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An Efficient Algorithm for Numerical Inversion of System of Generalized Abel Integral Equations

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Abstract

In this article a direct method is introduced, which is based on orthonormal Bernstein polynomials, to present an efficient and stable algorithm for numerical inversion of the system of singular integral equations of Abel type. The appropriateness of earlier numerical inversion methods was restricted to the one portion of singular integral equations of Abel type. The proposed method is absolutely accurate, and numerical illustrations are given to show the convergence and utilization of the suggested method and comparisons are made with some other existing numerical solution.

Keywords: Bernstein polynomials; Abel inversion; Operational matrix; System of generalized Abel integral equations

MSC 2010 No.: 26A33, 41A10, 45A05

1. Introduction

Many attention grabbing real life problems of physics and mechanics result in an integral equation in which the kernel of the integral equation is of convolution type. A similar type integral equation with singular kernel named singular integral equation of Abel type which emerges in many branches of science and engineering. Commonly, physical quantities available to assessment are actually often related to physically essential but experimentally unattainable ones by integral equations of generalized Abel type.

There are certain numerical approximation methods for solving Volterra integral equations and integral equations of Abel kernel and generalized Abel kernel like, homotopy analysis transform technique, Galerkin method, Taylor series method and Taylor polynomials Methods, (Kumar et al. (2014), Burton (2005), Maleknejad and Aghazadeh (2005) and Yalsinbas (2002)). In recent past a number of methods have been contemplated based on Chebyshev polynomials (Malekkejad et al. (2007)), Homotopy Perturbation method (Kumar and Singh O. P. (2010), Kumar et al. (2011)), Laplace transform method (Kumar et al. (2015)), Bernstein polynomials (Pandey et al. (2009)), Power Series method (Tahmasbi and Fard (2008)), Laguerre polynomial approximation (Singha and Nahak (2019)) and almost Bernstein operational matrix method (Dixit et al. (2011), Dixit et al. (2012)) to find the numerical inversion of integral equation with Abel type kernel.

Recently Mandal and Chakrabarti (2016) achieved a numerical inversion of the system of generalized singular integral equations of Abel type kernel using order 8 Bernstein polynomials but the proposed method is less accurate when the coefficients to equations are polynomial functions compared to the case when the coefficients are constants. Irfan et al. (2014) used the concept Bernstein operational matrices to solve integro-differential equation. Recently Kumar et al. (2020) used Bernstein wavelets to solve fractional predator – prey dynamical system and fractional SIR epidemic model respectively. This article is devoted to obtaining a class of almost operational matrices based on Bernstein type of orthogonal polynomials to find the numerical inversion of system of integral equation with generalized Abel type.

I have organized my paper as follows. In section 2, system of Abel integral equation with variable coefficients is given. Section 3 contains orthonormal Bernstein polynomials. In section 4 function approximation given to apply the technique. Section 5 contains method of solution to find inversion. Section 6 contains numerical examples to illustrate the stability and the accuracy of our algorithm

2. System of Abel Integral Equations

A system of singular integral equations of Abel type was studied by Lowengrub (1976) and Walton (1979). As told in J Walton (1979), a few mixed boundary value based problems arising in the classical elasticity theory diminish to the research problem of determining functions μ_1 and μ_2 gratifying generalized singular integral equations of the form,

$$q_{11}(y)\int_0^y \frac{\mu_1(z)dz}{(y^d - z^d)^b} + q_{12}(y)\int_y^1 \frac{\mu_2(z)dz}{(z^d - y^d)^b} = h_1(y),$$

$$q_{21}(y)\int_0^y \frac{\mu_1(z)dz}{(z^d - y^d)^b} + q_{22}(y)\int_y^1 \frac{\mu_2(z)dz}{(y^d - z^d)^b} = h_2(y), \tag{1}$$

where $y \in (0,1), 0 < b < 1$, $q_{ij}(y)$ are continuous on $[0,1]$, h_1 and h_2 are the known functions. Since only $d = 1,2$ occur in physics related problems, we impede our consideration to these values only.

Mandal *et al.* (1996), solved this physical problem analytically for the particular case $d=1, b=1/2, q_{ij}(y)=1$, using the concept of fractional calculus. Recently Mandal and Chakrabarti (2016) achieved a numerical inversion of the given system (1) using order 8 Bernstein polynomials

$$\int_0^y \frac{\mu(z)dz}{(y-z)^{1/2}} = h(y) \qquad \text{and} \qquad \int_y^1 \frac{\mu(z)dz}{(z-y)^{1/2}} = h(y). \tag{2}$$

But the method proposed by Mandal and Chakrabarti (2016) is less accurate when the coefficients $q_{ij}(y)$ are polynomial functions compared to the case when the coefficients are constants.

