Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 2 (December 2020), pp. 1323 - 1335

A New Approach to the Q-Conjugacy Character Tables of Finite Groups

Ali Moghani

Department of Mathematics and Computer Science
Seton Hall University
South Orange, NJ 07079 USA
E-mail: ali.moghani@shu.edu

Received: April 5, 2020; Accepted: July 2, 2020

Abstract

In this paper, we study the Q-conjugacy character table of an arbitrary finite group and introduce a general relation between the degrees of Q-conjugacy characters with their corresponding reductions. This could be accomplished by using the Hermitian symmetric form. We provide a useful technique to calculate the character table of a finite group when its corresponding Q-conjugacy character table is given. Then, we evaluate our results in some useful examples. Finally, by using GAP (Groups, Algorithms and Programming) package, we calculate all the dominant classes of the sporadic Conway group Co_2 enabling us to find all possible the integer-valued characters for the Conway group Co_2.

Keywords: Finite group; Conjugacy class; Q-conjugacy; symmetry; Sporadic group; Character; Unmatured; Conway group

MSC 2010: 20D99, 20C15, 20B40

1. Introduction

The group theory has drawn wide attention from researchers in mathematics, physics, and chemistry. The mathematical chemistry, the symmetry and isomers, and the topological cycle index have been researched using the computational group theory. They include not only the diverse properties of finite groups, but also their wide-ranging connections with many applied sciences, such as Nanoscience, Chemical Physics and Quantum Chemistry, see Cotton (1971), Hargittai et al. (1986), Moghani (2009), Ladd (1989), Harris et al. (1989) and Liu (2012).

The inspiration for this study is to provide a useful technique to calculate the character table of a finite group when its corresponding *Q*-conjugacy character table is given, and Moghani (2016) in the Journal of Applications and Applied Mathematics. The reader is encouraged to consult Aschbacher (1997), Kerber (1999), Lusztig (2014), Moghani et al. (2016), and Moghani (2010) for background materials as well as basic computational techniques.

In Section 2, we introduce some necessary concepts and results, such as the maturity and Q-conjugacy character of a finite group. In Section 3, we introduce in our main theorem that by considering the *Q*-conjugacy character table there is a useful and general relation between these characters, their corresponding degrees, and the order of the group.

In some papers, we calculated the *Q*-conjugacy character table of a finite group by knowing its corresponding character table, see Moghani (2009), Moghani (2013), and Moghani (2016) but there was no way to do conversely so far. We provide a useful technique to calculate the character table of a group when its corresponding *Q*-conjugacy character table is given. Then, we evaluate our results in some useful examples.

2. Q-Conjugacy Relation

Definition 2.1.

Let G be an arbitrary finite group and $z_1, z_2 \in G$. We say z_1 and z_2 are Q-conjugate if there exists $t \in G$ such that $t^{-1} < z_1 > t = < z_2 >$. Fujita showed that it is an equivalence relation on group G and generates equivalence classes that are called dominant classes. Therefore, G is partitioned into dominant classes, see Fujita (2007).

Throughout this paper we adopt the same notations as in ATLAS of finite groups. For instance, we will use for an arbitrary conjugacy class of elements of order n the notation nX, where $X = a$, b, c, …. See Conway et al. (1985).

Definition 2.2.

(a) A class function on a finite group G over field F, is a function from G into F which is constant in conjugacy classes.

(b) A dominant class is defined as a disjoint union of conjugacy classes corresponding to the same cyclic subgroups, which is selected as a representative of conjugate cyclic subgroups. Moreover, the cyclic (dominant) subgroup selected from a non-redundant set of cyclic subgroups of G is used to compute the Q-conjugacy characters of G as demonstrated by Fujita.

Let G_i be a representative cyclic subgroup corresponding to a dominant class K of G, then, $|K| = (|G|\varphi(|G_i|))/|N_G(G_i)|$, where φ is the Euler function. All Q-conjugacy characters of a finite group are class functions. Furthermore, they are constant on dominant classes too, see Fujita (2007).

(c) Denote by *cl*(*G*) the set of all class functions on *G*, a Hermitian symmetric form (,) on *cl*(*G*) concerning the complex conjugation map on $F = \mathbb{C}$:

$$(\lambda,\chi) = \frac{\left(\sum_{g\in G} \lambda(g)\overline{\chi(g)}\right)}{|G|}, \text{ for } \lambda \text{ and } \chi \in cl(G).$$

A character χ is irreducible if and only if (χ, χ) = 1. The number of irreducible representations of a group is equal to the number of conjugacy classes for a group, see Narasimha et al. (2018).

(d) Suppose $\chi_1, \ldots,$ and χ_k are all the irreducible characters of a group *G*.

Let g_i be representatives of the conjugacy classes of G for i = 1, 2, …, k. Then,

$$\sum_{l=1}^{k} \chi_l\,(g_i)\overline{\chi_l(g_j)} = \delta_{ij}\frac{|G|}{c(g_j)} \text{ for all } g_i, g_j, \text{ and any } \chi_l.$$

The irreducible characters form an orthonormal basis for the space of class functions and an orthogonality relation for the rows of the character table, see Kerber (1999).

(e) The character table of a finite group *G* with irreducible characters λ_i for i = 1, 2, … *n*, satisfies the following relations:

(i) For each character χ and each $g \in G$: $\overline{\chi}(g) = \chi(g^{-1})$.

(ii) $|G| = \sum_{i=1}^{n} \lambda_i(1)^2$, wherein $\lambda_i(1)$ is called the degree of the irreducible character λ_i, see Narasimha et al. (2018) and Kerber (1999).

Definition 2.3.

Suppose H be a cyclic subgroup of a finite group G. Then, the maturity discriminant of H denoted by *m*(*H*), is an integer delineated by $|N_G(H): C_G(H)|$.

