n\}.$$

See Anderson (1954) and Feller (1959).

A recent generalization of Takács (Theorem 3) by Mercancosk, Nair and Soet (2001) which the authors call Batch Ballot Theorem.

Theorem 7: Batch Ballot Theorem (*Theorem 2 from Mercancosk, Nair and Soet (2001))*

Let $n_1, n_2, \cdots, n_k$ be nonnegative integers such that $n_1 + n_2 + \cdots + n_k = n < km$. For $0 \le d \le m-1$, let C_d denote the number of cyclic permutations of $(n_1, n_2, \cdots, n_k)$ for which the sum of the first *s* elements is less than $sm - d$ for $1 \le s \le k$. Then,

$$C_0 + C_1 + \cdots + C_{m-1} = km - n.$$

Theorem 8: (*Theorem 3 from Mercancosk, Nair and Soet (2001))*

Let $\nu_1, \nu_2, \cdots, \nu_k$ be cyclically interchangeable random variables taking on nonnegative integral values. Set $N_s = \nu_1, \nu_2, \cdots, \nu_k$, for $1 \le s \le k$, $N_0 = 0$. Then, we have

$$\sum_{d=0}^{m-1} P\{N_s < sm - d, \text{ for } 1 \le s \le k \,|\, N_k = n\}$$

$$= \begin{cases} \frac{km-n}{k}, & \text{if } 0 \le n \le km, \\ 0, & otherwise. \end{cases}$$

These results are useful in handling *M/G/1*- type queues as proposed by Neuts (1989).

$$P\{\zeta = 0 | \zeta_n = i\} = \sum_{j=0}^{n-i} \left(1 - \frac{j}{n}\right) P\{N_n = j\},\ i \ge 0.$$

Interesting corollaries:

As a corollary of Theorem 3, we obtain the number of paths from the origin to $(n_0, \cdots, n_r)$ that do not touch the hyper-plane

$$x_0 = \sum_{i=1}^{r} \mu_i x_i$$

is given by

$$\frac{\alpha}{\alpha + \sum_{i=1}^{r}(\mu_i + 1)n_i} \binom{\alpha + \sum_{i=1}^{r}(\mu_i + 1)n_i}{n_1, \cdots, n_r},$$

where

$$\alpha = n_0 - \sum_{i=1}^{r} \mu_i n_i$$

and the $\mu_i s (\ge 0)$ are all different.

Another simple corollary of Theorem 5, which finds applications in batch queues is the following:

$$P\left(\sum_{i=1}^{r} X_i < \left[\frac{r-1}{m}\right] + 1,\ r = 1, \cdots, n \middle| X_r = k \right)$$

$$= \begin{cases} 1 - \frac{mk}{n}, & 0 \le mk \le n, \\ 0, & \text{otherwise.} \end{cases}$$

See Mohanty (1979).

Applications:

In his celebrated 1967 book, Takács has discussed applications of these ballot related combinatorial results to processes arising in queues, dams, storage and risk and to order statistics.

We give an **example** from a queueing process. (*Pages 94-95 of Takács, L. (1967).*

Denote by $\nu_1, \nu_2, \cdots, \nu_r, \cdots$ the number of customers joining the queue during the 1st, 2nd, …, *r*th, … services, respectively, and write

$$N_r = \nu_1 + \cdots + \nu_r,\ r = 1, 2, \cdots,\ N_0 = 0,$$

and $\zeta_n, , n = 1, 2, \cdots$, the queue size immediately after the *n*th departure; and ζ_0 is the initial queue size. In this case we speak about a queueing process of type

$$Q = \{\zeta_0;\ N_r,\ r = 0, 1, 2, \cdots\}.$$

Theorem 9: (*Theorem 1, page* 99 *of Takács, L. (1967))*

If $\nu_1, \nu_2, \cdots, \nu_n$ are interchangeable random variables, then:

$$\begin{aligned} P\{\zeta_n \le k | \zeta_0 = i\} &= P\{N_n \le n + k - i\} \\ &- \sum_{j=1}^{n-i} \sum_{l=0}^{n-i-j} \left(1 - \frac{1}{n-j}\right) P\{N_j = j + k,\ N_n = j + k + l\},\ i \ge 1, \end{aligned}$$

$$P\{\zeta_n \le k | \zeta_0 = 0\} = P\{\zeta_n \le k | \zeta_0 = 1\},$$

and, in particular,

$$P\{\zeta_n = 0 | \zeta_n = i\} = \sum_{j=0}^{n-i} \left(1 - \frac{j}{n}\right) P\{N_n = j\},\ i \ge 0.$$

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E-mail communications between L. Takács and Aliakabar montazer Haghighi during May to July of 2015, all are archived on PVAMU's server under amhaghighi@pvamu.edu.

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