3. The Orthonormal Polynomials

The Bernstein polynomials (B polynomials) characterized over the interval $[0,1]$ are given as;

$$B_{i,\alpha}(z) = \binom{\alpha}{i} z^i (1-z)^{\alpha-i}, \qquad \forall\ i = 0, 1, 2, ..., \alpha .$$

By using orthonormalization process given by Gram- Schmidt method on $B_{i\alpha}$, we attain a family of orthonormal polynomials from B polynomials. We denote these polynomials by $b_{0\alpha}, b_{1\alpha}, \cdots, b_{\alpha\alpha}$. After all, for $\alpha = 5$, the six orthonormal B polynomials were enumerated and given:

$$b_{05}(z) = \sqrt{11}\left(1-z\right)^5,$$

$$b_{15}(z) = 3(1-z)^4\left[11z-1\right],$$

$$b_{25}(z) = \sqrt{7}(1-z)^3\left[55z^2-20z+1\right],$$

$$b_{35}(z) = \frac{28}{\sqrt{5}}\left[10(1-z)^2z^3 - 15(1-z)^3z^2 + \frac{30}{7}(1-z)^4y - \frac{5}{28}(1-z)^5\right],$$

$$b_{45}(z) = 7\sqrt{3}\left[5(1-z)z^4 - 20(1-z)^2z^3 + 18(1-z)^3z^2 - 4(1-z)^4z + \frac{1}{7}(1-z)^5\right],$$

$$b_{55}(z) = 6\left[z^5 - \frac{25}{2}(1-z)z^4 + \frac{100}{3}(1-z)^2z^3 - 25(1-z)^3z^2 + 5(1-z)^4z - \frac{1}{6}(1-z)^5\right].$$

4. Function Approximation

Any given function $h \in L^2[0,1]$ be allowed to express as

$$h(z) = \lim_{\alpha \to \infty} \sum_{i=0}^{\alpha} c_{i\alpha}\, b_{i\alpha}(z)\,, \tag{3}$$

where $c_{i\alpha} = \langle h, b_{i\alpha} \rangle$ and $\langle .,. \rangle$ is the notation for inner product on $L^2[0,1]$.

If the given series (3) is truncated at $n = \alpha$, then we get

$$h \cong \sum_{i=0}^{\alpha} c_{i\alpha}\, b_{i\alpha} \;=\; C^T B(z)\,, \tag{4}$$

where C and $B(z)$ are $(\alpha+1)\times 1$ matrices given by

$$C = [c_{0\alpha}, c_{1\alpha}, \cdots, c_{\alpha\alpha}]^T, \tag{5}$$

and

$$B(z) = [b_{0\alpha}(z), b_{1\alpha}(z), \cdots, b_{\alpha\alpha}(z)]^T. \tag{6}$$

5. Method of Solution

Under this heading we solve integral equation of generalized Abel kernel by orthonormal B polynomials. Using Equation (4), we approximate $\mu_1(z), \mu_2(z), h_1(y)$ and $h_2(y)$ as

$$\tilde{\mu}_1(z) = C_1^{\,T} B(z), \tilde{\mu}_2(z) = C_2^{\,T} B(z), \tilde{h}_1(y) = H_1^{\,T} B(y), \tilde{h}_2(y) = H_2^{\,T} B(y). \tag{7}$$

The matrices H_1, H_2 are known and the matrices C_1, C_2 are to be found. From Equations (1) and (7), we have

$$\begin{aligned} q_{11}(y)\int_0^y \frac{C_1^{\,T} B(z)dz}{(y^d - z^d)^b} + q_{12}(y)\int_y^1 \frac{C_2^{\,T} B(z)dz}{(z^d - y^d)^b} &= H_1^{\,T} B(y), \\ q_{21}(y)\int_0^y \frac{C_1^{\,T} B(z)dz}{(z^d - y^d)^b} + q_{22}(y)\int_y^1 \frac{C_2^{\,T} B(z)dz}{(y^d - z^d)^b} &= H_2^{\,T} B(y), \end{aligned} \tag{8}$$

Using Equation (6) and the following formulae,

$$\int_0^y \frac{z^n dz}{(y^d - z^d)^b} = \frac{\pi y^n \left(d/y\right)^{b-1} (y^{d-1} d)^{-b} \operatorname{cosec}(\pi b) \Gamma\left(n + 1/d\right)}{\Gamma(b)\Gamma\left(n + d - d\,b + 1/d\right)}, \tag{9}$$

$$\int_y^1 \frac{z^n dz}{(z^d - y^d)^b} = \left[\frac{(y^{-d})^{-\frac{n+1}{d}} (y^d)^{-b} \left[\Gamma(1-b)\Gamma\left(b - n + 1/d\right)\right]}{d\Gamma(-n + d - 1/\alpha)} - \frac{(y^{-d})^{-\frac{n+1}{d}} (y^d)^{-b} \left[B_{y^d}(b - n + 1/d, 1-b)\Gamma(-n + d - 1/d)\right]}{d\Gamma(-n + d - 1/d)} \right], \tag{10}$$

where $B_y(p,q) = \int_0^y u^{p-1}(1-u)^{q-1} du$ is incomplete beta function. Thus, we obtain

$$\int_0^y \frac{B(z)}{(y^d - z^d)^b} dz = U B(y), \quad \int_y^1 \frac{B(z)}{(z^d - y^d)^b} dz = V B(y), \tag{11}$$

where U and V are $(\alpha+1)\times(\alpha+1)$ order matrices, which we call as AOMI for singular integral equation with generalized kernel.

Replacing the values of Equation (11) in Equation (8), we have

$$\begin{aligned} C_1^T &= -\left[\frac{H_2^T q_{12}(y) V - H_1^T q_{22}(y) U}{q_{11}(y) q_{22}(y) U^2 - q_{21}(y) q_{12}(y) V^2} \right], \\ C_2^T &= -\left[\frac{H_1^T q_{21}(y) V - H_2^T q_{11}(y) U}{q_{11}(y) q_{22}(y) U^2 - q_{21}(y) q_{12}(y) V^2} \right]. \end{aligned} \tag{12}$$

Hence, the approximate solutions $\tilde{\mu}_1(z), \tilde{\mu}_2(z)$ for singular integral equation (1) are attained by placing the value of C_1^T, C_2^T from Equation (12) in Equation (1).