The dominant class of $K \cap H$ in the normalizer $N_G(H)$ is the union of $t = \frac{\varphi(n)}{m(H)}$ conjugacy classes of G, where φ is the Euler function, see Fujita (2007).

Definition 2.4.

Let $C(G)$ be the character table for an arbitrary finite group *G* with irreducible characters λ_i for i = 1, 2, …, *n*. The *Q*-Conjugacy character table of *G* denoted by C_G^Q with m number of integer-valued characters is calculated:

(a) The *Q*-Conjugacy character table of *G* is exactly equal with its corresponding character table i.e., $C_G^Q = C(G)$, if n = m, or t = 1 in Definition 2.3 and group *G* is called a matured group.

(b) If $C_G^Q \neq C(G)$, then G is called an unmatured group such that $m < n$. There is at least one integer-valued Q-conjugacy character like ω in the Q-conjugacy character table of G such that $\omega = \sum_{k=1}^{t_\omega} \lambda_k$ with reduction $t_\omega \geq 2$ in Definition 2.3 and $\lambda_1(1) = \lambda_2(1) = \ldots = \lambda_{t_\omega}(1)$.

From now on t_ω is called the reduction corresponding to the reducible character ω, see Fujita (1998) and Moghani (2016). Furthermore, the author introduced the following theorems to study the maturity property in the finite groups:

(c) The wreath product (similarly semi-direct product) of the matured groups again is a matured group, but the wreath product (similarly semi-direct product) of groups is an unmatured group, if at least one of the groups is unmatured, see Moghani (2009).

3. Theory

In this section, we introduce a new general relation between the Q-conjugacy characters and their corresponding reductions for an arbitrary finite group. Hence, the well-known theorem in the Character Tables in Section e (ii) of Definition 2.2 would be the special case of our main theorem.

Theorem 3.1.

Let G be a finite group and k be the number of unmatured characters in the Q-conjugacy character table of G (i.e., C_G^Q) with dimension m, like ω_j for $1 \leq j \leq k \leq m$, with reduction r_{i_j}, i.e., $\omega_j = \sum_{j=1}^{r_{i_j}} \lambda_{i_j}$, then,

$$\sum_{t=1}^{m} \omega_t(1)^2 = |G| + r_{i_1}(r_{i_1} - 1)\lambda_{i_1}{}^2(1) + r_{i_2}(r_{i_2} - 1)\lambda_{i_2}{}^2(1) + \ldots + r_{i_k}(r_{i_k} - 1)\lambda_{i_k}{}^2(1).$$

Proof:

First consider G to be a matured group, $k = 0$. According to Definition 2.4, for each character, the reduction is one and each Q-conjugacy character is an irreducible character and $m=n$, so the right side of our result is $|G| + 0$, so we have a well-known property in character table of finite group:

$$\sum_{i=1}^{n} \lambda_i(1)^2 = |G|.$$

Now let G be an unmatured group i.e., $C_G^Q \neq C(G)$. Hence, G is a finite group with irreducible characters λ_i in the character table $C(G)$ for $1 \leq i \leq n$ and the Q-conjugacy characters ω_l for $1 \leq l \leq m$ in the Q-conjugacy character table C_G^Q such that
$m < n$.

We prove the theorem by induction on k, the number of unmatured characters. Let $k = 1$, there are n-irreducible characters λ_i in $C(G)$ for $1 \leq i \leq n$ such that in C_G^Q: $\lambda_1, \lambda_2 \ldots \lambda_p$ are matured characters but $\omega = \sum_{i=p+1}^{n} \lambda_i$ with the reduction $r_\omega = n\text{-}p$ is an unmatured character such that $\lambda_{p+1}(1) = \lambda_{p+2}(1) = \ldots = \lambda_n(1)$.

In this case, according to the Definition 2.2 part e (ii), we try to simplify the left side of the equation:

$$\sum_{t=1}^{m} \omega_t(1)^2 = \lambda_1(1)^2 + \lambda_2(1)^2 + \ldots + \lambda_p(1)^2 + \omega(1)^2 = \sum_{i=1}^{n} \lambda_i(1)^2 + 2\sum_{p+1<i<j<n} \lambda_i(1)\lambda_j(1) = |G| + 2\sum_{p+1<i<j<n} \lambda_{p+1}^2(1) = |G| + 2\lambda_{p+1}^2(1)(n-p)(n-p+1)/2 = |G| + r_\omega(r_\omega - 1)\lambda_{p+1}^2(1).$$

Suppose for the number of k number of the unmatured characters like ω_j for $1 \leq j \leq k$ where, r_{i_j} is the reduction of $\omega_j = \sum_{j=1}^{r_{i_j}} \lambda_{i_j}$ such that $\lambda_{i_1}(1) = \lambda_{i_2}(1) = \cdots = \lambda_{i_{r_{i_j}}}(1)$ we have the following relation:

$$\sum_{t=1}^{m} \omega_t(1)^2 = |G| + r_{i_1}(r_{i_1} - 1)\lambda_{i_1}{}^2(1) + r_{i_2}(r_{i_2} - 1)\lambda_{i_2}{}^2(1) + \ldots + r_{i_k}(r_{i_k} - 1)\lambda_{i_k}{}^2(1).$$

Now, we prove the above relationship for the number of k+1 unmatured ω_j for $1 \leq j \leq k+1$.

