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Explicit Formulas for Solutions Of Maxwell's Equations in Conducting Media

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Abstract

A new explicit presentation of the fundamental solution of the time-dependent Maxwell's equations in conducting isotropic media is derived by Hadamard techniques through the fundamental solution of the telegraph operator. This presentation is used to obtain explicit formulas for generalized solutions of the initial value problem for Maxwell's equations. A new explicit Kirchhoff's formula for the classical solution of the initial value problem for the Maxwell equations in conducting media is derived. The obtained explicit formulas can be used in the boundary integral method, Green's functions method and for computation of electric and magnetic fields in conducting media and materials.

Keywords: Maxwell's equations; Conducting media; Fundamental solution; Explicit formula

MSC 2010 No.: 78A25, 46F10, 35L45

1. Introduction

The electrical conductivity is an elegant experimental tool to probe the structural defects and internal purity of crystalline materials (see, for example, Gowri and Sahaya Shajan (2006) and Angel Mary Greena et al. (2010)). The wave propagation in these materials and media is described by the time-dependent Maxwell's equations containing the permittivity, permeability and conductivity.

The solutions of these equations are important issues in material science, geophysics, biology, etc. Let us point out some methods for solving these Maxwell's equations. A general solution for the time-dependent Maxwell's equations for an infinite isotropic medium with constant permittivity, permeability and conductivity has been studied in Moses and Prosser (1990). The method of the paper Moses and Prosser (1990) is based on the technique of the eigenfunctions of the curl operator. In the work Oster and Turbe (1993), the solution of the Maxwell's equations is constructed in a composite isotropic structure using the expansion technique. Moreover, numerical experiments given in Oster and Turbe (1993) for the optical medium present a real interest. Several methods of computation of solutions for Maxwell's equations are presented in works Yakhno and Ersoy (2015), Yakhno and Altunkaynak (2016), and Yakhno and Altunkaynak (2018).

The definition of a fundamental solution of a linear differential operator was given by L. Schwartz in 1950 (see, for example, Schwartz (1966)). More generally, Schwartz (1966) defined fundamental matrices (fundamental solutions) for systems of differential operators. As mentioned by Ortner and Wagner (2015), "The reason for this more general definition lies in the importance of such systems in the natural sciences: Physical phenomena are in general described by vector or tensor fields (as, e.g., displacements, electric and magnetic fields etc.) instead of by single scalar quantities, as e.g., the temperature." The content of the Malgrange–Ehrenpreis Theorem is that every non-trivial linear differential operator, as well as every system of differential operators, with constant coefficients has a fundamental solution. For some systems of differential operators of elastodynamics and crystal optics with constant coefficients the construction of the fundamental solutions can be done by the direct and inverse Fourier transform (or the Laplace transform), invariance methods and the Herglotz-Petrovsky formula (see, for example, Ortner and Wagner (2015), Burridge and Qian (2006)). The general principles in deriving analytical representations of the fundamental solutions by Fourier-Laplace transform, which are "as simple as possible," can be found in Ortner and Wagner (2015). The derivation of the formulas for the fundamental solutions by analytical computation of the direct and inverse Fourier-Laplace transform has been done for many scalar differential operators (Laplace, wave, Helmholtz, heat, diffusion and other operators, see Ortner and Wagner (2015), Vladimirov (1971), Kanwal (1983)) and for some particular cases of systems of differential operators with constant coefficients (Lame system of isotropic elasticity, crystal optics, see Ortner and Wagner (2015)). We note that for the case of the time-dependent Maxwell's system of bi-axial crystal optics an explicit formula of the singular part of the fundamental solution only has been derived in Wagner (2011). The complete presentation of the fundamental solution for this system has been done by means of Herglotz-Petrovskii formula in the form of integrals around real loops in Burridge and Qian (2006). This presentation does not have a simple form and demonstrates complicated structure of the fundamental solution, even when coefficients of the Maxwell's operator of bi-axial crystal optics are constants. Finding explicit presentations of the fundamental solutions for the systems of partial differential operators (in particular Maxwell's operators) is an important issue for many applications (for example, boundary integral method, Green's function method and others).

In the present paper the fundamental solution for the Maxwell's operators in conducting media is derived in a new explicit form containing singular terms, such as the Dirac delta functions and their derivatives with supports on the fronts of electromagnetic waves, as well as regular terms described by Bessel's functions and their derivatives. This form does not contain any integrals and Fourier-Laplace transforms. The derivation of this explicit form has been obtained by Hadamard technique. We note that Hadamard (1953) (see, also, Courant and Hilbert (1962)) has described the idea how to find the structure of the solution of the Cauchy's problem for the scalar hyperbolic equations of the second order. This idea has been realized in Romanov (1987), Romanov (2002), and Yakhno (2001) for finding the structure of the fundamental solution of a second order hyperbolic operator. In the present paper, we state also the generalized initial value problem for the Maxwell's equations, when the non-homogeneous term (density of current) is a vector function, whose components belong to the space of generalized functions. A generalized solution of this problem is found in the form of the convolution of the fundamental matrix with non-homogeneous term. Moreover, a new Kirchhoff's formula for the classical solution of Maxwell's equations with zero initial data is obtained by the direct calculation of the convolution.

The paper is organized as follows. The explicit formula of the fundamental solution for the telegraph operator is described in Section 2. The derivation of columns of the fundamental solution (fundamental matrix) of the time-dependent Maxwell's operator (equations) in conducting media through the fundamental solution of 3D telegraph operator is given in Section 3. Section 4 is devoted to the statement and solution of a generalized initial value problem for Maxwell's equations. The explicit Kirchhoff's formula for the classical solution of the initial value problem for Maxwell's equations is derived in Section 5.

2. Fundamental Solution of the Telegraph Operator

In this section we apply techniques from Romanov (1987), Romanov (2002), and Yakhno (2001) to derive "as simple as possible" formula for the fundamental solution of the telegraph operator

$$\mathbf{L} = \frac{\partial^2}{\partial t^2} - c^2\Delta + 2M\frac{\partial}{\partial t}, \tag{1}$$

where $c > 0$, $M \in R$ are constants and Δ is the Laplace operator in R^3 defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Since the telegraph operator is hyperbolic, then there exists one and only one fundamental solution with support in $t \geq 0$, $x = (x_1, x_2, x_3) \in R^3$ (see proposition 2.4.11 from Ortner and Wagner (2015)). Therefore, the fundamental solution of the telegraph operator can be defined as a generalized function (distribution) $T(x,t)$ satisfying

$$\mathbf{L}T = \delta(x,t), \quad x = (x_1, x_2, x_3) \in R^3, \quad t \in R, \tag{2}$$

$$T|_{t<0} = 0, \tag{3}$$

where $\delta(x,t)$ is the Dirac delta function of four variables $x = (x_1, x_2, x_3) \in R^3$, $t \in R$.