6. Illustrative Examples

Under this heading, we examine the utilization of our suggested method and examine its accuracy and stability by implementing it on test functions with known analytical result of Abel inversion. For it is always required to check the demeanor of a numerical method using given simulated data, for which the exact solutions are known and thus creating a resemblance between the inverted solutions and the theoretical data is possible. We have chosen two test functions having various shapes for this objective.

The five numerical illustrations are given with added noise terms and without added noise terms to understand the efficiency as well as stability of our given algorithm. Note that in all five illustrated results to follow, the series in Equation (4) is truncated at level $\alpha = 5$ and hence the AOMI in Equation (11) are of order 6×6.

The accuracy of given method is established by considering the parameters of absolute error $\Delta\mu_j(z_i)$ and root mean square error (RMS) σ^j also known as average deviation. Both of them are calculated using the equations given below:

$$E_j(z_i) = \left|\mu_j(z_i) - \tilde{\mu}_j(z_i)\right|,\ \sigma^j{}_N = \{\frac{1}{N}\sum_{i=1}^{N}[\mu_j(z_i) - \tilde{\mu}_j(z_i)]^2\}^{1/2}$$
$$= \{\frac{1}{N}\sum_{i=1}^{N}\Delta\mu_j{}^2(z_i)\}^{1/2} = \left\|\Delta\mu_j\right\|_2, \tag{13}$$

where $\mu_j(z_i)$ represent the exact solution and $\tilde{\mu}_j(z_i)$ represent approximate numerical result calculated at discrete point z_i corresponding to the point, $j = 1, 2$. For computational accessibility σ^j is stand for $\sigma^j{}_N$, the discrete l^2-norm of the absolute error $\Delta\mu_j$ represented by $\left\|\Delta\mu_j\right\|_2$.

Note that the root mean square error results of $\sigma^j{}_N$ is found by taking N =1000, 500 in Equation (13). In all five numerical examples, the exact profile is represented by $h_j(y)$ and the profiles with added noises are represented by $h_j^{\delta}(y)$, where $h_j^{\delta}(y)$ is obtained by adding a noise term δ to $h_j(y)$ so that $h_j^{\delta}(y_i) = h_j(y_i) + \delta\theta_i$, where $y_i = ih,\ i = 1,...,N,\ Nh = 1$ and θ_i is the discrete random variable uniformly distributed with values in the interval [-1, 1] such that

$$\underset{1\le i\le N}{Max}\left|h_{ji}{}^{\delta}(y) - h_{ji}(y)\right| \le \delta.$$

The given five numerical illustrations are determined without noise and with noise to learn the efficiency and stability of our given algorithm by deciding two distinct values of the noises δ_k as $\delta_0 = 0,\ \delta_1 = \sigma_N^j$.

In all five numerical illustrations to follow, we measure the corresponding errors $E_j(z)$ and express them as $E1(z), E2(z)$ respectively. Then small perturbation (noise term) δ_1 is added in the known forcing term $h_j(y)$ and for $N = 1000, 500$ the errors associated to μ_1 and μ_2 are labeled by $E3(z), E4(z)$ and $E5(z), E6(z)$, respectively. For correct scaling some $E_j(z)$ are multiplied by 10. Also, we listed the approximate and exact values for $z = 0.0, 0.2, ..., 0.8, 1.0$.

In the first two numerical examples, $d = 1$ whereas we have taken $d = 2$ in the last three examples.

Example 1:

In the first example, take

$$q_{11}(y)=1\,, q_{12}(y)=1/4,\; q_{21}(y)=1/2,\; q_{22}(y)=3\,,\; b=1/3$$

with

$$h_1(y)=\frac{243}{1760}(1-y)^{2/3}y^3+\frac{27}{40}y^{8/3}+\frac{459}{1760}y^2(1-y)^{2/3}+\frac{333}{1760}y(1-y)^{2/3}+\frac{57}{352}(1-y)^{2/3},$$

and

$$h_2(y)=\frac{81}{440}(9y+11)(y)^{8/3}+\frac{3}{80}(1-y)^{2/3}(9y^2+6y+5).$$

This generalized system has the solution

$$\mu_1(z)=z^2 \text{ and } \mu_2(z)=z^2+z^3.$$

The corresponding values of $C_1^T, C_2^T, \sigma_{1000}^1, \sigma_{500}^1, \sigma_{1000}^2$ and σ_{500}^2 are given as

$$C_1^T=[0.026317, 0.130948, 0.304484, 0.457853, 0.49879, 0.333334],$$

$$C_2^T=[0.019736, 0.089282, 0.185845, 0.252881, 0.255692, 0.166667],$$

$$\sigma_{1000}^1=1.8\times10^{-5}, \qquad \sigma_{500}^1=1.804\times10^{-5},$$

$$\sigma_{1000}^2=1.846\times10^{-5}, \qquad \sigma_{500}^2=1.851\times10^{-5}.$$

Table 1. Approximate and exact values of μ for Example 1

Z	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{\mu}_1(z)$	0.000053	0.039999	0.160017	0.360003	0.640000	1.00003
$\mu_1(z)$	0.0	0.040000	0.16000	0.360000	0.640000	1.000000
$\mu_2(z)$	0.000061	0.047969	0.224018	0.576003	1.151998	2.000004
$\tilde{\mu}_2(z)$	0.0	0.048000	0.224000	0.576000	1.152000	2.00000