A similar discussion shows that by plugging the induction hypothesis:

$$\sum_{t=1}^{m} \omega_t(1)^2 = |G| + r_{i_1}(r_{i_1} - 1)\lambda_{i_1}{}^2(1) + r_{i_2}(r_{i_2} - 1)\lambda_{i_2}{}^2(1) + \ldots + r_{i_k}(r_{i_k} - 1)\lambda_{i_k}{}^2(1) + 2\sum_{on\ \#\ of\ r_{i_{k+1}}} \lambda_{i_{k+1}}{}^2(1).$$

$$\sum_{t=1}^{m} \omega_t(1)^2 = |G| + r_{i_1}(r_{i_1} - 1)\lambda_{i_1}{}^2(1) + r_{i_2}(r_{i_2} - 1)\lambda_{i_2}{}^2(1) + \ldots + r_{i_{K+1}}(r_{i_{K+1}} - 1)\lambda_{i_{K+1}}{}^2(1).$$

We provide an additional useful tool for study on the *Q*-conjugacy characters. The method exactly works like what we often apply to calculate for the character table of an arbitrary finite group, especially we provide a first step to calculate directly the character table of a finite group *G* when its *Q*- conjugacy character table is given. Our future work would be research to find more techniques for the above target such as the orthogonality properties in the *Q*-conjugacy character tables similar to the well-known property in the Definition 2.2 part (d).

In this section, we consider some finite groups to evaluate our main result in Theorem 3.1.

Example 3.2.

The *Q*-conjugacy character table of C_p, the cyclic group of order *p* (*p* prime) introduced by Fujita (1998).
Consider Table 1, ω_2 is an unmatured character with the reduction *p* -1, and the corresponding dominant class D is the union of *p* -1 irreducible characters of degrees 1.

Table 1. The Q-conjugacy Character Table of C_p, Fujita (1998)

$C^{Q}_{C_p}$	*1a*	$D = \cup_{i=1}^{p-1} C_i^p$
ω_1	1	1
ω_2	p-1	-1

Hence, we have $\sum_{t=1}^{2} \omega_t(1)^2 = 1 + (p-1)^2 = p + (p-1)(p-2)$. It satisfies our result in Theorem 3.1.

Example 3.3:

The *Q*-conjugacy character of the projective linear group $L_2(8)$ of order 504 is introduced by the author Moghani (2016) in the Journal of Mathematics Research, see Table 2.

There are only two unmatured characters ω_3 and ω_5 with the reductions of three, corresponding two dominant classes $D_1 = 7a \cup 7b \cup 7c$ and $D_2 = 9a \cup 9b \cup 9c$.

Here, $|L_2(8)| = 504$. We have $\sum_{t=1}^{5} \omega_t(1)^2 = 1284 = 504 + 3(3-1)(7)^2 + 3(3-1)(9)^2$,

Table 2. The *Q*-conjugacy Character Table of $L_2(8)$

$C^{Q}_{L_2(8)}$	*1a*	*2a*	*3a*	*D₁*	*D₂*
ω_1	1	1	1	1	1
ω_2	7	-1	-2	0	1
ω_3	21	-3	3	0	0
ω_4	8	0	-1	1	-1
ω_5	27	3	0	-1	0

It satisfies our main result.

Example 3.4.

The Conway group Co_2 of order 495766656000 has sixty irreducible characters, see Conway (1985). According to Moghani (2016) in the Journal of Applications and Applied Mathematics, the Conway group Co_2 is an unmatured group.

For arbitrary $z_1, z_2 \in Co_2$. We say z_1 and z_2 are Q-conjugate if there exists $t \in Co_2$ such that $t^{-1} < z_1 > t = < z_2 >$. This is an equivalence relation on the group Co_2 and generates equivalence classes, called dominant classes.

By using the GAP (1995) program:

```
LogTo("Co2.txt");
        CharCo2:= CharacterTable("co2");
```

```
M:= Display(TableOfMarksConwayGroup(2));
Print("M");
Cong:=List(ConjugacyClassesSubgroups(co2),x->Elements(x));
Len:=Length(Cong);cyclic:=[];
for i in [1,2..Len]do
if IsCyclic(Cong[i][1])then Add(cyclic,i);
fi;od;
Display(CharCo2);Display(cyclic);
LogTo();
```

Hence, Co_2 is partitioned into the fifty-six dominant classes as follow:

$D_1 = 1a$, $D_2 = 2a$, $D_3 = 2b$, $D_4 = 2c$, $D_5 = 3a$, $D_6 = 3b$, $D_7 = 4a$, $D_8 = 4b$, $D_9 = 4c$, $D_{10} = 4d$, $D_{11} = 4e$, $D_{12} = 4f$, $D_{13} = 4g$, $D_{14} = 5a$, $D_{15} = 5b$, $D_{16} = 6a$, $D_{17} = 6b$, $D_{18} = 6c$, $D_{19} = 6d$, $D_{20} = 6e$, $D_{21} = 6f$, $D_{22} = 7a$, $D_{23} = 8a$, $D_{24} = 8b$, $D_{25} = 8c$, $D_{26} = 8d$, $D_{27} = 8e$, $D_{28} = 8f$, $D_{29} = 9a$, $D_{30} = 10a$, $D_{31} = 10b$, $D_{32} = 10c$, $D_{33} = 11a$, $D_{34} = 12a$, $D_{35} = 12b$, $D_{36} = 12c$, $D_{37} = 12d$, $D_{38} = 12e$, $D_{39} = 12f$, $D_{40} = 12g$, $D_{41} = 12h$, $D_{42} = 14a$, $D_{43} = 14b \cup 14c$, $D_{44} = 15a$, $D_{45} = 15b \cup 15c$, $D_{46} = 16a$, $D_{47} = 16b$, $D_{48} = 18a$, $D_{49} = 20a$, $D_{50} = 20a$, $D_{51} = 23a \cup 23b$, $D_{52} = 24a$, $D_{53} = 24b$, $D_{54} = 28a$, $D_{55} = 30a$, and $D_{56} = 30b \cup 30c$, such that $Co_2 = \bigcup_{i=1}^{56} D_i$.

Furthermore, there are exactly the fifty-six -conjugacy characters corresponding to the above dominant classes with the following degrees:

1, 23, 253, 275, 1771, 2024, 2277, 4025, 7084, 19250, 20790, 12650, 23000, 31625, 31625, 31878, 37422, 44275, 63256, 182250, 113850, 129536, 177100, 184437, 212520, 221375, 2266888, 478170, 245916, 253000, 284625, 312984, 368874, 398475, 398475, 430353, 442750, 462000, 467775, 558900, 637560, 664125, 664125, 664125, 853875, 1288000, 1291059, 1771000, 1771000, 1835008, 1943040, 1992375, 2004750, 2040192, 2072576, and 2095875.