We will show that the fundamental solution $T(x,t)$ of the telegraph operator can be presented by the formula

$$T(x,t) = \frac{1}{2\pi c^3}\exp(-Mt)\left(\theta(t)\delta(\Gamma) + \frac{M}{2\sqrt{\Gamma}}\theta(t - \frac{|x|}{c})I_1(\frac{M\sqrt{\Gamma}}{2})\right), \quad (4)$$

where

$$\Gamma = t^2 - \frac{|x|^2}{c^2}, \quad I_1(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{k!(k+1)!}.$$

We note that $I_1(z)$ is known as the modified Bessel function (or the hyperbolic Bessel function) of the first kind of the first order.

To derive the above formula, we search for the fundamental solution for the telegraph operator in the form

$$T(x,t) = \theta(t)\sum_{k=-1}^{\infty} \alpha_k(x,t)\,\theta_k(\Gamma), \quad (5)$$

where $\theta(t)$ is the Heaviside step function (i.e., $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for $t < 0$), $\alpha_k(x,t)$ are unknown functions, $\theta_{-1}(\Gamma) = \delta(\Gamma)$ is the Dirac delta function of one variable Γ, $\theta_0(\Gamma) = \theta(\Gamma)$, and $\theta_k(\Gamma) = \frac{1}{k!}\Gamma^k H(\Gamma),\ k \geq 1$.

We note that the following equalities are realized,

$$\frac{\partial \Gamma}{\partial t} = 2t, \quad \nabla_x \Gamma = -\frac{2x}{c^2}, \quad \left(\frac{\partial \Gamma}{\partial t}\right)^2 - c^2|\nabla_x \Gamma|^2 = 4\Gamma,$$

$$\Gamma\theta_{k-2}(\Gamma) = (k-1)\theta_{k-1}(\Gamma), \quad \theta_k'(\Gamma) = \theta_{k-1}(\Gamma),$$

and the following properties of $\theta_k(\Gamma)$ are satisfied (see, for example, Yakhno (2001)),

$$-\theta'(t)\cdot\theta_{-1}(\Gamma) = 2\pi c^3\delta(x,t),\ -\theta'(t)\cdot\theta_k(\Gamma) = 0\,(k \geq 0),\ \theta(t)\cdot\theta_k(\Gamma) = 0\,(k \geq -1).$$

Moreover, the following expressions for $\partial T/\partial t$, $\partial^2 T/\partial t^2$, ∇T, ΔT take place:

$$\frac{\partial T}{\partial t} = \theta(t)\sum_{k=-1}^{\infty} \alpha_k\,\theta_k(\Gamma) + \theta(t)\frac{\partial}{\partial t}\sum_{k=-1}^{\infty} \alpha_k\,\theta_k(\Gamma)$$

$$= \theta(t)\sum_{k=-1}^{\infty}\left[\frac{\partial \alpha_{k-1}}{\partial t} + \frac{\partial \Gamma}{\partial t}\alpha_k\right]\theta_{k-1}(\Gamma),$$

$$\frac{\partial^2 T}{\partial t^2} = -\theta'(t)\sum_{k=-1}^{\infty} \alpha_k\,\theta_k(\Gamma) + 2\frac{\partial}{\partial t}\left[\delta(t)\sum_{k=-1}^{\infty} \alpha_k\,\theta_k(\Gamma)\right]$$

$$+H(t)\frac{\partial^2}{\partial t^2}\sum_{k=-1}^{\infty} \alpha_k\,\theta_k(\Gamma) = 2\pi c^3\alpha_{-1}(0,0)\delta(x,t)$$

$$+\theta(t)\sum_{k=-1}^{\infty}\left[\frac{\partial^2\alpha_{k-1}}{\partial t^2}+2\frac{\partial\alpha_k}{\partial t}\frac{\partial\Gamma}{\partial t}+\alpha_k\frac{\partial^2\Gamma}{\partial t^2}+\left(\frac{\partial\Gamma}{\partial t}\right)^2\alpha_k\frac{(k-1)}{\Gamma}\right]\theta_{k-1}(\Gamma),$$

$$\nabla T=\theta(t)\sum_{k=-1}^{\infty}\left[\nabla_x\alpha_{k-1}+\alpha_k\nabla_x\Gamma\right]\theta_{k-1}(\Gamma),$$

$$\Delta T=\theta(t)\sum_{k=-1}^{\infty}\left[\Delta\alpha_{k-1}+2\nabla\alpha_k\nabla\Gamma+\alpha_k\Delta\Gamma\right.$$

$$\left.+\alpha_k\left(\nabla\Gamma\right)^2\frac{(k-1)}{\Gamma}\right]\theta_{k-1}(\Gamma),$$

where $\alpha_2\equiv 0$.

Substituting (5) into (2) and using above mentioned equalities, properties and expressions we find

$$\left[2\pi c^3\alpha_{-1}(0,0))-1\right]\delta(x,t)+\theta_0(t)\sum_{k=-1}^{\infty}\left[\mathbf{L}\alpha_{k-1}+2\frac{\partial\alpha_k}{\partial t}\frac{\partial\Gamma}{\partial t}-2c^2\nabla_x\alpha_k\nabla_x\Gamma\right.$$

$$\left.+\left(\mathbf{L}\Gamma+4(k-1)\right)\alpha_k\right]\theta_{k-1}(\Gamma)=0.$$

Equating coefficients to zero for $\delta(x,t)$ and $\theta_{k-1}(\Gamma)$ we find

$$\alpha_{-1}(0,0)=\frac{1}{2\pi c^3},$$

$$t\frac{\partial\alpha_k}{\partial t}+x\nabla\alpha_k+\left(4(k+1)+Mt\right)\alpha_k=-\frac{1}{4}\mathbf{L}\alpha^{k-1}=0,\qquad k=-1,0,1,2,.... \tag{6}$$

Let us consider the curve $x=x(\tau)$, $t=p\tau$ defined by equations

$$\frac{dx}{d\tau}=\frac{x(\tau)}{\tau},\quad t=p\tau, \tag{7}$$

where p is a constant, τ is a real parameter. Equation (6) along this curve can be written in the form

$$\frac{d}{d\tau}\left[\tau^{k+1}\alpha_k(x(\tau),p\tau)\right]+Mp\tau^{k+1}\alpha_k(x(\tau),p\tau)=-\frac{\tau^k}{4}\mathbf{L}\alpha_{k-1}|_{x=x(\tau),\,t=p\tau}. \tag{8}$$

Integrating (8), we find

$$\alpha_{-1}(x,t)=\exp(-Mt)\alpha_{-1}(0,0), \tag{9}$$

$$\tau^{k+1}\alpha_k(x(\tau),p\tau)=-\frac{1}{4}\exp(-Mp\tau)\int_0^{\tau}z^k\exp(Mpz)\mathbf{L}\alpha_{k-1}|_{x=x(z),pz}dz,\ k\geq 0. \tag{10}$$

Using (9) and (10) we find

$$\mathbf{L}\alpha_{-1}=-M^2\exp(-Mt)\alpha_{-1}(0,0),\quad \alpha_0=\left(\frac{M}{2}\right)^2\exp(-Mt)\alpha_{-1}(0,0),$$

$$\alpha_k=\frac{1}{(k+1)!}\left(\frac{M}{2}\right)^{2k+2}\exp(-Mt)\alpha_{-1}(0,0),\quad k=1,2,3,....$$

Substituting formulae for α_k into (5) we obtain (4).

