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On an Ecological Model of Mutualisim Between Two Species With a Mortal Predator

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Abstract

In this paper, we study an ecological model of a three-space food chain consists of two logically growing mutual species and third species acts as a predator to second mutual species with Holling type II functional response. This model is constituted by a system of nonlinear decoupled ordinary differential equations. By using perturbed method, we identify the nature of the system at each equilibrium point and also global stability is investigated for this model using Lypanov function at the possible equilibrium points.

Keywords: Prey-predator; Holling type-II response function; Local and global stability; Hopf bifurcation

MSC 2010 No.: 93A30, 37B25

1. Introduction

Many applications of real world problems are expressed in ecological models. The study of ecological models has become a central role of mathematics and created much interest among authors. The classical models of food chain with only two trophic levels are discussed by many researchers and scientists. The two species models are consider in many applications with only one type of interaction at a time: prey-predation, mutualism or competition and they show these models are to be insufficient to produce realistic dynamic (Chauvet et al. (2002); Hsu et al. (2003); Freedman and Waltman (1977); Hastings and Powell (1991); Klebanoff and Hastings (1994); Mada et al. (2011); Thota (2019); Thota (2020)). The prey-predator models with Holling types I, II, III and IV are discussed in Chen et al. (2012); Seo and DeAngelis (2011); Peng et al. (2009); Yu (2012); Huang et al. (2011); Naji and Shalan (2011). Therefore, in this paper, we focused on the dynamics of a three-space food chain consists of two logically growing competing species and third species acts as a predator with Holling type II functional response.

The rest of the paper is organized as follows: Section 2 presents the mathematical formulation of the proposed model; in Section 3, we discuss the stability analysis of the model in both locally and globally.

There are different types of interactions available in nature. For example, the Prey-Predation, Competition, Mutualism, Commensalism, Ammensalism, Parasitism and so on. Among all species around whose individuals have a different life style that can be divided into different stages. Since then great work have been done on mutualism interaction or multi-interactions of the species which includes the mutualism interaction among two or three species in ecological systems.

A mutualistic association between two or more species represents a relationship in which all of them experience a positive effect from their interactions, consisting in an increase of their ability to survive, grow or reproduce. Ants, butterflies, caterpillars and Acacia (small tree) are examples of a three way interaction. Ants, butterflies, caterpillars and an Acacia are beneficial to each other, i.e., some protections are provided for both Acacia plant and caterpillars by ants, Acacia flowers are helped by the ants and caterpillars in pollination, the caterpillars have nectar organs from which the ants drink nectar, a number of benefits are provided to the ants by Acacia in terms of shelter, protection and nectar, and also nectar to caterpillars (Dhakne and Munde (2012); Suresh Kumar et al. (2019)).

2. Model Formulation

In the proposed model, suppose the non-dimensional population density of the prey is X at time t, the population density of the predator is Y at time t and the population density of the host is Z at time t. Now the ecological setup of food chain involving three species is shown in Figure 1. The species X and Y are the two logistically growing competing species with intrinsic growth rates and carrying capacities r_i and k_i $(i = 1, 2)$, respectively. The species Z is predating over Y with Holling type-II response. The proportion p of species Y is refuge from predation. The coefficients a_{ij} $(i, j = 1, 2, i \neq j)$ is the inter-species competition coefficients.

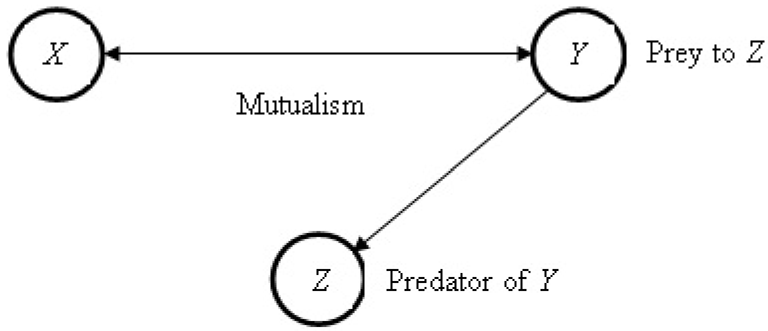

Figure 1. Schematic diagram of 3 spaces food-chain system

2.1. Assumptions

Suppose $X(T), Y(T)$ and $Z(T)$ are the densities of the two mutual species and a predator respectively at any time T. The predator Z feeds on the first mutual species X according to Holling type-II functional response.