In Table 3, all the integer-valued Q-conjugacy characters for the Conway group Co_2 are stored, see Safarisabet et al. (2013) for more details.

In Table 3, only φ_{10}, φ_{11}, φ_{20}, and φ_{28} are reducible unmatured characters with the same reductions that is two, we have

$$\sum_{t=1}^{56} \varphi_t(1)^2 = |Co_2| + 2\,(19250)^2 + 2\,(20790)^2 + 2\,(182250)^2 + 2\,(478170)^2.$$

It again satisfies our main result in theorem 3.1.

4. Conclusion

In this paper, we study the *Q*-conjugacy character table of an arbitrary finite group and introduce a general relation between degrees of the *Q*-conjugacy characters with their corresponding reductions. This could be accomplished by using the Hermitian symmetric form.

We provide a useful tool for study on the Q-conjugacy characters. The method exactly works like what we often apply to calculate for the character table of an arbitrary finite group, especially we provide a first step to calculate directly the character table of a finite group G when its Q-conjugacy character table is given. Our future work would be research to find more techniques for the above target such as the orthogonality properties in the Q-conjugacy character tables similar to the well-known property in the Definition 2.2 part (d).

Furthermore, we evaluated our main result on some useful examples like the projective linear group $L_2(8)$ and the sporadic Conway group Co_2 of orders 504 and 495766656000, respectively. Finally, by using GAP program, we computed all the dominant classes of the sporadic Conway group Co_2, enabling us to find all possible the Q-conjugacy characters for the Conway group Co_2.

Acknowledgment

The author is grateful for partial research support from William Paterson University by dear Dr. John Najarian, and referees for useful discussions and suggestions.

REFERENCES

Aschbacher, Michael (1997). Sporadic Groups, Cambridge University Press, UK.

Conway, John H., Curtis, Robert T., Norton, Simon P., Parker, Richard A. and Wilson, Robert A. (1985). ATLAS of finite groups. Oxford, Oxford Univ. Press UK.

Cotton, F.Albert (1971). Chemical Applications of Group Theory, Wiley-International, New York.

Fujita, Shinsaku (1991). Symmetry and Combinatorial Enumeration in Chemistry, Springer-Verlag, Berlin–Heidelberg.

Fujita, Shinsaku (1998). Inherent Automorphism and Q-Conjugacy Character Tables of Finite Groups. An Application to Combinatorial Enumeration of Isomers, Bulletin of the Chemical Society of Japan, Vol. 71. http://www2.chemistry.or.jp/journals/bcsj/bc-cont/bc71-10

Fujita, Shinsaku (2007). Diagrammatical Approach to Molecular Symmetry and Enumeration of Stereoisomers, MCM, Kragujevac.

GAP, (1995). Groups, Algorithms and Programming, Lehrstuhl De für Mathematik, RWTH, Aachen.

Hargittai, István. and Hargittai, Magdolna (1986). Symmetry through the Eyes of a Chemist, VCH, Weinheim.

Harris, Daniel C. and Bertolucci, Michael D. (1989).Symmetry and Spectroscopy, Dover, New York.

Kerber Adalbert (1999). Applied Finite Group Actions, Springer-Verlag, Berlin.

Kettle, Sidney F. A. (1985). Symmetry and Structure, Wiley, Chichester.

Ladd, Mark F. C. (1989). Symmetry in Molecules and Crystals, Ellis Horwood, Chichester.

Liu, Dongwen (2012). Q-Conjugacy classes, Proceedings of the American Mathematical Society vol. 40 (9). https://www.ams.org/journals/proc/2012-140-09/S0002-9939-2012-11213-4/S0002-9939-2012-11213-4

Lusztig, George (2014). On the character of certain modular irreducible representations, Representation Theory of the American Mathematical Society 19(2).

https://www.ams.org/journals/ert/2015-19-02/S1088-4165-2015-00463-7/

Moghani, Ali (2009). A New Simple Method for Maturity of Finite Groups and Application to Fullerenes and Fluxional Molecules, Bulletin of the Chemical Society of Japan, Vol 82. http://www.ci.nii.ac.jp/naid/130004152662

Moghani, Ali (2010). Study of Symmetries on some Chemical Nanostructures, Journal of Nano Research Vol 11, No.7. http://www.scientific.net/JNanoR.11.7

Moghani, Ali (2016). Study on the Q-Conjugacy Relations for the Janko Groups, International Journal of Applications and Applied Mathematics. Vol. 11, No 2 http://www.pvamu.edu/aam/previous-issues/vol-11-issue-2-december-2016/

Moghani, Ali (2016). Computing of Z-valued Characters for the special linear groups $L_2(2^m)$ and the Conway group Co_3, Journal of Mathematics Research Vol 8, No. 3. http://dx.doi.org/10.5539/jmr.v8n3p61

Moghani Ali and Najarian John, P. (2016). The Z-valued Characters for the Huge Symmetry of Hexamethylethane, Open Journal of Discrete Mathematics Vol 6, No.4. http://dx.doi.org/10.4236/ojdm.2016.64026

Narasimha Sastry, N.S. and Manoj Kumar, Y. (2018). Group Theory and Computation, Springer.