Remark 2.1.

The fundamental solution of the Klein - Gordon operator can be constructed by other standard technique (see, for example, Kanwal (1983), pp. 276 – 282). We note also that the method, described in this section, is significant because it can be generalized on the case of the telegraph equation (or the acoustic equation) whose coefficients are functions depending on the space variables $x = (x_1, x_2, x_3) \in R^3$ (see Yakhno (2001)).

3. Fundamental Solution of Maxwell's Equations

3.1. The fundamental matrix of the Maxwell's equations

The fundamental matrix (fundamental solution) of the Maxwell system is a 6×3 matrix whose columns

$$\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix} = (E_1^k, E_2^k, E_3^k, H_1^k, H_2^k, H_3^k)^t, \quad \text{(the superscript "}t\text{" means "transpose")}$$

satisfy the following equations,

$$\operatorname{curl} \mathbf{H}^k = \varepsilon \frac{\partial \mathbf{E}^k}{\partial t} + \sigma \mathbf{E}^k + \mathbf{e}^k \, \delta(x,t), \tag{11}$$

$$\operatorname{curl} \mathbf{E}^k = -\mu \frac{\partial \mathbf{H}^k}{\partial t}, \tag{12}$$

where $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ is the $3D$ space variable; $t \in \mathbf{R}$ is the time variable; $\mathbf{E}^k$, $\mathbf{H}^k$ are vector functions depending on x and t, i.e., $\mathbf{E}^k(x,t) = (E_1^k(x,t), E_2^k(x,t), E_3^k(x,t))^t$, $\mathbf{H}^k(x,t) = (H_1^k(x,t), H_2^k(x,t), H_3^k(x,t))^t$; $\mathbf{e}^1 = (1,0,0)^t$, $\mathbf{e}^2 = (0,1,0)^t$, $\mathbf{e}^3 = (0,0,1)^t$ are basis vectors of $\mathbf{R}^3$; $\delta(x,t)$ is the Dirac delta function concentrated at $x = 0$, $t = 0$; curl is the differential operator defined by

$$\operatorname{curl} \mathbf{E}(x) = \left(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}, \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}, \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right),$$

for a vector function column $\mathbf{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))^t$.

Since the system of Equations (11), (12) is hyperbolic, then there exists one and only one fundamental solution whose columns satisfy (Proposition 2.4.11 of Ortner and Wagner (2015))

$$\mathbf{E}^k|_{t<0} = 0, \quad \mathbf{H}^k|_{t<0} = 0. \tag{13}$$

Our goal is to derive a presentation of this fundamental solution in the explicit form. To reach this aim we start to describe some properties of the fundamental matrix (fundamental solution) of Maxwell's equations in conducting media. These properties are formulated in terms of lemmas and will be used for the construction of the explicit formulas.

Lemma 3.1.
Let $\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix}$ be $k-$column of the fundamental matrix of Maxwell's equations. Then,

$$\operatorname{div}\mathbf{H}^k = 0, \quad \operatorname{div}\mathbf{E}^k = \rho_k(x,t), \tag{14}$$

where

$$\rho_k(x,t) = -\theta(t)\exp(-\frac{\sigma}{\varepsilon}t)\operatorname{div}(\mathbf{e}^k\,\delta(x)).$$

Proof:

Applying operator div to (12) we find

$$\mu\frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{H}^k\right) = 0.$$

Using inequality $\mu > 0$ and the second equality of (13) we find the first equality of (14).

Applying div to (11) we have

$$\varepsilon\frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{E}^k\right) + \sigma\operatorname{div}\mathbf{E}^k + \operatorname{div}(\mathbf{e}^k\,\delta(x,t)) = 0.$$

The last equality can be written as follows

$$\frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{E}^k\right) + \frac{\sigma}{\varepsilon}\operatorname{div}\mathbf{E}^k = -\frac{1}{\varepsilon}\operatorname{div}(\mathbf{e}^k\,\delta(x))\delta(t).$$

Using the first condition of (13) and applying the standard calculus of generalized functions (see, for example, Vladimirov (1971), pp. 143 – 147) we obtain the second equality of (14). ■

Corollary 3.1.
Let $\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix}$ be $k-$column of the fundamental matrix of Maxwell's equations. Then, $\mathbf{J}^k = \mathbf{e}^k\,\delta(x,t)$ and $\rho_k(x,t)$, defined in the statement of Lemma 3.1, satisfy the conservation law

$$\varepsilon\frac{d\rho_k}{dt} + \sigma\rho_k + \operatorname{div}\mathbf{J}^k = 0.$$

Proof:

The confirmation of the corollary follows from the second equality of (14) and the last equality of the proof of Lemma 3.1. ■

Lemma 3.2.
Let $\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix}$ be $k-$column of the fundamental matrix of Maxwell's equations and $\mathbf{L}$ be the telegraph operator defined by (1). Then, the following equality takes place,

$$\mathbf{L}\mathbf{H}^k = c^2\operatorname{curl}[\mathbf{e}^k\,\delta(x,t)], \tag{15}$$

where $\mathbf{L}\mathbf{H}^k = (\mathbf{L}H_1^k, \mathbf{L}H_2^k, \mathbf{L}H_3^k)^t$.

Proof:

Applying curl to (11) we have

$$\operatorname{curl}\operatorname{curl}\mathbf{H}^k = \varepsilon\frac{\partial}{\partial t}\operatorname{curl}\mathbf{E}^k + \sigma\operatorname{curl}\mathbf{E}^k + \operatorname{curl}[\mathbf{e}^k\,\delta(x,t)]. \tag{16}$$

Using the first equality of (14) we obtain that $\operatorname{curl}\operatorname{curl}\mathbf{H}^k = -\Delta\mathbf{H}^k$. Therefore, taking into consideration equation (12), equality (16) can be written in the form

$$-\Delta\mathbf{H}^k = -\varepsilon\mu\frac{\partial^2}{\partial t^2}\mathbf{H}^k - \sigma\mu\frac{\partial}{\partial t}\mathbf{H}^k + \operatorname{curl}[\mathbf{e}^k\,\delta(x,t)].$$

The last relation is rewritten as (15), where

$$c^2 = \frac{1}{\varepsilon\mu}, \quad 2M = \frac{\sigma}{\varepsilon}. \tag{17}$$

■

3.2. Explicit formula for columns of the fundamental matrix

Let $\begin{pmatrix}\mathbf{E}^k\\ \mathbf{H}^k\end{pmatrix}$ be $k-$column of the fundamental matrix of the Maxwell's equations. It follows from Lemma 3.2 that $\mathbf{H}^k$ can be found as a solution of (15) satisfying the second condition of (13). We note that this solution can be written as

$$\mathbf{H}^k(x,t) = c^2\operatorname{curl}(\mathbf{e}^k T(x,t)), \tag{18}$$

where $T(x,t)$ is defined by (4) with M and c satisfying (17). Really, applying the operator $\mathbf{L}$ to (18) we have

$$\mathbf{L}\mathbf{H}^k = \mathbf{L}[c^2\operatorname{curl}(\mathbf{e}^k T)] = c^2\operatorname{curl}(\mathbf{L}[\mathbf{e}^k T]) = c^2\operatorname{curl}(\mathbf{e}^k\mathbf{L}T) = c^2\operatorname{curl}(\mathbf{e}^k\delta(x,t)).$$

Remark 3.1.