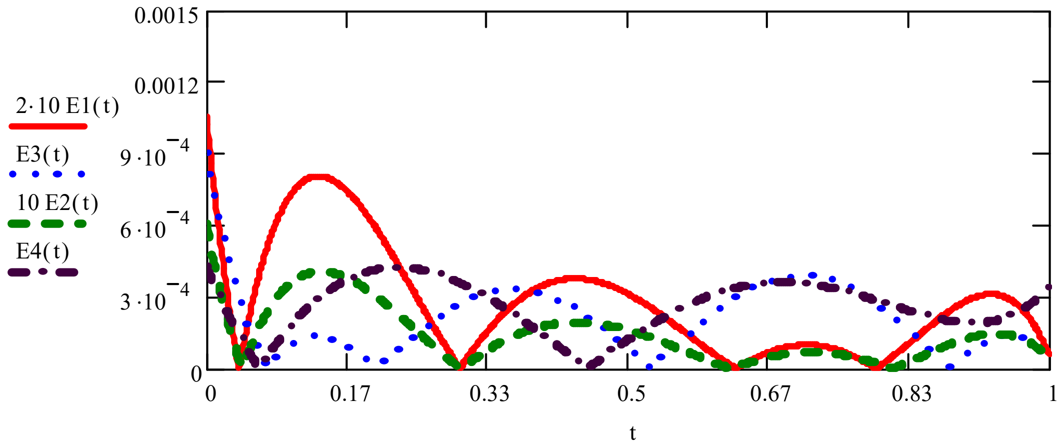

Figure 1. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N=1000$

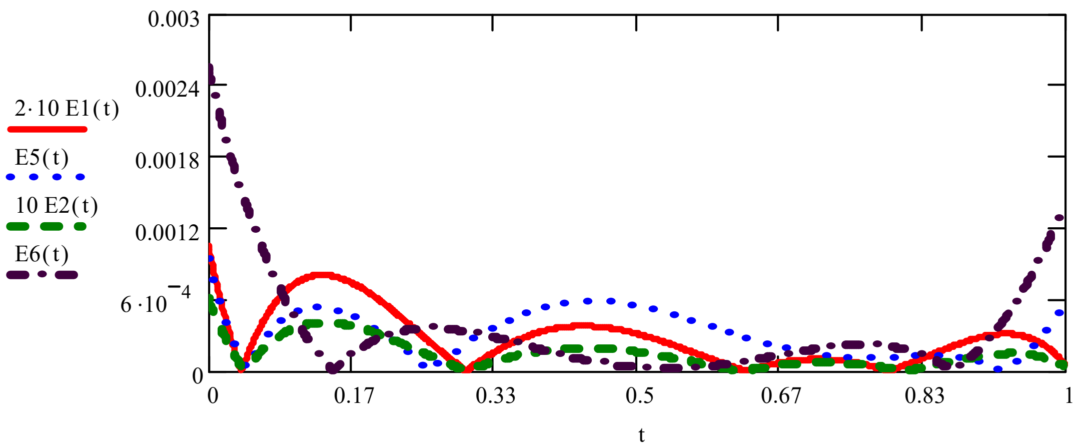

Figure 2. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N=500$

Example 2.

In second example, take $q_{11}(y)=y^2+1$, $q_{12}(y)=(y+1)/4$, $q_{21}(y)=y^2/2$, $q_{22}(y)=y$, $b=1/2$ and

$$h_1(y)=\left[1/4\,(y+1)(4\sqrt{y+1}\sin^{-1}(\sqrt{1-y}/\sqrt{2})+\sqrt{1-y}(-4+\log 4))+e^y\sqrt{\pi}\,erf(\sqrt{y})(y^2+1)\right],$$

$$h_2(y)=1/2\,y\left(8\sqrt{y+1}\sinh^{-1}(\sqrt{y})+\frac{e^y y\sqrt{\pi}\sqrt{1-y}\,erf(\sqrt{y-1})}{\sqrt{y-1}}-8\sqrt{y}\right),$$

where $erf(y)$ is error function, defined by $erf(y)=2/\sqrt{\pi}\int_0^y e^{-t^2}dt$. This has the exact solution $\mu_1(z)=e^z$ and $\mu_2(z)=\log(z+1)$.

The corresponding values of $C_1^T, C_2^T, \sigma_{1000}^1, \sigma_{500}^1, \sigma_{1000}^2$ and σ_{500}^2 are given as

$$C_1^T = [0.030749, 0.078086, 0.097141, 0.097892, 0.083314, 0.050181],$$
$$C_2^T = [0.642824, 0.766876, 0.847149, 0.85023, 0.739863, 0.453045],$$
$$\sigma_{1000}^1 = 5.968\times10^{-6}, \qquad \sigma_{500}^1 = 6.001\times10^{-6},$$
$$\sigma_{1000}^2 = 1.102\times10^{-5}, \qquad \sigma_{500}^2 = 1.109\times10^{-5}.$$

Table 2. Approximate and exact values of μ for Example 2

z	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{\mu}_1(z)$	0.999978	1.221404	1.491823	1.822111	2.225541	2.718264
$\mu_1(z)$	1.000000	1.221403	1.491825	1.822119	2.225541	2.718271
$\mu_2(z)$	-0.000022	0.079194	0.146128	0.20413	0.255279	0.301079
$\tilde{\mu}_2(z)$	0.000000	0.079181	0.146128	0.20412	0.255273	0.30103