Namazi Javad, Moghani Ali and Najarian John P. (2019). Computational algebraic geometry theory for chemical structures. AMS, Joint Mathematics Meeting 2019. https://jointmathematicsmeetings.org/amsmtgs/2217_abstracts/1145-20-750.pdf

Safarisabet Shaban, A., Moghani Ali and Ghaforiadl Naser (2013). A Study on the Q-Conjugacy Characters of Some Finite Groups, International Journal of Theoretical Physics, Group Theory, and Nonlinear Optics Vol. 17, No.1. https://www.novapublishers.com/catalog/product_info.php?products_id=44982

APPENDIX

Table 3. The Q-conjugacy character table for the Conway group Co_2

$C^Q_{Co_2}$	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}
φ_1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	23	-9	7	-1	-4	5	7	-5	3	-1	3	-1
φ_3	253	29	13	-11	10	10	29	9	1	5	1	-3
φ_4	275	51	35	11	5	14	19	15	7	-5	7	3
φ_5	1771	-21	-21	11	-11	16	91	-5	-5	-5	-5	3
φ_6	2024	232	104	40	-1	26	8	24	24	8	8	8
φ_7	2277	-219	133	-11	9	36	21	-35	13	5	13	-3
φ_8	4025	-231	105	1	-25	29	105	-35	5	1	5	1
φ_9	7084	-84	-84	44	10	19	140	-4	-4	12	-4	-4
φ_{10}	19250	-910	210	-30	80	-10	-14	10	58	2	-6	-14
φ_{11}	20790	630	-42	-90	54	0	-42	-18	30	38	-2	6
φ_{12}	12650	554	330	26	-40	59	-6	50	2	10	18	2
φ_{13}	23000	600	280	120	50	5	184	40	8	24	8	8
φ_{14}	31625	265	-55	-55	35	35	377	25	-7	-7	-7	1
φ_{15}	31625	1385	505	145	35	35	41	45	53	1	5	25
φ_{16}	31878	-378	518	-26	45	45	-42	-26	-26	-10	22	-2

φ_{17}	37422	1134	462	-66	0	81	126	86	6	-18	6	-10
φ_{18}	44275	-1869	595	-29	-5	94	35	-85	59	-13	11	-5
φ_{19}	63256	-110	210	-110	-65	-20	322	-30	2	2	2	10
φ_{20}	182250	810	-630	90	0	0	-54	-30	18	-102	18	-6
φ_{21}	113850	954	-70	154	45	-9	138	10	10	-22	10	2
φ_{22}	129536	512	-512	0	-64	44	512	0	0	0	0	0
φ_{23}	177100	-1204	140	44	-20	-29	588	-20	12	-20	12	-20
φ_{24}	184437	2997	405	261	0	0	-27	-3	45	21	-3	-3
φ_{25}	212520	-2520	1064	40	111	120	-56	-104	-40	8	8	8
φ_{26}	221375	4095	735	15	-160	-25	-49	-25	87	-1	-9	7
φ_{27}	2266888	4480	896	128	-4	68	0	64	64	0	0	0
φ_{28}	478170	-5670	-294	90	-216	0	-294	90	90	-38	-6	-6
φ_{29}	245916	3996	1308	156	0	81	-36	76	12	-36	12	-4
φ_{30}	253000	2120	-440	200	-125	10	104	-40	24	40	8	-24
φ_{31}	284625	-3855	1505	-55	-90	45	273	-115	5	-7	5	1
φ_{32}	312984	-1512	-168	120	0	81	504	-56	-24	24	-24	8
φ_{33}	368874	810	-630	-198	0	0	378	-30	18	42	18	18
φ_{34}	398475	-6741	1435	-61	-45	36	-21	-25	95	19	-1	-21
φ_{35}	398475	4011	1435	259	-45	36	-21	55	-17	19	-1	27
φ_{36}	430353	-5103	273	177	0	81	-63	-31	33	33	-15	-7
φ_{37}	442750	-770	1470	-130	-185	40	-210	30	-66	46	14	6
φ_{38}	462000	5040	560	-400	30	120	112	80	16	48	-16	-16
φ_{39}	467775	-3969	735	111	0	-81	63	55	39	15	-9	-17
φ_{40}	558900	2484	1140	276	0	-81	468	20	-12	-12	-12	4
φ_{41}	637560	5880	952	280	90	-45	-168	-56	40	56	8	8
φ_{42}	664125	-1155	-35	-275	195	-30	637	85	5	13	5	-3
φ_{43}	664125	-1155	-35	365	195	-30	-35	5	-11	45	5	-3
φ_{44}	664125	2205	1645	-315	195	-30	77	-15	-39	37	-7	-11
φ_{45}	853875	7155	435	-45	135	0	-237	-45	51	-77	-13	3
φ_{46}	1288000	-2240	-2240	320	100	100	448	0	0	-64	0	0
φ_{47}	1291059	2835	1827	27	0	0	-189	39	-81	-69	15	-21
φ_{48}	1771000	-12040	1400	-200	205	-20	-168	40	40	-40	-8	24
φ_{49}	1771000	1400	-840	-40	-200	115	-168	40	-56	-8	-24	8
φ_{50}	1835008	0	0	0	-128	-128	0	0	0	0	0	0
φ_{51}	1943040	7680	512	0	-96	-60	512	0	0	0	0	0
φ_{52}	1992375	3255	-1225	215	180	45	-441	-25	-25	39	-9	-17
φ_{53}	2004750	8910	-1170	-450	0	0	-162	-90	54	-18	6	6
φ_{54}	2040192	-2688	896	128	-36	-36	0	-64	-64	0	0	0
φ_{55}	2072576	-8192	0	0	-160	-16	0	0	0	0	0	0
φ_{56}	2095875	-3645	-2205	-45	0	0	-189	75	27	51	27	3