Let us discuss the uniqueness of the solution of (15) in the class of generalized functions $\mathcal{D}'(\mathbf{R}^4)$ which are equal to zero for $t < 0$. Let T, defined by (4), be the fundamental solution of the telegraph operator $\mathbf{L}$ and $f(x,t) \in \mathcal{D}'(\mathbf{R}^4)$ be a generalized function such that $f(x,t) = 0$ for $t < 0$. We note that $\operatorname{supp} T \subset \Sigma^+$. Applying property 5 of the convolution (see Appendix A and Vladimirov (1971), pp. 159–163, Section 11.2) we find that the convolution $T * f$ exists in the class $\mathcal{D}'(\mathbf{R}^4)$. Using the property of the fundamental solution (see Appendix A) we find that the generalized function $u = T * f$ is a generalized solution of $\mathbf{L}\,u = f$, and this solution is unique in the class of generalized functions belonging to $\mathcal{D}'(\mathbf{R}^4)$ for which the convolution with T exists. According to property 5 of the convolution (see Appendix A) the convolution $T * u$ will exist in $\mathcal{D}'(\mathbf{R}^4)$ if $\operatorname{supp} T \subset \Sigma^+$ and $u = 0$ for $t < 0$. Therefore, using the property of the fundamental solution (see Appendix A and Vladimirov (1971), pp. 143-144, Section 10.3) we find that the generalized solution of $\mathbf{L}\,u = f$ is unique in the class of generalized functions $\mathcal{D}'(\mathbf{R}^4)$ which are equal to zero for $t < 0$. Moreover, Equation (15) can be written as follows

$$\mathbf{L}\,H_j^k = f_j^k, \quad j = 1,2,3.$$

Here $f_1^k = \frac{\partial}{\partial x_2}\left[\delta_{3k}\delta(x,t)\right] - \frac{\partial}{\partial x_3}\left[\delta_{2k}\delta(x,t)\right]$, $f_2^k = \frac{\partial}{\partial x_3}\left[\delta_{1k}\delta(x,t)\right] - \frac{\partial}{\partial x_1}\left[\delta_{3k}\delta(x,t)\right]$,
$f_3^k = \frac{\partial}{\partial x_1}\left[\delta_{2k}\delta(x,t)\right] - \frac{\partial}{\partial x_2}\left[\delta_{1k}\delta(x,t)\right]$, where $\delta_{kk} = 1$, $\delta_{ik} = 0$ for $i \neq k$.

The generalized solutions $H_j^k = T * f_j^k$ of the above equations exist in $\mathcal{D}'(\mathbf{R}^4)$ and these solutions are unique in the class of generalized functions $\mathcal{D}'(\mathbf{R}^4)$ which are equal to zero for $t < 0$. This means that $\mathbf{H}^k(x,t)$, defined by (18), is the generalized solution of (15) which is unique in the class of vector functions $\mathbf{H} = (H_1, H_2, H_3)$, whose components are generalized functions from $\mathcal{D}'(\mathbf{R}^4)$ and equal to zero for $t < 0$.

To find the explicit formula for $\mathbf{E}^k(x,t)$ we note that $\mathbf{E}^k(x,t)$ satisfies Equation (11) and the first condition of (13). Moreover, Equation (11) can be written in the form

$$\frac{\partial \mathbf{E}^k}{\partial t} + \frac{\sigma}{\varepsilon}\mathbf{E}^k = \frac{1}{\varepsilon}\mathrm{curl}\,\mathbf{H}^k - \frac{1}{\varepsilon}\mathbf{e}^k\,\delta(x,t). \tag{19}$$

Substituting (18) into (19) we have

$$\frac{\partial \mathbf{E}^k}{\partial t} + \frac{\sigma}{\varepsilon}\mathbf{E}^k = \frac{1}{\varepsilon}\left\{c^2\mathrm{curl}\,\mathrm{curl}\left[\mathbf{e}^k T(x,t)\right] - \mathbf{e}^k\,\delta(x,t)\right\}. \tag{20}$$

Applying tools of Vladimirov (1971) (see pp. 139–147) we obtain that a solution of (20) satisfying the first condition of (13) can be presented in the form

$$\mathbf{E}^k(x,t) = \frac{\theta(t)}{\varepsilon}\left\{c^2\int_0^t \exp(-\frac{\sigma}{\varepsilon}(t-z))\mathrm{curl}\,\mathrm{curl}\,\left[\mathbf{e}^k T(x,z)\right]dz \right.$$
$$\left. - \exp(-\frac{\sigma}{\varepsilon}t)\,\mathbf{e}^k\,\delta(x)\right\}. \tag{21}$$

Theorem 3.1.

Let $\varepsilon > 0$, $\mu > 0$, $\sigma \in R$ be given constants; $M = \frac{\sigma}{2\varepsilon}$, $c = \frac{1}{\sqrt{\varepsilon\mu}}$; $\begin{pmatrix}\mathbf{E}^k \\ \mathbf{H}^k\end{pmatrix}$ be $k-$column of the fundamental matrix of Maxwell's equations (11) and (12). Then, $\mathbf{H}^k$ and $\mathbf{E}^k$ are presented by explicit formulas (18) and (21), where $T(x,t)$ is defined by (4).

Proof:

The confirmation of the theorem follows from above mentioned reasoning and equalities (18) - (21). ■

Corollary 3.2.

Let $\varepsilon > 0$, $\mu > 0$ be given constants, $\sigma = 0$, $c = \frac{1}{\sqrt{\varepsilon\mu}}$; $\begin{pmatrix}\mathbf{E}^k \\ \mathbf{H}^k\end{pmatrix}$ be $k-$column of the fundamental matrix of Maxwell's equations (11), (12). Then, $\mathbf{H}^k$ and $\mathbf{E}^k$ are presented by the following explicit formulas:

$$\mathbf{H}^k(x,t) = \frac{1}{2\pi c}\theta(t)\mathrm{curl}\,\left[\mathbf{e}^k\delta(t^2 - \frac{|x|^2}{c^2})\right], \tag{22}$$

$$\mathbf{E}^k(x,t) = \frac{\theta(t)}{\varepsilon}\left\{\frac{1}{2\pi c}\int_0^t \mathrm{curl}\,\mathrm{curl}\,\left[\mathbf{e}^k\delta(z^2 - \frac{|x|^2}{c^2})\right]dz - \mathbf{e}^k\,\delta(x)\right\}. \tag{23}$$

Proof:

The confirmation of the corollary follows from equalities (18), (21) and (4) for $\sigma = 0$. We note that, using the well known formula

$$\delta(t^2 - \frac{|x|^2}{c^2}) = \frac{c}{2|x|}\delta\left(t - \frac{|x|}{c}\right),$$

formulas (22) and (23) can be written in the form

$$\mathbf{H}^k(x,t) = \mathrm{curl}\,\left[\theta(t)\frac{\mathbf{e}^k}{4\pi}\delta\left(t - \frac{|x|}{c}\right)\right], \tag{24}$$

$$\mathbf{E}^k(x,t) = -\frac{\theta(t)}{\varepsilon}\mathbf{e}^k\delta(x) + \mathrm{curl}\,\mathrm{curl}\,\left[\frac{\mathbf{e}^k}{4\pi\varepsilon|x|}\theta\left(t - \frac{|x|}{c}\right)\right]. \tag{25}$$

■

3.3. Maxwell's matrix operator

We define the Maxwell's matrix operator $\mathcal{M}$ as the 6×6 matrix of the form

$$\mathcal{M} = \begin{pmatrix} -\varepsilon\mathbf{I}_3\frac{\partial}{\partial t} - \sigma\mathbf{I}_3 & \mathrm{curl} \\ \mathrm{curl} & \mu\mathbf{I}_3\frac{\partial}{\partial t} \end{pmatrix}.$$

Here $\varepsilon > 0$, $\mu > 0$, σ are constants; $\mathbf{I}_3$ is the 3×3 identity matrix;

$$\mathrm{curl}\; = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{pmatrix}.$$

Applying results of Section 3.1 we find that if $\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix}$ is the $k-$column of the fundamental matrix, then

$$\mathcal{M}\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix} = \begin{pmatrix} \mathbf{e}^k\delta(x,t) \\ \mathbf{0} \end{pmatrix},$$

where $\mathbf{0} = (0,0,0)^t$.

4. Solution of a Generalized Initial Value Problem for Maxwell's Equations

4.1. Convolution of the fundamental matrix of the Maxwell's operator and vector generalized function

Let $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}$; $\mathcal{D}(\mathbf{R}^4)$ be the space of test functions and $\mathcal{D}'(\mathbf{R}^4)$ be the space of generalized functions (distributions) (see, for example, Vladimirov (1971)). For the given generalized functions $f \in \mathcal{D}'(\mathbf{R}^4)$ and $g \in \mathcal{D}'(\mathbf{R}^4)$ convolution of f and g is denoted as $f * g$

similar to Vladimirov (1971), pp. 102 – 110, (see also Appendix A of the present paper). Let $\mathbf{E} = (E_1, E_2, E_3)$, $\mathbf{H} = (H_1, H_2, H_3)$ be vector functions whose components be generalized functions from $\mathcal{D}'(\mathbf{R}^4)$ and f be a generalized function from $\mathcal{D}'(\mathbf{R}^4)$. Then, the convolutions $\mathbf{E} * f$, $f * \mathbf{E}$ are defined by $\mathbf{E} * f = (E_1 * f, E_2 * f, E_3 * f)$ and $f * \mathbf{E} = (f * E_1, f * E_2, f * E_3)$, respectively. Similar we define

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} * f = (E_1 * f, E_2 * f, E_3 * f, H_1 * f, H_2 * f, H_3 * f)^t,$$

$$f * \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = (f * E_1, f * E_2, f * E_3, f * H_1, f * H_2, f * H_3)^t,$$

if $E_j * f$ and $H_j * f$ exist for $j = 1, 2, 3$.

Let $\mathcal{E}$ be the fundamental matrix of the Maxwell's operator $\mathcal{M}$. The k-column of $\mathcal{E}$ is denoted as

$$\begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix} = (E_1^k, E_2^k, E_3^k, H_1^k, H_2^k, H_3^k)^t,$$

where $\mathbf{E}^k$, $\mathbf{H}^k$ are defined by (18), (21) for $\sigma \neq 0$ and (24), (25) for $\sigma = 0$.

Applying property 5 of the convolution (see Appendix A and Vladimirov (1971), pp. 159 – 163), we find that the following convolutions

$$\theta(t)\delta(t^2 - \frac{|x|^2}{c^2}) * f, \quad \theta(t)\delta(x) * f, \quad \theta(t - \frac{|x|}{c}) \frac{I_1(\frac{\sigma}{\varepsilon}\sqrt{t^2 - \frac{|x|^2}{c^2}})}{\sqrt{t^2 - \frac{|x|^2}{c^2}}} * f$$

exist for any generalized function $f \in \mathcal{D}'(\mathbf{R}^4)$ satisfying $f|_{t<0} = 0$. Hence, using property 4 of convolution (see Appendix A) we obtain that $E_j^k * f$ and $H_j^k * f$ exist for $j = 1, 2, 3$; $k = 1, 2, 3$ and for any generalized function $f \in \mathcal{D}'(\mathbf{R}^4)$ satisfying $f|_{t<0} = 0$. Therefore, there exists the convolution $\mathcal{E}$ with any vector function $\mathbf{J} = (J_1, J_2, J_3)^t$, whose components are generalized functions from $\mathcal{D}'(\mathbf{R}^4)$ satisfying $J_m|_{t<0} = 0$, $m = 1, 2, 3$. We define this convolution by

$$\mathcal{E} * \mathbf{J} = \sum_{k=1}^{3} \begin{pmatrix} \mathbf{E}^k \\ \mathbf{H}^k \end{pmatrix} * J_k$$

$$= (\sum_{k=1}^{3} E_1^k * J_k, \sum_{k=1}^{3} E_2^k * J_k, \sum_{k=1}^{3} E_3^k * J_k, \sum_{k=1}^{3} H_1^k * J_k, \sum_{k=1}^{3} H_2^k * J_k, \sum_{k=1}^{3} H_3^k * J_k)^t.$$

4.2. A generalized solution of the initial value problem for Maxwell's equations

The main result of this section is the following theorem about the generalized solution of the initial value problem for Maxwell's equations.

Theorem 4.1.

Let components of $\tilde{J} = (J_1, J_2, J_3)$ be generalized functions from $\mathcal{D}'(\mathbf{R}^4)$ *satisfying* $J_m|_{t<0} = 0$, $m = 1, 2, 3$. Then,

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathcal{E} * \tilde{J}$$

is a generalized solution of the initial value problem

$$\mathcal{M} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \tilde{J} \\ \mathbf{0} \end{pmatrix}, \tag{26}$$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \Big|_{t<0} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{27}$$

Proof:

Using Section 4.1 we find that components of the vector $\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathcal{E} * \tilde{J}$ exist in the space of generalized functions $\mathcal{D}'(\mathbf{R}^4)$ for any $\tilde{J} = (J_1, J_2, J_3)$ with components from $\mathcal{D}'(\mathbf{R}^4)$ satisfying $J_m|_{t<0} = 0$, $m = 1, 2, 3$. Moreover, we have

$$\begin{aligned}
\mathcal{M}\left(\mathcal{E} * \mathbf{J}\right) &= \sum_{k=1}^{3} \mathcal{M} \begin{pmatrix} \mathbf{E}^k * J_k \\ \mathbf{H}^k * J_k \end{pmatrix} = \sum_{k=1}^{3} \Bigg(-\varepsilon \frac{\partial(\mathbf{E}^k * J_k)}{\partial t} \\
&- \sigma(\mathbf{E}^k * J_k) + \operatorname{curl}(\mathbf{H}^k * J_k), \mu \frac{\partial(\mathbf{H}^k * J_k)}{\partial t} + \operatorname{curl}(\mathbf{E}^k * J_k) \Bigg)^t \\
&= \sum_{k=1}^{3} \left((-\varepsilon \frac{\partial \mathbf{E}^k}{\partial t} - \sigma \mathbf{E} + \operatorname{curl} \mathbf{H}^k) * J_k, (\mu \frac{\partial \mathbf{H}^k}{\partial t} + \operatorname{curl} \mathbf{E}^k) * J_k \right)^t \\
&= \sum_{k=1}^{3} \left((\mathbf{e}^k \delta(x,t)) * J_k, 0, 0, 0 \right)^t = \sum_{k=1}^{3} \left((\mathbf{e}^k J_k, 0, 0, 0) \right)^t = \begin{pmatrix} \tilde{J} \\ \mathbf{0} \end{pmatrix}.
\end{aligned}$$

Equality (26) is proved. Since $\mathcal{E}|_{t<0} = 0$ and $\tilde{J}|_{t<0} = 0$, then

$$\begin{pmatrix} \tilde{J} \\ \mathbf{0} \end{pmatrix} |_{t<0} = \mathcal{E} * \tilde{J}|_{t<0} = 0.$$

■

Remark 4.1.

Equation (26) can be written in the form

$$-\varepsilon \frac{\partial \mathbf{E}}{\partial t} - \sigma \mathbf{E} + \operatorname{curl} \mathbf{H} = \tilde{J}, \tag{28}$$

$$\mu \frac{\partial \mathbf{H}}{\partial t} + \operatorname{curl} \mathbf{E} = \mathbf{0}. \tag{29}$$

Equations (28) and (29) are equalities in the space of generalized functions $\mathcal{D}'(\mathbf{R}^4)$. We note also that if the vector functions $\mathbf{E}$ and $\mathbf{H}$ are a solution of equations (28) and (29) and satisfy conditions

(27), then they are the solution of Maxwell's equations (see Appendix B). We note that there are only sources of electromagnetic waves in real phenomena which are modeled by (28) and (29) (more exactly by (40) – (43), see Appendix B). Here the inhomogeneous term of (29) is always equal to zero. This gave us an opportunity to define the fundamental solution like we made in Section 3.1.

4.3. An explicit formula for the classical solution of the initial value problem for Maxwell's equations

Let us consider the initial value problem for equation (28) and (29) subject to initial data

$$\mathbf{E}(x,+0)=\mathbf{0},\quad \mathbf{H}(x,+0)=\mathbf{0}, \tag{30}$$

where components of $\mathbf{J}(x,t)$ are classical functions from the class of continuous functions $C(\mathbf{R}^3\times[0,+\infty))$. Equations (28) and (29) can be written in the operator form (26) (see Remark 4.1). Let us suppose that there is a classical solution $\mathbf{E}(x,t)$, $\mathbf{H}(x,t)$ of (28), (29), (30). This means that components of $\mathbf{E}(x,t)$, $\mathbf{H}(x,t)$ are functions from the class $C^1(\mathbf{R}^3\times(0,+\infty))\cap C(\mathbf{R}^3\times[0,+\infty))$ satisfying (30) as $t\to+0$. We shall continue vector functions $\mathbf{E}(x,t)$, $\mathbf{H}(x,t)$, $\mathbf{J}(x,t)$ as zero for $t<0$, supposing

$$\tilde{E}=\begin{cases}\mathbf{E}(x,t),\ t\geq 0\\ \mathbf{0},\ t<0\end{cases},\ \tilde{H}=\begin{cases}\mathbf{H}(x,t),\ t\geq 0\\ \mathbf{0},\ t<0\end{cases},\ \tilde{J}=\begin{cases}\mathbf{J}(x,t),\ t\geq 0\\ \mathbf{0},\ t<0\end{cases}. \tag{31}$$

Using $\tilde{E}$, $\tilde{H}$, $\tilde{J}$ we can define a regular generalized vector function by the standard rule (see, for example, Vladimirov (1971)). We will use the same notations $\tilde{E}$, $\tilde{H}$, $\tilde{J}$ for these regular generalized vector functions. Let us consider the matrix operator $\mathcal{M}^*$ which is defined by

$$\mathcal{M}^*=\begin{pmatrix}\varepsilon\mathbf{I}_3\frac{\partial}{\partial t}-\sigma\mathbf{I}_3 & \mathrm{curl}\\ \mathrm{curl} & -\mu\mathbf{I}_3\frac{\partial}{\partial t}\end{pmatrix}.$$

Remark 4.2.

Let $\vec{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$ be a vector function whose components from $\mathcal{D}(\mathbf{R}^4)$. Then,

$$\int_{\mathbf{R}^3}\mathrm{curl}\,\tilde{H}\cdot\vec{\varphi}\,dx$$

$$\begin{aligned}&=\int_{\mathbf{R}^3}\left((\frac{\partial H_3}{\partial x_2}-\frac{\partial H_2}{\partial x_3})\varphi_1+(\frac{\partial H_1}{\partial x_3}-\frac{\partial H_3}{\partial x_1})\varphi_2+(\frac{\partial H_2}{\partial x_1}-\frac{\partial H_1}{\partial x_2})\varphi_3\right)dx\\&=\int_{\mathbf{R}^3}\left((\frac{\partial \varphi_3}{\partial x_2}-\frac{\partial \varphi_2}{\partial x_3})H_1+(\frac{\partial \varphi_1}{\partial x_3}-\frac{\partial \varphi_3}{\partial x_1})H_2+(\frac{\partial \varphi_2}{\partial x_1}-\frac{\partial \varphi_1}{\partial x_2})H_3\right)dx\\&=\int_{\mathbf{R}^3}\tilde{H}\cdot\,curl\,\vec{\varphi}\,dx,\end{aligned}$$

where $\cdot$ is the operation of the 'dot' product in $\mathbf{R}^3$.

Remark 4.3.

Let

$$\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix} = (\varphi_1,\varphi_2,\varphi_3,\psi_1,\psi_2,\psi_3)^t, \quad \varphi_k,\psi_k \in \mathcal{D}(\mathbf{R}^4), k=1,2,3.$$

Using Remark 4.2 we can show that

$$\left\langle \mathcal{M}\begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}, \begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}\right\rangle = \int_{\mathbf{R}^3} \mathcal{M}\begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}\cdot\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}\,dx\,dt$$
$$= \int_{\mathbf{R}^3}\begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}\cdot\mathcal{M}^*\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}\,dx\,dt = \left\langle \begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}, \mathcal{M}^*\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}\right\rangle.$$

This means that $\mathcal{M}^*$ is adjoint to $\mathcal{M}$.