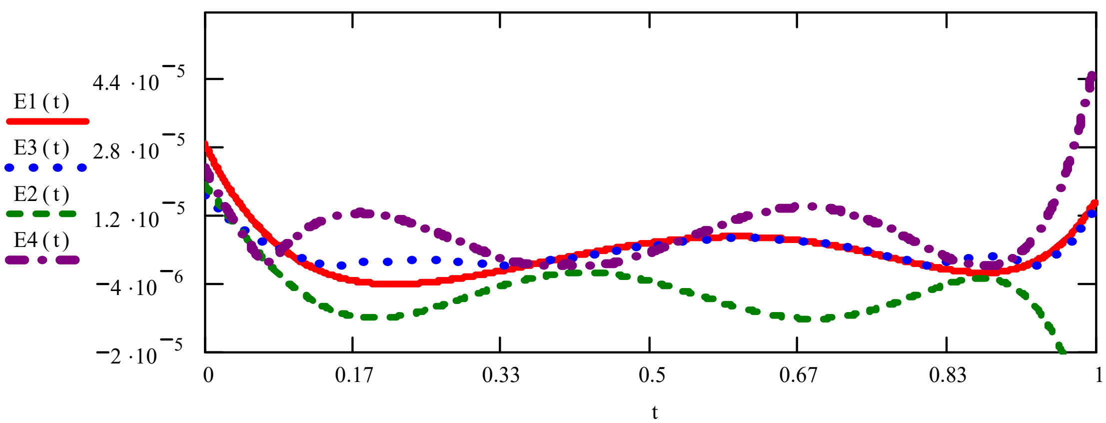

Figure 3. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N = 1000$

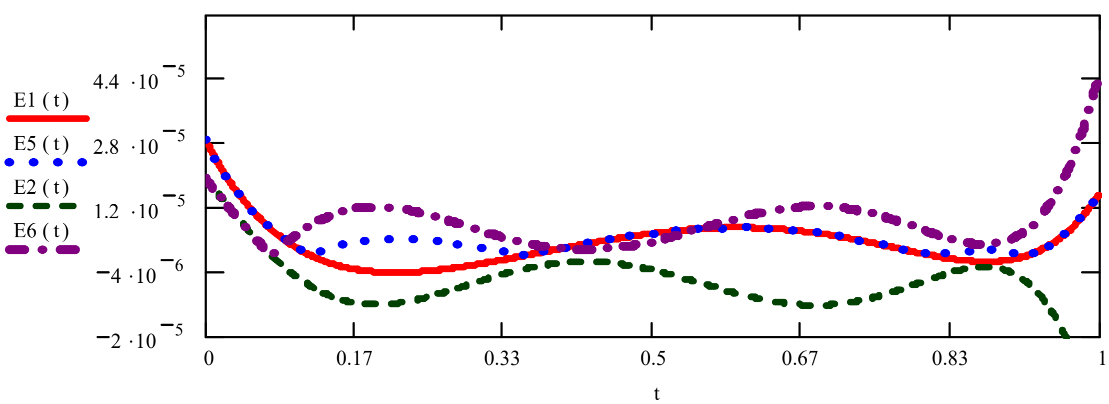

Figure 4. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N = 500$

Example 3.

This example presents the problem (1) with

$$q_{11}(y)=e^{2y}+1,\ q_{12}(y)=e^{y},\ q_{21}(y)=e^{y}/2,\ q_{22}(y)=e^{3y},\ \mu=1/4 \text{ and}$$

$$h_1(y)=\Big[e^{y}\Big(2(3\sqrt{(1-y)y}+4\sqrt{(1-y)y^3}+8\sqrt{(1-y)y^5}\Big/15\sqrt{y}\Big)$$
$$+(1+e^{2y})(e^{y}(\Gamma(3/4)-\Gamma(3/4,y))+4y^{3/4}/3)\Big],$$
$$h_2(y)=\left[e^{3y}(16/15\,y^{5/2})+\frac{e^{2y}(1-y)^{3/4}((\Gamma(3/4)-\Gamma(3/4,y-1))}{2(y-1)^{3/4}}\right].$$

This has the exact solution $\mu_1(z)=e^{z}+1$ and $\mu_2(z)=z^2$. The corresponding values of $C_1^T, C_2^T, \sigma_{1000}^1, \sigma_{500}^1, \sigma_{1000}^2$ and σ_{500}^2 are given as

$$C_1^T=[0.019755, 0.089266, 0.185858, 0.25288, 0.255705, 0.166674],$$

$$C_2^T=[1.195592, 1.266867, 1.288123, 1.222903, 1.028541, 0.619684],$$

$$\sigma_{1000}^1=3.259\times10^{-5},\qquad \sigma_{500}^1=3.284\times10^{-5},$$

$$\sigma_{1000}^2=4.651\times10^{-5},\qquad \sigma_{500}^2=4.692\times10^{-5}.$$

Table 3. Approximate and exact values of μ for Example 3

z	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{\mu}_1(z)$	2.000092	2.221366	2.491837	2.822111	3.225568	3.718126
$\mu_1(z)$	2.000000	2.221403	2.491825	2.822119	3.225541	3.718271
$\mu_2(z)$	0.000233	0.039951	0.160026	0.359993	0.640037	0.99986
$\tilde{\mu}_2(z)$	0.00000	0.040000	0.160000	0.360000	0.640000	1.0000