$C^{Q}_{Co_2}$	D_{13}	D_{14}	D_{15}	D_{16}	D_{17}	D_{18}	D_{19}	D_{20}	D_{21}	D_{22}	D_{23}	D_{24}	D_{25}	D_{26}
φ_{1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_{2}	-1	-2	3	4	0	3	-3	1	-1	2	-1	-3	3	-1
φ_{3}	1	3	3	10	2	2	2	-2	-2	1	-3	3	5	1
φ_{4}	-1	0	5	5	-3	6	6	2	2	2	3	5	3	-1
φ_{5}	-1	-4	1	21	-3	-6	0	0	2	0	3	-1	7	-1
φ_{6}	0	-1	4	-1	7	4	10	2	4	1	0	4	0	0
φ_{7}	1	2	7	1	-3	6	-12	4	-2	2	-3	-7	1	1
φ_{8}	1	0	5	15	3	9	-3	-3	1	0	1	-5	5	1
φ_{9}	-4	9	-1	18	6	-9	3	3	-1	0	4	0	4	4
φ_{10}	10	0	0	0	8	-10	-10	6	6	0	2	6	2	-6
φ_{11}	-10	-10	0	6	-18	0	0	0	0	0	6	2	-2	6
φ_{12}	6	0	5	0	-4	-1	11	3	-1	1	2	6	-2	-2
φ_{13}	0	0	0	10	6	15	-3	1	3	-2	8	0	0	0
φ_{14}	5	0	0	35	-5	-5	-5	-1	-1	-1	-7	1	5	-3
φ_{15}	5	0	0	-5	-1	5	11	7	1	-1	1	-1	1	5
φ_{16}	-6	3	3	5	9	-15	-3	5	1	0	-2	-2	-6	2
φ_{17}	-6	-3	7	0	0	9	9	-3	-3	0	-2	6	2	2
φ_{18}	-1	0	5	-5	3	6	-18	-2	-2	0	3	-5	-1	-1
φ_{19}	-10	0	0	15	7	10	4	0	-2	-2	-6	2	-2	6
φ_{20}	10	0	0	0	0	0	0	0	0	-2	-6	6	2	10

φ_{21}	6	0	-5	5	9	9	-9	-1	1	2	2	-2	-2	6
φ_{22}	0	-14	-4	16	8	-16	-4	4	0	1	0	0	0	0
φ_{23}	4	0	-5	20	-16	11	11	-1	-1	0	4	0	-4	-4
φ_{24}	9	12	-3	0	0	0	0	0	0	1	-3	-3	1	1
φ_{25}	0	-5	0	-1	-9	0	0	-4	4	0	0	-4	0	0
φ_{26}	-5	0	0	0	0	-15	15	3	-3	0	-1	-9	3	3
φ_{27}	0	13	-2	-4	-20	10	4	-4	2	0	0	0	0	0
φ_{28}	10	20	0	24	0	0	0	0	0	0	-6	-6	-6	-6
φ_{29}	-4	-9	1	0	0	-9	9	-3	3	-1	-4	0	-4	-4
φ_{30}	0	0	0	-5	23	-10	2	-2	2	-1	0	4	0	0
φ_{31}	5	0	0	-10	6	15	-3	5	-1	-2	-7	3	1	-3
φ_{32}	0	9	4	0	0	-9	9	-3	3	0	8	0	0	0
φ_{33}	-10	24	-6	0	0	0	0	0	0	2	-6	6	-2	-2
φ_{34}	-9	0	-5	-5	-9	-6	-12	4	2	0	3	5	3	-1
φ_{35}	-1	0	-5	-5	15	6	-12	4	-2	0	3	1	3	-1
φ_{36}	5	3	-2	0	0	9	9	-3	-3	0	1	-3	-3	5
φ_{37}	10	0	0	15	-5	-20	-8	0	-4	0	6	2	-6	2
φ_{38}	0	0	0	-10	18	0	0	-4	-4	0	0	0	0	0
φ_{39}	-1	0	5	0	0	-9	-9	3	3	0	-1	3	-1	7
φ_{40}	-4	0	5	0	0	9	-9	3	-3	-1	4	0	-4	-4
φ_{41}	0	-15	0	10	-6	-15	3	-5	1	0	-8	0	0	0
φ_{42}	5	0	0	-5	15	0	-6	-2	4	0	-3	-7	-3	5
φ_{43}	5	0	0	-5	15	0	-6	-2	-4	0	-3	5	5	-3
φ_{44}	5	0	0	-5	-9	0	18	10	0	0	-3	-5	1	-3
φ_{45}	-5	0	0	15	27	0	0	0	0	1	3	-1	-5	-5
φ_{46}	0	0	0	-20	-8	-20	4	4	-4	0	0	0	0	0
φ_{47}	-5	9	-6	0	0	0	0	0	0	0	3	-3	7	3
φ_{48}	0	0	0	5	-7	-10	-4	-4	-2	0	0	4	0	0
φ_{49}	0	0	0	0	-4	5	13	3	5	0	-8	0	0	0
φ_{50}	0	8	8	0	0	0	0	0	0	0	0	0	0	0
φ_{51}	0	-10	0	-16	-24	0	-12	-4	0	1	0	0	0	0
φ_{52}	-5	0	0	20	-12	15	-3	5	-1	0	-1	3	-5	3
φ_{53}	10	0	0	0	0	0	0	0	0	-1	6	6	2	2
φ_{54}	0	17	2	-4	12	-6	12	-4	2	0	0	0	0	0
φ_{55}	0	-24	-4	0	16	16	16	0	0	2	0	0	0	0
φ_{56}	-5	0	0	0	0	0	0	0	0	-2	3	-9	3	-5

Table 3 (Cont.)