Now we show that $\tilde{E}$ and $\tilde{H}$ satisfy equations (28) and (29) in the sense of generalized functions. In fact for each test vector function $\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix} = (\varphi_1,\varphi_2,\varphi_3,\psi_1,\psi_2,\psi_3)^t$ such that φ_m,ψ_m, $m=1,2,3$, we have

$$\left\langle \mathcal{M}\begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}, \begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}, \mathcal{M}^*\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}\right\rangle$$

$$= \int_0^{+\infty}\int_{\mathbf{R}^3}\left((\varepsilon\frac{\partial\vec{\varphi}}{\partial t} - \sigma\vec{\varphi} + \operatorname{curl}\vec{\psi})\cdot\tilde{E}(x,t) + (\operatorname{curl}\vec{\varphi} - \mu\frac{\partial\vec{\psi}}{\partial t})\cdot\tilde{H}(x,t)\right)dx\,dt$$
$$= \int_0^{+\infty}\int_{\mathbf{R}^3}\left((-\varepsilon\frac{\partial\tilde{E}}{\partial t} - \sigma\tilde{E} + \operatorname{curl}\tilde{H})\cdot\vec{\varphi}(x,t) + (\operatorname{curl}\tilde{E} + \mu\frac{\partial\tilde{H}}{\partial t})\cdot\vec{\psi}(x,t)\right)dx\,dt$$
$$+ \int_{\mathbf{R}^3}\left(\varepsilon\vec{\varphi}(x,+0)\tilde{E}(x,+0) - \mu\vec{\psi}(x,+0)\tilde{H}(x,+0)\right)dx$$
$$= \int_0^{+\infty}\int_{\mathbf{R}^3}\mathcal{M}\begin{pmatrix}\tilde{E}\\ \tilde{H}\end{pmatrix}\cdot\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}dx\,dt = \int_0^{+\infty}\int_{\mathbf{R}^3}\begin{pmatrix}\tilde{J}\\ \mathbf{0}\end{pmatrix}\cdot\begin{pmatrix}\vec{\varphi}\\ \vec{\psi}\end{pmatrix}dx\,dt.$$

This means that $\tilde{E}$ and $\tilde{H}$ satisfy equations (28) and (29) in space of generalized functions $\mathcal{D}'(\mathbf{R}^4)$. As a result we can conclude that the classical solutions of the initial value problem (28), (29), (30) are among these generalized solutions which become zero when $t<0$. Using Theorem 3.1 we find that

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \mathcal{E} * \tilde{J},$$

is a solution of (28) and (29) satisfying (27) when components of $\mathbf{J}(x,t)$ are classical continuous (or continuously differentiable) functions for $t \geq 0$, $x \in \mathbf{R}^3$. As a result of explicit evaluation of $\mathcal{E} * \tilde{J}$ we obtain the following theorem about explicit formulas for the classical solution of the initial value problem for Maxwell's equations.

Theorem 4.2.

Let components of $\mathbf{J}(x,t)$ belong to $C^4(\mathbf{R}^3 \times [0,+\infty))$. Then,

$$\mathbf{E}(x,t) = -\frac{1}{\varepsilon}\int_0^t \exp(-\frac{\sigma}{\varepsilon}(t-\tau))\mathbf{J}(0,\tau)\,d\tau$$

$$+\int_0^t \left[\int\int\int_{|x-\xi|\le c(t-\tau)} \left(\frac{1}{4\pi\varepsilon\,|x-\xi|}\exp(-\frac{\sigma}{\varepsilon}(t-\tau-\frac{|x-\xi|}{2c}))\right.\right.$$

$$\left.\left.+\frac{c^2\sigma}{4\varepsilon^2}\int_{|x-\xi|/c}^{t-\tau} \frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{z^2-\frac{|x-\xi|^2}{c^2}})}{\sqrt{z^2-\frac{|x-\xi|^2}{c^2}}}\,dz\right)\mathrm{curl}_\xi\,\mathrm{curl}_\xi\,\mathbf{J}(\xi,\tau)\,d\xi\right]\,d\tau, \tag{32}$$

$$\mathbf{H}(x,t) = \frac{1}{4\pi}\int\int\int_{|x-\xi|\le ct}$$

$$\exp(-\frac{\sigma|x-\xi|}{2c\varepsilon})\,\frac{\mathrm{curl}_\xi\,\mathbf{J}(\xi,z)}{|x-\xi|}\Big|_{z=t-\frac{|x-\xi|}{c}}\,d\xi + \frac{\sigma}{8\pi c\varepsilon}\int_0^t$$

$$\left(\int\int\int_{|x-\xi|\le c(t-\tau)} \frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{(t-\tau)^2-\frac{|x-\xi|^2}{c^2}})}{\sqrt{(t-\tau)^2-\frac{|x-\xi|^2}{c^2}}}\mathrm{curl}_\xi\tilde{J}(\xi,\tau)d\xi\right)\,d\tau, \tag{33}$$

are a classical solution of the initial value problem (28), (29), (30).

Proof:

We note that for the case when T , J_m, $m=1,2,3$ are regular or singular generalized functions for which the convolutions $T*J_m$, $m=1,2,3$ exist, we have

$$\mathrm{curl}\,[T\,\mathbf{e}^m]*\tilde{J} = \sum_{m=1}^{3}\left(\delta_{3m}\frac{\partial T}{\partial x_2}*J_m\right.$$

$$\left.-\delta_{2m}\frac{\partial T}{\partial x_3}*J_m, \delta_{1m}\frac{\partial T}{\partial x_3}*J_m-\delta_{3m}\frac{\partial T}{\partial x_1}*J_m, \delta_{2m}\frac{\partial T}{\partial x_1}*J_m-\delta_{1m}\frac{\partial T}{\partial x_2}*J_m\right)$$

$$=\sum_{m=1}^{3}\left(T*(\delta_{3m}\frac{\partial J_m}{\partial x_2}-\delta_{2m}\frac{\partial J_m}{\partial x_3}),\right.$$

$$\left.T*(\delta_{1m}\frac{\partial J_m}{\partial x_3}-\delta_{3m}\frac{\partial J_m}{\partial x_1}), T*(\delta_{2m}\frac{\partial J_m}{\partial x_1}*J_m-\delta_{1m}\frac{\partial J_m}{\partial x_2})\right)$$

$$=\sum_{m=1}^{3}T*\mathrm{curl}\,[\mathbf{e}^m J_m] = T*\mathrm{curl}\,[\sum_{m=1}^{3}\mathbf{e}^m J_m] = T*\mathrm{curl}\,\tilde{J}.$$