(1) The parameters r_1 and r_2 are intrinsic growth rates of two mutual species X and Y, and also assumed that the growth of the mutual species are logistic.
(2) The parameters k_1 and k_2 are the carrying capacities of two mutual species X and Y respectively.
(3) The parameters a_{12} and a_{21} are the supporting constants for both mutual species X and Y respectively.
(4) The parameter ca denotes the attack rate of the single predator search for single mutual species.
(5) The parameter p of species X is refuge from predator.
(6) The parameter b is half saturation level of the predator over the first mutual species.
(7) The parameter e is the natural death rate coefficient of the predator.
(8) The term represents the functional response for grazing of the first mutual species by the predator. This functional response is called Holling type-II functional response represents the rate at which the predator consumes the mutual species X. Here, b is half saturation constant for a Holling type-II.

Now, the model equation for the multi-interaction among three species (in which two species interacting mutually themselves and a mortal predator, which consumes the first mutual species in terms of Holling type-II functional response) is given by the following system of non-linear decoupled ordinary differential equations:

$$\begin{aligned} \frac{dX}{dT} &= r_1 X \left(1 - \frac{X}{k_1} + \frac{a_{12}Y}{k_1}\right), \\ \frac{dY}{dT} &= r_2 Y \left(1 - \frac{Y}{k_2} + \frac{a_{21}X}{k_2}\right), \\ \frac{dZ}{dT} &= Z \left(-e + \frac{ca(1-p)Y}{b+(1-p)Y}\right), \end{aligned} \tag{1}$$

with the non-negative initial conditions are $X(0) \geq 0, Y(0) \geq 0, Z(0) \geq 0$. The following non-dimensional variables and parameters will help us to write the above set of equations in a simple form:

$$t = r_1 T,\ \ x = \frac{X}{k_1},\ \ y = \frac{Y}{k_2},\ \ z = \frac{aZ}{r_1 k_1},\ \ \alpha_{12} = \frac{a_{12}k_2}{k_1}, \alpha_{21} = \frac{a_{21}k_1}{k_2},\ \ v_1 = \frac{b}{k_1},$$
$$v_2 = \frac{e}{r_1},\ \ v_3 = \frac{ac}{r_1},\ \ r = \frac{r_2}{r_1}.$$

The corresponding non-dimensional equations of the above system (1) and associated initial conditions are

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - x + \alpha_{12} y\right) = xF_1(x,y,z), \\ \frac{dy}{dt} &= ry\left(1 - y + \alpha_{21} x\right) - \frac{(1-p)yz}{v_1 + (1-p)y} = yF_2(x,y,z), \\ \frac{dz}{dt} &= z\left(-v_2 + \frac{v_3(1-p)y}{v_1 + (1-p)y}\right) = zF_3(x,y,z), \end{aligned} \tag{2}$$

and $x(0) \geq 0, y(0) \geq 0, z(0) \geq 0$.

3. Analysis

In this section, we discuss the local and global stability analysis of the proposed model in Equation (2).

3.1. Equilibrium Points

The equilibrium points of the system are necessary for the purpose of studying the local stability nature of the ecological model. The system under investigation has the following six equilibrium points:

(1) Fully washed state or extent state: $E_1 = (0,0,0)$,
(2) Only first mutual species survival state: $E_2 = (0,1,0)$,
(3) Only second mutual species survival state: $E_3 = (1,0,0)$,
(4) Predator washed state: $E_4 = (\widehat{x}, \widehat{y}, 0)$, where

$$\widehat{x} = \frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}},\quad \widehat{y} = \frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}.$$

This exists only when $\alpha_{12}\alpha_{21} < 1$.

(5) Second mutual species washed state: $E_5 = (0,\ \bar{y}, \bar{z})$ where

$$\begin{aligned} \bar{y} &= \frac{v_1 v_2}{(1-p)(v_3 - v_2)}, \\ \bar{z} &= \frac{r v_1 v_3[(1-p)(v_3 - v_2) - v_1 v_2]}{(1-p)^2 (v_3 - v_2)^2}. \end{aligned}$$

This exists when $v_3 > v_2$ and $(v_3 - v_2) > v_1 v_2$.