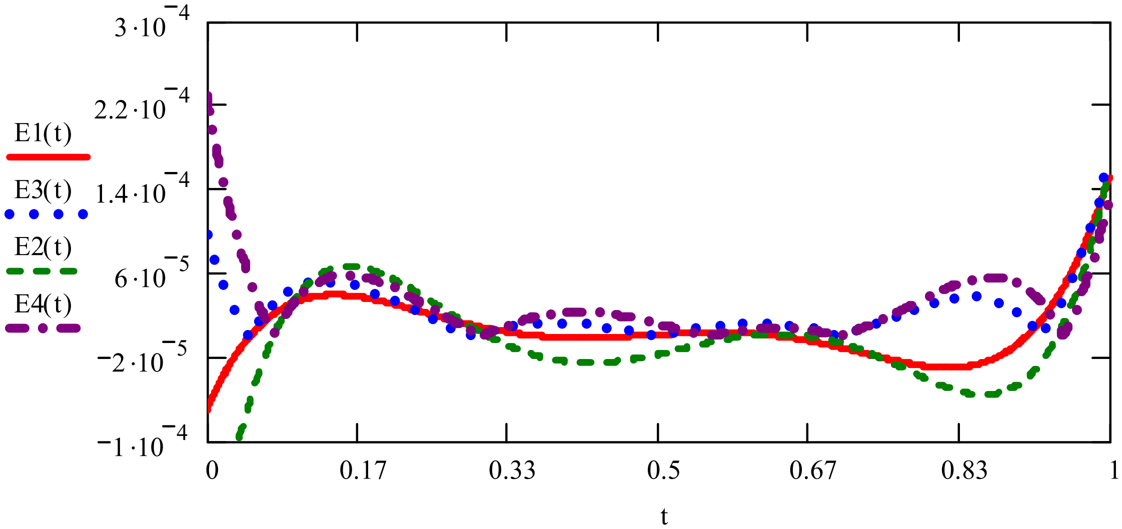

Figure 5. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N = 1000$

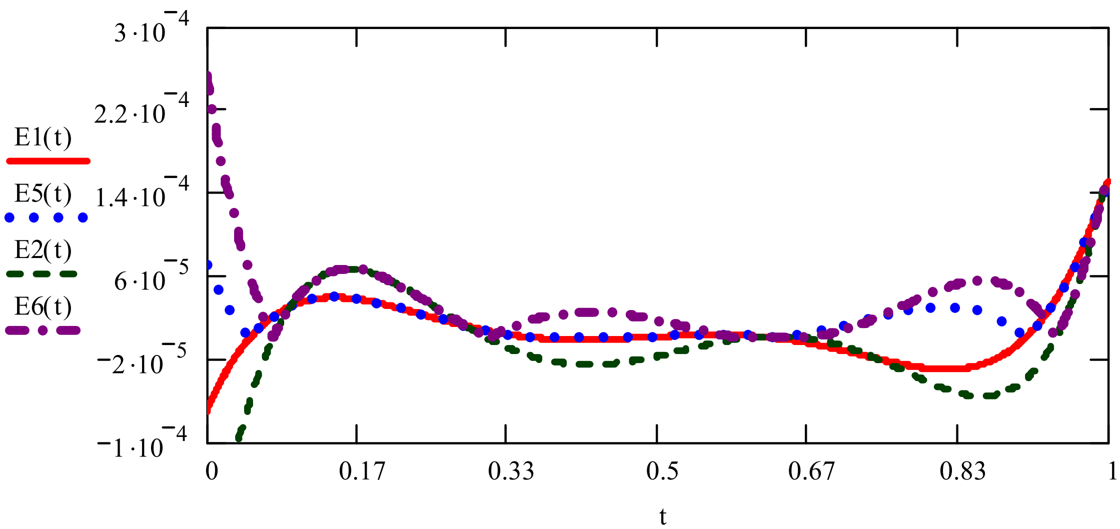

Figure 6. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N = 500$

Example 4:

Consider the system of generalized Abel integral equation with $q_{11}(y) = \sin y$,

$$q_{12}(y) = \cos 2y,\ q_{21}(y) = 2\sin y,\ \ q_{22}(y) = \sin y/2,\ b = 1/2$$

and

$$h_1(y) = \left[y\sin y + \frac{1}{2}\cos 2y\left(\ln\left(\frac{1+\sqrt{1-y^2}}{y} \right) y^2 + \sqrt{1-y^2} \right) \right],$$

$$h_2(y) = \left[\frac{1}{4}(\pi y^2 + 16(1-y^2)^{1/2})\sin y\right].$$

This has the exact solution $\mu_1(z) = z$ and $\mu_2(z) = z^2$. The corresponding values of $C_1^T, C_2^T, \sigma_{1000}^1, \sigma_{500}^1, \sigma_{1000}^2$ and σ_{500}^2 are given as

$$C_1^T = [0.01974, 0.089287, 0.185834, 0.252889, 0.255684, 0.166674],$$

$$C_2^T = [0.078967, 0.214286, 0.289776, 0.30879, 0.27218, 0.166680],$$

$$\sigma_{1000}^1 = 1.288 \times 10^{-5}, \qquad \sigma_{500}^1 = 1.299 \times 10^{-5},$$

$$\sigma_{1000}^2 = 7.711 \times 10^{-5}, \qquad \sigma_{500}^2 = 7.75 \times 10^{-5}.$$