$C^{Q}_{Co_2}$	D_{27}	D_{28}	D_{29}	D_{30}	D_{31}	D_{32}	D_{33}	D_{34}	D_{35}	D_{36}	D_{37}	D_{38}	D_{39}	D_{40}
φ_{1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_{2}	1	1	2	2	1	-1	1	-2	0	1	-3	2	1	0
φ_{3}	-1	-1	1	3	-1	-1	0	2	-2	2	4	2	0	-2
φ_{4}	1	1	2	0	1	1	0	1	1	-2	4	1	0	1
φ_{5}	-1	-1	-2	4	-1	1	0	1	1	4	-2	1	-2	1
φ_{6}	4	0	-1	-1	2	0	0	-1	3	2	0	-1	0	-1
φ_{7}	1	1	0	-2	1	-1	0	3	1	0	-2	-1	-2	1
φ_{8}	-1	-1	2	0	-1	1	-1	-3	-1	-3	-7	1	1	-1
φ_{9}	0	0	-2	1	1	-1	0	-4	2	-1	-1	0	-1	2
φ_{10}	-2	-2	2	0	0	0	0	4	-8	-2	-2	-4	-2	0
φ_{11}	-6	2	0	-2	0	0	0	-6	6	0	0	2	0	-1
φ_{12}	-2	2	-1	0	-1	1	0	-6	-4	3	-1	-2	-1	0
φ_{13}	0	0	2	0	0	0	-1	4	2	1	5	0	1	2
φ_{14}	1	1	-1	0	0	0	0	-1	-1	-1	5	-1	1	-1
φ_{15}	3	-1	-1	0	0	0	0	5	-1	-1	-1	1	3	-1
φ_{16}	-2	2	0	3	-3	-1	0	3	1	-3	1	-1	1	1
φ_{17}	-2	-2	0	-3	-1	-1	0	0	0	-3	3	0	-1	0
φ_{18}	3	-1	-2	0	1	1	0	-1	-1	2	2	-1	2	-1
φ_{19}	2	2	1	0	0	0	0	7	-1	4	-4	-1	0	-1

φ_{20}	-2	-2	0	0	0	0	2	0	0	0	0	0	0	0
φ_{21}	-2	-2	0	0	-1	-1	0	3	1	3	1	-1	1	1
φ_{22}	0	0	-1	-2	2	0	0	8	0	-4	0	0	0	0
φ_{23}	0	0	1	0	1	-1	0	-6	0	3	-3	-2	1	0
φ_{24}	5	1	0	0	-3	1	0	0	0	0	0	0	0	0
φ_{25}	-4	0	0	-1	0	0	0	7	-1	4	2	-1	-2	-1
φ_{26}	-1	-1	2	0	0	0	0	-4	0	-1	3	-4	-1	0
φ_{27}	0	0	-1	1	0	-2	0	0	4	0	-2	0	-2	0
φ_{28}	-6	2	0	-4	0	0	0	12	0	0	0	4	0	0
φ_{29}	0	0	0	3	1	1	0	0	0	-3	-3	0	1	0
φ_{30}	4	0	1	0	0	0	0	5	3	2	0	1	-4	-1
φ_{31}	-1	-1	0	0	0	0	0	-6	2	-3	-1	2	-1	2
φ_{32}	0	0	0	-3	-2	0	1	0	0	-3	-3	0	1	0
φ_{33}	-2	-2	0	0	0	2	0	0	0	0	0	0	0	0
φ_{34}	1	1	0	0	-1	-1	0	-3	-1	0	2	1	2	-1
φ_{35}	-3	1	0	0	1	-1	0	-3	-5	0	-2	1	-2	-1
φ_{36}	-3	1	0	3	2	2	0	0	0	-3	3	0	-1	0
φ_{37}	2	-2	1	0	0	0	0	-3	3	0	0	1	0	-1
φ_{38}	0	0	0	0	0	0	0	4	-2	4	-2	0	2	2
φ_{39}	-5	-1	0	0	1	1	0	0	0	3	-3	0	1	0
φ_{40}	0	0	0	0	-1	1	1	0	0	3	3	0	-1	0
φ_{41}	0	0	0	-3	0	0	0	-6	-2	3	1	2	1	2
φ_{42}	1	1	0	0	0	0	0	-11	-1	-2	2	1	-2	-1
φ_{43}	-3	1	0	0	0	0	0	1	7	-2	-2	-3	2	-1
φ_{44}	-1	-1	0	0	0	0	0	5	3	2	0	1	0	-1
φ_{45}	-1	-1	0	0	0	0	0	-3	3	0	0	1	0	-1
φ_{46}	0	0	1	0	0	0	1	-2	0	4	0	2	0	0
φ_{47}	1	1	0	-3	0	2	0	0	0	0	0	0	0	0
φ_{48}	4	0	1	0	0	0	0	3	1	0	-2	-1	-2	-1
φ_{49}	0	0	1	0	0	0	0	-6	4	3	1	-2	1	0
φ_{50}	0	0	-2	0	0	0	-1	0	0	0	0	0	0	0
φ_{51}	0	0	0	2	0	0	0	8	0	-4	0	0	0	0
φ_{52}	3	-1	0	0	0	0	0	0	-4	-3	-1	0	1	0
φ_{53}	-2	0	0	0	0	0	0	0	0	0	0	0	0	0
φ_{54}	0	0	0	1	2	-2	0	0	-4	0	2	0	2	0
φ_{55}	0	0	-1	0	-2	0	0	0	0	0	0	0	0	0
φ_{56}	-1	-1	0	0	0	0	1	0	0	0	0	0	0	0

Table 3 (Cont.)