(6) Coexistence state: $E_6 = (\,x^*, y^*, z^*)$ where

$$x^* = 1 + \frac{\alpha_{12} v_1 v_2}{(1-p)(v_3 - v_2)},$$
$$y^* = \frac{v_1 v_2}{(1-p)(v_3 - v_2)},$$
$$z^* = \frac{y v_1 v_3}{(1-p)^2 (v_3 - v_2)^2}((1-p)(v_3 - v_2)(\alpha_{21}+1) - v_1 v_2 (1 - \alpha_{12}\alpha_{21})).$$

This exists when $(1-p)(v_3 - v_2)(\alpha_{21}+1) > v_1 v_2 (1 - \alpha_{12}\alpha_{21})$ and $v_3 > v_2$.

3.2. Existence and Stability Analysis of Equilibrium Points

The Jacobin matrix for the system (2) at equilibrium point $E = (x, y, z)$ is given by

$$J_E = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & J_{23} \\ 0 & J_{32} & J_{33} \end{pmatrix}, \tag{3}$$

where

$$J_{11} = \alpha_{12} y - 2x + 1, \quad J_{12} = x\alpha_{12}, \quad J_{21} = ry\alpha_{21},$$
$$J_{22} = r\left(\alpha_{21} x - y + 1\right) - ry - \frac{(1-p)\,z}{v_1 + (1-p)\,y} + \frac{(1-p)^2 yz}{(v_1 + (1-p)\,y)^2},$$
$$J_{23} = -\frac{(1-p)\,y}{v_1 + (1-p)\,y},$$
$$J_{32} = \frac{v_3\,(1-p)}{v_1 + (1-p)\,y} - \frac{v_3 (1-p)^2 y}{(v_1 + (1-p)\,y)^2},$$
$$J_{33} = -v_2 + \frac{v_3\,(1-p)\,y}{v_1 + (1-p)\,y}.$$

Theorem 3.1.

The system is always exists and unstable at the equilibrium points E_1, E_2, E_3.

Proof:

(1) The Eigenvalues for the extinct equilibrium point $E_1 = (0, 0, 0)$ are $1, r, -v_2$ and hence the equilibrium point is saddle point. Therefore, the dynamical system is unstable.
(2) The second axial equilibrium point $E_2 = (0, 1, 0)$ is also saddle point because of one Eigenvalue, $1 + \alpha_{12}$, is positive and remaining two, $-r$, $\frac{v_1 v_2 - (1-p)(v_3 - v_2)}{p - 1 - v_1}$, are negative.
(3) The axial equilibrium point $E_3 = (1, 0, 0)$ always exists and its Eigenvalues are $-1, -v_2, r(1+\alpha_{21})$, and so it also is a saddle point in any case. The system is also unstable locally at this fixed point.

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Theorem 3.2.

The boundary steady state E_4 is always stable in xy-direction.

Proof:

In the absence of species z (which is predator for y species), the system reduces to

$$\begin{aligned}\frac{dx}{dt} &= x\left(1 - x + \alpha_{12} y\right),\\ \frac{dy}{dt} &= ry\left(1 - y + \alpha_{21} x\right),\end{aligned} \tag{4}$$

and $x(0) \geq 0, y(0) \geq 0$. Local stability analysis for the system (4) gives the following results. The equilibrium points of system (4) under investigation are

$$(0,0), (0,1), (1,0), \text{ and } \left(\frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}}, \frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}\right),$$

and the Jacobin matrix for the system (4) at equilibrium point $E = (x, y)$ is

$$J = \begin{pmatrix} 1 - 2x + \alpha_{12} & x\alpha_{12} \\ ry\alpha_{21} & r(\alpha_{21}x - y + 1) - ry \end{pmatrix}.$$

The fully washed state $(0,0)$ is always exists and it is a unstable point, and the equilibrium points $(0,1), (1,0)$ are always exist and these are always saddle points. The nontrivial positive equilibrium point is $(\widehat{x}, \widehat{y})$, where

$$\widehat{x} = \frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}}, \quad \widehat{y} = \frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}.$$