Table 4. Approximate and exact values of μ for example 4

z	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{\mu}_1(z)$	-0.000004	0.199996	0.400001	0.600011	0.799993	1.000073
$\mu_1(z)$	0.0	0.20000	0.40000	0.60000	0.80000	1.0000
$\mu_2(z)$	-0.000016	0.039998	0.160001	0.360006	0.639996	1.000041
$\tilde{\mu}_2(z)$	0.0	0.04000	0.160000	0.360000	0.640000	1.00000

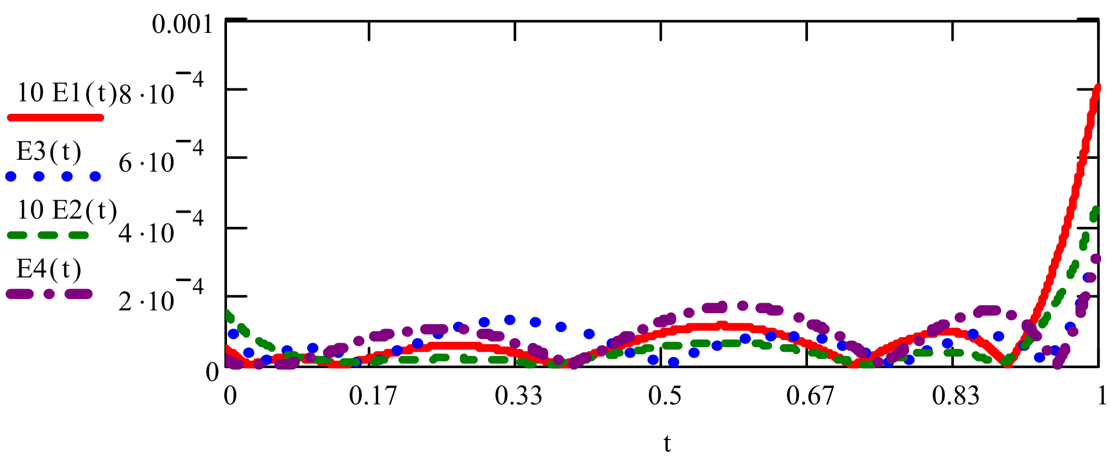

Figure 7. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N = 1000$

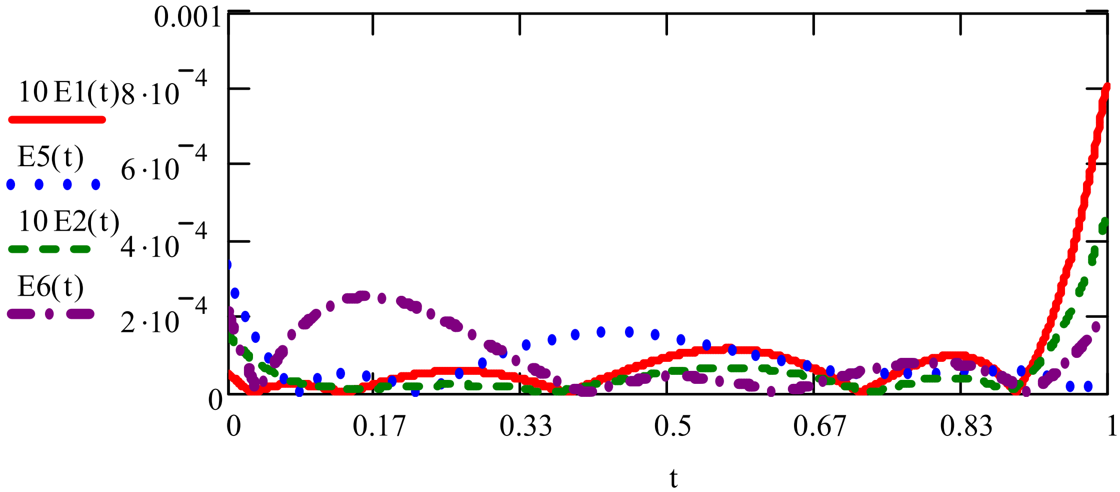

Figure 8. Absolute errors with added noise $\delta_1(=\sigma_N)$ for $N=500$

Example 5.

Consider the system of generalized Abel integral equation with $q_{ij}(y)=1$, for $i,j=1,2$, $b=1/3$ and

$$h_1(y)=\left[\frac{1}{4}\left(y\Gamma\left(\frac{2}{3}\right)\left(\frac{2\Gamma\left(\frac{-17}{12}\right)\sqrt{y}}{\Gamma\left(\frac{-3}{4}\right)}+\frac{\sqrt{\pi}}{\Gamma\left(\frac{13}{6}\right)}\right)+3\right)y^{4/3}+\frac{16}{17}\,{}_2F_1\left(\frac{-17}{12},\frac{1}{3},-\frac{5}{12},y^2\right)\right],$$

$$h_2(y)=\left[\frac{1}{4}\left(\Gamma\left(\frac{2}{3}\right)\left(\frac{2\sqrt{y}\,\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{29}{12}\right)}-\frac{\Gamma\left(-\frac{7}{6}\right)}{\sqrt{\pi}}\right)y^{7/3}+3(1-y^2)^{2/3}\right)+\frac{3}{7}\,{}_2F_1\left(-\frac{7}{6},\frac{1}{3},-\frac{1}{6},y^2\right)\right].$$