$C^{Q}_{Co_2}$	D_{41}	D_{42}	D_{43}	D_{44}	D_{45}	D_{46}	D_{47}	D_{48}	D_{49}	D_{50}	D_{51}	D_{52}	D_{53}	D_{54}	D_{55}	D_{56}
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	-1	-2	0	0	1	1	-1	0	0	-1	0	-1	0	0	0	-2
φ_3	0	1	-1	0	0	1	-1	-1	-1	1	0	0	0	0	1	2
φ_4	0	2	0	-1	0	-1	1	0	0	-1	-1	0	-1	-1	-2	1
φ_5	0	0	0	1	-1	1	1	0	0	-1	0	0	-1	-1	0	-1
φ_6	2	1	-1	1	-1	0	0	1	-1	0	0	0	1	1	1	-1
φ_7	0	-2	0	1	-1	-1	-1	0	0	1	0	0	-1	-1	0	1
φ_8	1	0	0	-1	0	-1	1	0	0	1	0	1	1	1	0	-1
φ_9	-1	0	0	-1	0	0	0	0	1	1	0	1	0	0	0	1
φ_{10}	-1	0	0	0	0	2	-2	1	0	0	-1	2	0	0	0	0
φ_{11}	0	0	0	0	-1	2	-2	0	2	0	-2	0	2	2	0	0
φ_{12}	-1	1	1	-1	0	0	0	-1	0	1	0	-1	0	0	1	-1
φ_{13}	-1	-2	0	0	0	0	0	0	0	0	0	-1	0	0	2	0
φ_{14}	1	-1	1	0	0	-1	-1	1	0	0	0	-1	1	1	-1	0
φ_{15}	1	-1	1	0	0	1	-1	-1	0	0	0	1	-1	-1	-1	0
φ_{16}	1	0	0	0	0	0	0	0	-1	-1	0	1	1	1	0	0
φ_{17}	-1	0	0	1	0	0	0	0	1	-1	1	1	0	0	0	-1
φ_{18}	-2	0	0	-1	0	1	1	0	0	-1	0	0	1	1	0	1
φ_{19}	-2	2	0	0	0	0	0	0	1	0	0	0	-1	-1	0	0

$\boldsymbol{\varphi_{20}}$	0	-2	0	0	0	-2	2	0	0	0	-2	0	0	0	2	0
$\boldsymbol{\varphi_{21}}$	-1	2	0	1	0	0	0	0	0	1	0	-1	1	1	-2	-1
$\boldsymbol{\varphi_{22}}$	0	1	-1	-1	1	0	0	-1	0	0	0	0	0	0	1	-1
$\boldsymbol{\varphi_{23}}$	1	0	0	1	0	0	0	-1	0	-1	0	1	0	0	0	1
$\boldsymbol{\varphi_{24}}$	0	1	-1	0	0	-1	-1	0	2	-1	0	0	0	0	1	0
$\boldsymbol{\varphi_{25}}$	2	0	0	0	1	0	0	0	1	0	0	0	-1	-1	0	0
$\boldsymbol{\varphi_{26}}$	1	0	0	0	0	1	1	0	0	0	0	-1	0	0	0	0
$\boldsymbol{\varphi_{27}}$	0	0	0	-2	1	0	0	1	-1	0	0	0	0	0	0	0
$\boldsymbol{\varphi_{28}}$	0	0	0	0	-1	2	2	0	0	0	0	0	0	0	0	0
$\boldsymbol{\varphi_{29}}$	-1	-1	-1	1	0	0	0	0	1	1	0	-1	0	0	-1	1
$\boldsymbol{\varphi_{30}}$	0	-1	1	0	0	0	0	-1	0	0	0	0	1	1	-1	0
$\boldsymbol{\varphi_{31}}$	1	2	0	0	0	1	-1	0	0	0	0	-1	0	0	0	0
$\boldsymbol{\varphi_{32}}$	-1	0	0	1	0	0	0	0	-1	0	0	-1	0	0	0	1
$\boldsymbol{\varphi_{33}}$	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\boldsymbol{\varphi_{34}}$	0	0	0	1	0	-1	1	0	0	1	0	0	-1	-1	0	-1
$\boldsymbol{\varphi_{35}}$	0	0	0	1	0	-1	1	0	0	-1	0	0	1	1	0	1
$\boldsymbol{\varphi_{36}}$	-1	0	0	1	0	-1	-1	0	-1	0	0	1	0	0	0	-1
$\boldsymbol{\varphi_{37}}$	0	0	0	0	0	0	0	1	0	0	0	0	-1	-1	0	0
$\boldsymbol{\varphi_{38}}$	2	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0
$\boldsymbol{\varphi_{39}}$	1	0	0	-1	0	-1	-1	0	0	-1	1	-1	0	0	0	1
$\boldsymbol{\varphi_{40}}$	1	-1	-1	-1	0	0	0	0	0	1	0	1	0	0	-1	-1
$\boldsymbol{\varphi_{41}}$	-1	0	0	0	0	0	0	0	-1	0	0	1	0	0	0	0
$\boldsymbol{\varphi_{42}}$	0	0	0	0	0	1	1	0	0	0	0	0	-1	-1	0	0
$\boldsymbol{\varphi_{43}}$	0	0	0	0	0	1	1	0	0	0	0	0	-1	-1	0	0
$\boldsymbol{\varphi_{44}}$	-2	0	0	0	0	-1	1	0	0	0	0	0	1	1	0	0
$\boldsymbol{\varphi_{45}}$	0	1	1	0	0	-1	-1	0	0	0	0	0	-1	-1	1	0
$\boldsymbol{\varphi_{46}}$	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
$\boldsymbol{\varphi_{47}}$	0	0	0	0	0	1	-1	0	-1	0	0	0	0	0	0	0
$\boldsymbol{\varphi_{48}}$	0	0	0	0	0	0	0	-1	0	0	0	0	1	1	0	0
$\boldsymbol{\varphi_{49}}$	-1	0	0	0	0	0	0	-1	0	0	0	1	0	0	0	0
$\boldsymbol{\varphi_{50}}$	0	0	0	2	2	0	0	0	0	0	-1	0	0	0	0	0
$\boldsymbol{\varphi_{51}}$	0	1	1	0	-1	0	0	0	0	0	0	0	0	0	1	0
$\boldsymbol{\varphi_{52}}$	1	0	0	0	0	1	1	0	0	0	0	-1	0	0	0	0
$\boldsymbol{\varphi_{53}}$	0	-1	-1	0	0	0	0	0	0	0	1	0	0	0	-1	0
$\boldsymbol{\varphi_{54}}$	0	0	0	-1	-1	0	0	0	1	0	0	0	0	0	0	-1
$\boldsymbol{\varphi_{55}}$	0	-2	0	-1	0	0	0	1	0	0	0	0	0	0	0	1
$\boldsymbol{\varphi_{56}}$	0	2	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0