The associated Jacobian matrix at $(\widehat{x}, \widehat{y})$ is

$$J_1 = \begin{pmatrix} -\frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}} & \frac{\alpha_{12}(1+\alpha_{12})}{1-\alpha_{12}\alpha_{21}} \\ \frac{r\alpha_{21}(1+\alpha_{21})}{1-\alpha_{12}\alpha_{21}} & -\frac{r(1+\alpha_{21})}{1-\alpha_{12}\alpha_{21}} \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 is

$$\lambda^2 + \frac{\alpha_{21}r + \alpha_{12} + r + 1}{1-\alpha_{12}\alpha_{21}}\lambda + \frac{r(1+\alpha_{21})(1+\alpha_{21})}{1-\alpha_{12}\alpha_{21}} = 0.$$

Comparing with $\lambda^2 + a_0\lambda + a_1 = 0$, we get

$$\begin{aligned} a_0 &= \frac{\alpha_{21}r + \alpha_{12} + r + 1}{1-\alpha_{12}\alpha_{21}} > 0, \text{ and} \\ a_1 &= \frac{r(1+\alpha_{21})(1+\alpha_{21})}{1-\alpha_{12}\alpha_{21}} > 0. \end{aligned}$$

According to Routh-Hurwitz criteria, the system is stable. Hence the system is unstable in z-direction and stable in xy-direction. ■

Theorem 3.3.

The equilibrium point $E_4 = (\widehat{x}, \widehat{y}, 0)$ is globally asymptotically stable in the interior R_+^2 of the $xy-$plane.

Proof:

For any vale in interior R_+^2 of the $xy-$plane, the system reduces to following subsystem in the interior of $xy-$plane

$$\begin{aligned}\frac{dx}{dt} &= x\left(1 - x + \alpha_{12}y\right) = g_1(x, y),\\ \frac{dy}{dt} &= ry\left(1 - y + \alpha_{21}x\right) = g_2(x, y).\end{aligned}$$

Now, assume that $M_1(x, y) = \frac{1}{xy}$, it is clear that $M_1(x, y) > 0$, for all (x, y) in interior of R_+^2.

$$\begin{aligned}\nabla.(M_1(x, y)).\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix} &= \nabla.\left(\frac{1}{xy}\right).\begin{pmatrix}g_1(x, y)\\ g_2(x, y)\end{pmatrix}\\ &= \frac{\partial}{\partial x}\left(\frac{1}{xy}\left(x\left(1 - x + \alpha_{12}y\right)\right)\right) + \frac{\partial}{\partial y}\left(\frac{1}{xy}\left(ry\left(1 - y + \alpha_{21}x\right)\right)\right)\\ &= \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{y} + \alpha_{12}\right) + \frac{\partial}{\partial y}\left(\frac{r}{x} - \frac{ry}{x} + r\alpha_{21}\right)\\ &= -\frac{x + ry}{xy} < 0, \quad \text{for all} \quad (x, y) \in R_+^2.\end{aligned}$$

So, according to Bendixson-Dulac criteria, there is no periodic solution in the interior R_+^2 of $xy-$plane. Since all the solutions of the system are bonded and E_4 is unique positive equilibrium point in the interior R_+^2 of the $xy-$plane, hence, by Poincare Bendixson-Dulac theorem, the equilibrium point $E_4 = (\widehat{x}, \widehat{y}, 0)$ is globally asymptotically stable in the interior of R_+^2. ■

Theorem 3.4.

The boundary equilibrium point $E_4 = (\widehat{x}, \widehat{y}, 0)$ is globally asymptotically stable in the interior R_+^3.