This has the exact solution $\mu_1(z)=z+z^2$ and $\mu_2(z)=z^{5/2}$. The corresponding values of $C_1^T, C_2^T, \sigma_{1000}^1, \sigma_{500}^1, \sigma_{1000}^2$ and σ_{500}^2 are given as

$$C_1^T=[0.011095, 0.060171, 0.148163, 0.227998, 0.247406, 0.166718],$$

$$C_2^T=[0.098659, 0.303574, 0.475602, 0.56168, 0.527859, 0.333320],$$

$$\sigma_{1000}^1=2.091\times10^{-5}, \qquad \sigma_{500}^1=2.104\times10^{-5},$$

$\sigma^2_{1000} = 5.503 \times 10^{-5}$, $\sigma^2_{500} = 5.65 \times 10^{-5}$.

Table 5. Approximate and exact values of μ for example 5

z	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{\mu}_1(z)$	-0.00006	0.239993	0.559996	0.959993	1.440007	1.999917
$\mu_1(z)$	0.0	0.24	0.56	0.96	1.44	2.0
$\mu_2(z)$	0.000271	0.01792	0.101171	0.278885	0.572426	1.000307
$\tilde{\mu}_2(z)$	0.0	0.017889	0.101193	0.278855	0.572433	1.0

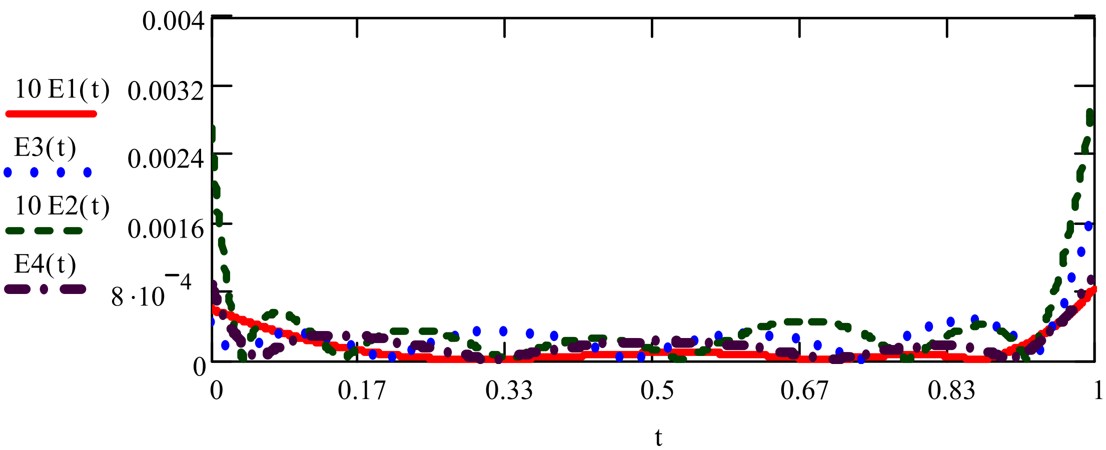

Figure 9. Absolute errors with added noise $\delta_1 (= \sigma_N)$ for $N = 1000$

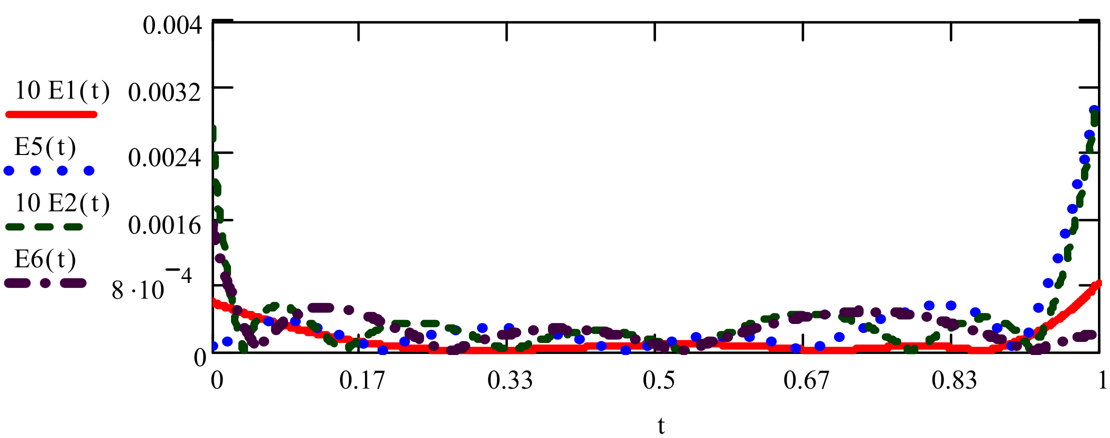

Figure 10. Absolute errors with added noise $\delta_1 (= \sigma_N)$ for $N = 500$

7. Conclusion

We have composed an almost Bernstein polynomials operational matrices of integration (AOMI) and used them to present an efficient and stable method for numerical approximation of system of singular integral equation with generalized Abel type kernel. The stability of method with relevancy to the data is restored, and a favorable result is attained, even for highly added noise modest sample intervals in the data. The selection of a small number of orthonormal polynomials (as shown for $\alpha = 5$) makes the method easy and straight forward to use. This algorithm can be applied to any system of integral equations to get the desired numerical inversion, which is one of the applications of the proposed method.

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