Proof:

Let the Lyapunov function for the nonlinear system be

$$V_1(x, y, z) = l_1\left(x - \widehat{x} - \widehat{x}\log\left(\frac{x}{\widehat{x}}\right)\right) + l_2\left(y - \widehat{y} - \widehat{y}\log\left(\frac{y}{\widehat{y}}\right)\right).$$

On differentiating with respect to t and substituting derivatives of x, y, z expressions, we get

$$\begin{aligned}\frac{dV_1}{dt} &= l_1\left(\frac{x-\widehat{x}}{x}\right)\frac{dx}{dt} + l_2\left(\frac{y-\widehat{y}}{y}\right)\frac{dy}{dt}\\ &= l_1\left(\frac{x-\widehat{x}}{x}\right)\left(x\left(1-x+\alpha_{12}y\right)\right) + l_2\left(\frac{y-\widehat{y}}{y}\right)\left(ry\left(1-y+\alpha_{21}x\right)\right)\\ &= l_1(x-\widehat{x})(-(x-\widehat{x})+\alpha_{12}(y-\widehat{y})) + l_2 r(y-\widehat{y})(-(y-\widehat{y})+\alpha_{21}(x-\widehat{x}))\\ &= -l_1(x-\widehat{x})^2 - l_2 r(y-\widehat{y})^2,\end{aligned}$$

for $l_1 = \frac{-r\alpha_{21}}{\alpha_{12}}$ and $l_2 = 1$, then $\frac{dV_1}{dt} < 0$ and hence by Lyapunov theorem, the system is globally stable at the boundary equilibrium point $E_4 = (\widehat{x}, \widehat{y}, 0)$. ■

Theorem 3.5.

If $v_1v_3 + v_1v_2 > (1-p)(v_3 - v_2)$, the boundary steady state $E_5 = (0, \bar{y}, \bar{z})$ is stable in $yz-$plane.

Proof:

In the absence of species x, which is commensal of z species, the system reduces to system:

$$\begin{aligned}\frac{dy}{dt} &= ry\left(1-y\right) - \frac{(1-p)yz}{v_1+(1-p)y},\\ \frac{dz}{dt} &= z\left(-v_2 + \frac{v_3(1-p)y}{v_1+(1-p)y}\right),\end{aligned} \tag{5}$$

under the condition

$$v_2 < \frac{v_3(1-p)}{v_1+(1-p)}. \tag{6}$$

Local stability analysis for the system (5) gives the following results. The equilibrium points of system (5) under investigation are

$$(0,0), (1,0), \text{ and } (\bar{y}, \bar{z}),$$

and the Jacobin matrix for the system (5) at equilibrium point $E = (y, z)$ is

$$J = \begin{pmatrix} r(1-y) - ry - \frac{(1-p)z}{v_1+(1-p)y} + \frac{(1-p)^2yz}{(v_1+(1-p)y)^2} & -\frac{(1-p)y}{v_1+(1-p)y} \\ z\left(\frac{v_3(1-p)}{v_1+(1-p)y} - \frac{v_3(1-p)^2y}{(v_1+(1-p)y)^2}\right) & 0 \end{pmatrix}.$$

The trivial equilibrium point $(0,0)$ is always exists and it is a saddle point and the equilibrium point $(1,0)$ is also exists always and it is a saddle point if $v_2 < \frac{v_3(1-p)}{v_1+(1-p)}$. The nontrivial positive equilibrium point is $(\bar{y}, \bar{z})$, where

$$\bar{y} = \frac{v_1v_2}{(1-p)(v_3-v_2)}, \quad \bar{z} = \frac{rv_1v_3[(1-p)(v_3-v_2) - v_1v_2]}{(1-p)^2(v_3-v_2)^2}.$$

The associated Jacobian matrix at $(\bar{y}, \bar{z})$ is

$$J_1 = \begin{pmatrix} \frac{rv_2((1-p)(v_3-v_2)-v_1(v_2+v_3))}{v_3(1-p)(v_3-v_2)} & -\frac{v_2}{v_3} \\ \frac{r((1-p)(v_3-v_2)-v_1v_2)}{1-p} & 0 \end{pmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 is

$$\lambda^2 - \frac{rv_2((1-p)(v_3-v_2)-v_1(v_2+v_3))}{v_3(1-p)(v_3-v_2)}\lambda + \frac{rv_2((1-p)(v_3-v_2)-v_1v_2)}{v_3(1-p)} = 0.$$

Compare with $\lambda^2 + a_0\lambda + a_1 = 0$, where

$$a_1 = \frac{rv_2((1-p)(v_3-v_2)-v_1v_2)}{v_3(1-p)} > 0, \quad \text{and}$$
$$a_0 = -\frac{rv_2((1-p)(v_3-v_2)-v_1(v_2+v_3))}{v_3(1-p)(v_3-v_2)}.$$

Here, $a_0 > 0$ if $v_1v_3 > ((1-p)(v_3-v_2)-v_1v_2)$ and $a_0 < 0$ if $v_1v_3 < ((1-p)(v_3-v_2)-v_1v_2)$. Hence, according to Routh-Hurwitz criteria, the system is stable in xy-direction if $v_1v_3 + v_1v_2 > (1-p)(v_3-v_2)$ and unstable in z-direction. ■

Theorem 3.6.

Along the condition stated in Theorem 3.5, the equilibrium point $E_5 = (0, \bar{y}, \bar{z})$ is globally asymptotically stable in the interior R^2_+ of the $yz-$plane.

Proof:

For any value in interior R^2_+ of the $yz-$plane, the system reduces to following subsystem in the interior of $yz-$plane

$$\frac{dx}{dt} = x\left(1 - x - \frac{(1-p)z}{v_1+(1-p)x}\right) = g_3(x,y),$$
$$\frac{dz}{dt} = z\left(-v_2 + \frac{v_3(1-p)x}{v_1+(1-p)x}\right) = g_4(x,y).$$

Assume that $M_2(y,z) = \frac{v_1+(1-p)y}{yz}$. It is clear that $M_2(y,z) > 0$ for all (y,z) in interior of R^2_+. Now

$$\begin{aligned}
\nabla.(M_2(y,z)).\begin{pmatrix}\frac{dy}{dt}\\ \frac{dz}{dt}\end{pmatrix} &= \nabla.\left(\left(\frac{v_1+(1-p)\,y}{yz}\right).\begin{pmatrix}g_3(y,z)\\ g_4(y,z)\end{pmatrix}\right)\\
&= \frac{\partial}{\partial y}\left(r\frac{v_1+(1-p)y}{z} - r\frac{v_1y+(1-p)y^2}{z} - (1-p)\right)\\
&\quad + \frac{\partial}{\partial z}\left(-v_2\frac{v_1+(1-p)y}{y} + v_3(1-p)\right)\\
&= \frac{r}{z}((1-p)(1-2y)-v_1) < 0, \text{ if } (1-p)(1-2y) < v_1.
\end{aligned}$$

Then from Bendixson-Dulac criteria, there is no periodic solution in the interior R^2_+ of $yz-$plane. Since all the solutions of the system are bonded and E_5 is unique positive equilibrium point in the interior R^2_+ of the $yz-$plane, hence, by Poincare Bendixson-Dulac theorem, the equilibrium point $E_5 = (0, \bar{y}, \bar{z})$ is globally asymptotically stable in the interior of R^2_+. ■

Theorem 3.7.

Assume that the condition in the Theorem 3.5, the planar equilibrium point $E_5 = (0, \bar{y}, \bar{z})$ is globally asymptotically stable in the interior of R^3_+.

Proof:

Consider the positive definite function

$$V_2(y,z) = l_1\left(y - \bar{y} - \bar{y}\log\left(\frac{y}{\bar{y}}\right)\right) + l_2\left(z - \bar{z} - \bar{z}\log\left(\frac{z}{\bar{z}}\right)\right),$$

where l_1, l_2 are the positive constants to be determined. On differentiating with respect to t and substituting derivatives of y, z expressions, and simplifying, we get

$$\begin{aligned}
\frac{dV_2}{dt} &= l_1\left(\frac{y-\bar{y}}{y}\right)\frac{dy}{dt} + l_2\left(\frac{z-\bar{z}}{z}\right)\frac{dz}{dt} \\
&= l_1(y-\bar{y})\left(r - ry - \frac{(1-p)z}{v_1+(1-p)y}\right) + l_2(z-\bar{z})\left(-v_2 + \frac{v_3(1-p)y}{v_1+(1-p)y}\right) \\
&= l_1(y-\bar{y})\left(-r(y-\bar{y}) - (1-p)\left(\frac{z}{v_1+(1-p)y} - \frac{\bar{z}}{v_1+(1-p)\bar{y}}\right)\right) \\
&\quad + l_2(z-\bar{z})\left(v_3(1-p)\left(\frac{y}{v_1+(1-p)y} - \frac{\bar{y}}{v_1+(1-p)\bar{y}}\right)\right) \\
&= -l_1 r(y-\bar{y})^2 - l_1 v_1(1-p)\frac{(y-\bar{y})(z-\bar{z})}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})} \\
&\quad - l_1(1-p)^2\frac{\bar{y}(y-\bar{y})(z-\bar{z})}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})} \\
&\quad + l_1(1-p)^2\frac{\bar{z}(y-\bar{y})^2}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})} \\
&\quad + l_2 v_1 v_3\frac{(y-\bar{y})(z-\bar{z})}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})} \\
&= -l_1(y-\bar{y})^2\left(r - \frac{(1-p)^2\bar{z}}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})}\right) \\
&\quad + \frac{(1-p)(y-\bar{y})(z-\bar{z})}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})}\left(-l_1 v_1 - l_1(1-p)\bar{y} + l_2 v_1 v_3\right).
\end{aligned}$$

By choosing non-negative constants $l_1 = 1$ and $l_2 = \frac{v_1+(1-p)\bar{y}}{v_1 v_3}$, then the above equation becomes

$$\begin{aligned}
\frac{dV_2}{dt} &= -l_1(y-\bar{y})^2\left(r - \frac{(1-p)^2\bar{z}}{(v_1+(1-p)y)(v_1+(1-p)\bar{y})}\right) \\
&< 0 \quad \text{if } r(v_1+(1-p)y)(v_1+(1-p)\bar{y}) > (1-p)^2\bar{z}.
\end{aligned}$$

Hence, by Lyapunov theorem the equilibrium point $E_5 = (0, \bar{y}, \bar{z})$ is globally stable, if

$$r(v_1+(1-p)y)(v_1+(1-p)\bar{y}) > (1-p)^2\bar{z}.$$

■

Theorem 3.8.

The interior equilibrium point $E_6 = (x^*, y^*, z^*)$ exists if $(1-p)(v_3 - v_2)(\alpha_{21} + 1) > v_1 v_2 (1 - \alpha_{12}\alpha_{21})$ and $v_3 > v_2$.

Proof:

Let x^*, y^*, z^* be the positive solutions of the equations

$$\begin{aligned} x^* \left(1 - x^* + \alpha_{12} y^*\right) &= 0, \\ r y^* \left(1 - y^* + \alpha_{21} x^*\right) - \frac{(1-p) y^* z^*}{v_1 + (1-p) y^*} &= 0, \\ z^* \left(-v_2 + \frac{v_3 (1-p) y^*}{v_1 + (1-p) y^*}\right) &= 0. \end{aligned}$$

By solving these equations, we obtain

$$\begin{aligned} x^* &= 1 + \frac{\alpha_{12} v_1 v_2}{(1-p)(v_3 - v_2)}, \qquad y^* = \frac{v_1 v_2}{(1-p)(v_3 - v_2)}, \\ z^* &= \frac{y v_1 v_3}{(1-p)^2 (v_3 - v_2)^2} ((1-p)(v_3 - v_2)(\alpha_{21} + 1) - v_1 v_2 (1 - \alpha_{12}\alpha_{21})). \end{aligned}$$

Hence, the interior equilibrium point $E_6 = (x^*, y^*, z^*)$ exists if $(1-p)(v_3 - v_2)(\alpha_{21} + 1) > v_1 v_2 (1 - \alpha_{12}\alpha_{21})$ and $v_3 > v_2$. ∎

Theorem 3.9.

The interior equilibrium point $E_6 = (x^*, y^*, z^*)$ is locally asymptotically stable if $a_0 > 0, a_2 > 0$ and $a_0 a_1 - a_2 > 0$ otherwise is unstable.

Proof:

The Jacobian matrix for the coexistent equilibrium state $E_6 = (x^*, y^*, z^*)$ is

$$J_{E_6} = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & H_{23} \\ 0 & H_{32} & 0 \end{pmatrix},$$

where

$$\begin{aligned} &H_{11} = -x^*, \quad H_{12} = \alpha_{12} x^*, \quad H_{22} = \left(1 - y^* + \alpha_{21} x^*\right) \left(r - 1 + \frac{v_2}{v_3}\right) - r y^*, \\ &H_{21} = r \alpha_{21} y^*, \quad H_{23} = -\frac{(1-p) y^*}{v_1 + (1-p) y^*}, \quad H_{32} = \frac{(1-p) z^*}{v_1 + (1-p) y^*} (v_3 - v_2). \end{aligned}$$

The characteristic equation is $\lambda^3 + a_0 \lambda^2 + a_1 \lambda + a_2 = 0$, with

$$\begin{aligned} a_0 &= -\left(H_{11} + H_{22}\right), \\ a_1 &= H_{11} H_{22} - \left(H_{12} H_{21} + H_{23} H_{32}\right), \\ a_2 &= H_{32} H_{23} H_{11}. \end{aligned}$$

According to Routh-Hurwitz criteria is locally asymptotically stable if $a_0 > 0, a_2 > 0$ and $a_0 a_1 - a_2 > 0$ otherwise is unstable. ■

Theorem 3.10.

Along with the conditions stated in the Theorem 3.9 and if

$$r(v_1 + (1-p)y)(v_1 + (1-p)y^*) > (1-p)^2 z^*,$$

the interior equilibrium point $E_6 = (x^*, y^*, z^*)$ is globally asymptotically stable.

Proof:

Let the Lyapunov function for the nonlinear system be

$$\begin{aligned} V(x,y,z) = x - x^* - x^* \log\left(\frac{x}{x^*}\right) \\ + l_1\left(y - y^* - y^* \log\left(\frac{y}{y^*}\right)\right) \\ + l_2\left(z - z^* - z^* \log\left(\frac{z}{z^*}\right)\right). \end{aligned}$$

On differentiating with respect to t and substituting derivatives of x, y, z expressions, we get

$$\begin{aligned} \frac{dV}{dt} &= (x - x^*)(1 - x + \alpha_{12} y) \\ &+ l_1(y - y^*)\left(r(1 - y + \alpha_{21} x) - \frac{(1-p)z}{v_1 + (1-p)y}\right) \\ &+ l_2(z - z^*)\left(-v_2 + \frac{v_3(1-p)y}{v_1 + (1-p)y}\right) \\ &= (x - x^*)(-(x - x^*) + \alpha_{12}(y - y^*)) \\ &+ l_1(y - y^*)[-r(y - y^*) + r\alpha_{21}(x - x^*) \\ &- (1-p)\left(\frac{z}{v_1 + (1-p)y} - \frac{z^*}{v_1 + (1-p)y^*}\right)] \\ &+ l_2 v_3(1-p)(z - z^*)\left(\frac{y}{v_1 + (1-p)y} - \frac{y^*}{v_1 + (1-p)y^*}\right) \\ &= -(x - x^*)^2 + (\alpha_{12} - l_1\alpha_{21} r)(x - x^*)(y - y^*) \\ &+ \frac{(y - y^*)(z - z^*)}{[v_1 + (1-p)y][v_1 + (1-p)y^*]} \\ &\left(l_2 v_1 v_3(1-p) - l_1(1-p)^2 y^* - l_1(1-p)v_1\right) \\ &- \left(l_1 r - \frac{l_1(1-p)^2 z^*}{(v_1 + (1-p)y)(v_1 + (1-p)y^*)}\right)(y - y^*)^2. \end{aligned}$$

If $l_1 = \frac{\alpha_{12}}{r\alpha_{21}}, l_2 = \frac{\alpha_{12}}{\alpha_{21} r v_1 v_3} \left(v_1 + (1-p)y^*\right)$ and $r > \frac{(1-p)^2 z^*}{[v_1+(1-p)y]\,[v_1+(1-p)y^*]}$, then $\frac{dV}{dt} < 0$, and hence, by known theorem, the system is globally stable. ■

4. Conclusion

In this paper, we focused on an ecological model of a three-space food chain that consists of two logically growing mutual species and a third species acts as a predator to second mutual species with Holling type II functional response. A system of nonlinear decoupled ordinary differential equations formed using the model and we identify the nature of the system at each equilibrium point by perturbed method. We also investigated the global stability at the possible equilibrium points for this model using Lypanov function.

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