Let us evaluate explicitly the convolution $\mathcal{E}*\tilde{J}$, where $\tilde{J}$ is defined by (31). We have

$$\mathcal{E}*\tilde{J} = \sum_{k=1}^{3}\begin{pmatrix}\mathbf{E}^k*J_k\\ \mathbf{H}^k*J_k\end{pmatrix} = \begin{pmatrix}\sum_{k=1}^{3}\mathbf{E}^k*J_k\\ \sum_{k=1}^{3}\mathbf{H}^k*J_k\end{pmatrix},$$

and explicit formulas (18) and (21) for $\mathbf{E}^k$ and $\mathbf{H}^k$. Using these formulas we find

$$\mathcal{E} * \tilde{J} = \begin{pmatrix} -\frac{1}{\varepsilon}\sum_{k=1}^{3}\theta(t)\exp(-\frac{\sigma}{\varepsilon}t)\mathbf{e}^k\,\delta(x) *\ J_k \\ \mathbf{0} \end{pmatrix}$$

$$+\begin{pmatrix} \sum_{k=1}^{3}\operatorname{curl}\operatorname{curl}[\theta(t)\mathbf{e}^k\,P] *\ J_k \\ \sum_{k=1}^{3}\operatorname{curl}[\mathbf{e}^k\,c^2T] *\ J_k \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\theta(t)}{\varepsilon}\int_0^t \exp(-\frac{\sigma}{\varepsilon}(t-\tau))\tilde{J}(0,\tau)\,d\tau \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \theta(t)P * \operatorname{curl}\operatorname{curl}\tilde{J} \\ \theta(t)c^2T * \operatorname{curl}\tilde{J} \end{pmatrix},$$

where

$$P(x,t) = \frac{c^2}{\varepsilon}\int_0^t \exp(-\frac{\sigma}{\varepsilon}(t-z))\,T(x,z)\,dz, \tag{34}$$

and $T(x,t)$ is defined by (4). Now we evaluate explicitly

$$\theta(t)P * \operatorname{curl}\operatorname{curl}\tilde{J}, \quad \theta(t)c^2T * \operatorname{curl}\tilde{J}.$$

Using (4) we find

$$\theta(t)c^2T * \operatorname{curl}\tilde{J} = \frac{\theta(t)}{4\pi}\int\int\int_{|x-\xi|\le ct}$$

$$\exp(-\frac{\sigma|x-\xi|}{2c\varepsilon})\,\frac{\operatorname{curl}_\xi \tilde{J}(\xi,z)}{|x-\xi|}\Big|_{z=t-\frac{|x-\xi|}{c}}\,d\xi + \frac{\theta(t)\sigma}{8\pi c\varepsilon}\int_0^t$$

$$\left(\int\int\int_{|x-\xi|\le c(t-\tau)} \frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{(t-\tau)^2-\frac{|x-\xi|^2}{c^2}})}{\sqrt{(t-\tau)^2-\frac{|x-\xi|^2}{c^2}}}\operatorname{curl}_\xi \tilde{J}(\xi,\tau)d\xi\right)d\tau. \tag{35}$$

Using (34) we have

$$\theta(t)P(x,t) = \frac{c^2\theta(t)}{\varepsilon}\int_0^t \exp(-\frac{\sigma}{\varepsilon}(t-z))\,T(x,z)\,dz$$

$$= \frac{c^2\theta(t)}{\varepsilon}\int_0^t \exp(-\frac{\sigma}{\varepsilon}(t-z))\left[\frac{1}{4\pi c^2|x|}\exp(-\frac{\sigma}{2\varepsilon}z)\delta(z-|x|/c)\right.$$

$$\left.+\frac{\sigma}{4\varepsilon}\theta(t-\frac{|x|}{c})\,\frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{z^2-\frac{|x|^2}{c^2}})}{\sqrt{z^2-\frac{|x|^2}{c^2}}}\right]dz = \theta(t-\frac{|x|}{c})\left[\frac{1}{\pi\varepsilon|x|}\exp(-\frac{\sigma}{\varepsilon}(t-\frac{|x|}{2c}))\right.$$

$$\left.+\quad \frac{c^2\sigma}{4\varepsilon^2}\int_{|x|/c}^t \frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{z^2-\frac{|x|^2}{c^2}})}{\sqrt{z^2-\frac{|x|^2}{c^2}}}\right]dz. \tag{36}$$

Using (36) we have

$$\theta(t)P(x,t) * \operatorname{curl}\operatorname{curl}\tilde{J} = \theta(t-\frac{|x|}{c})\left[\frac{1}{\pi\varepsilon|x|}\exp(-\frac{\sigma}{\varepsilon}(t-\frac{|x|}{2c}))\right.$$

$$+\frac{c^2\sigma}{4\varepsilon^2}\int_{|x|/c}^{t}\frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{z^2-\frac{|x|^2}{c^2}})}{\sqrt{z^2-\frac{|x|^2}{c^2}}}\,dz\Bigg]*\operatorname{curl}\operatorname{curl}\tilde{J}$$

$$=\theta(t)\int_0^t\Bigg[\int\int\int_{|x-\xi|\le c(t-\tau)}\Bigg(\frac{1}{4\pi\varepsilon\,|x-\xi|}\exp(-\frac{\sigma}{\varepsilon}(t-\tau-\frac{|x-\xi|}{2c}))$$

$$+\frac{c^2\sigma}{4\varepsilon^2}\int_{|x-\xi|/c}^{t-\tau}\frac{I_1(\frac{\sigma}{4\varepsilon}\sqrt{z^2-\frac{|x-\xi|^2}{c^2}})}{\sqrt{z^2-\frac{|x-\xi|^2}{c^2}}}\,dz\Bigg)\operatorname{curl}_\xi\operatorname{curl}_\xi\tilde{J}(\xi,\tau)\,d\xi\Bigg]\,d\tau. \tag{37}$$

Formulas (32), (33) follow from (34) – (37). Moreover, if components of $\operatorname{curl}\operatorname{curl}\mathbf{J}(x,t)$ are from $C^2(\mathbf{R}^3\times[0,+\infty))$, then components of the solution defined by (32), (33) belong to the class $C^2(\mathbf{R}^3\times[0,+\infty))\cap C^1(\mathbf{R}^3\times[0,+\infty))$. ■

Corollary 4.1.

Let $\sigma=0$ and components of $\mathbf{J}(x,t)$ belong to $C^4(\mathbf{R}^3\times[0,+\infty))$. Then,

$$\mathbf{E}(x,t)=-\frac{1}{\varepsilon}\int_0^t\mathbf{J}(0,\tau)\,d\tau$$

$$+\frac{1}{4\pi\varepsilon\,|x-\xi|}\int_0^t\Bigg[\int\int\int_{|x-\xi|\le c(t-\tau)}\frac{\operatorname{curl}_\xi\operatorname{curl}_\xi\mathbf{J}(\xi,\tau)}{|x-\xi|}\,d\xi\Bigg]\,d\tau, \tag{38}$$

$$\mathbf{H}(x,t)=\frac{1}{4\pi}\int\int\int_{|x-\xi|\le ct}\frac{\operatorname{curl}_\xi\mathbf{J}(\xi,z)}{|x-\xi|}\Big|_{z=t-\frac{|x-\xi|}{c}}\,d\xi. \tag{39}$$

are a classical solution of the initial value problem (28), (29), (30).

Proof:

The confirmation of the corollary follows from equalities (32) and (33) for $\sigma=0$. ■

5. Conclusion

The coefficient σ (in Maxwell's equations) is called the conductivity of the medium. If σ is different from zero, then the medium is called conducting, and if $\sigma=0$, then medium is nonconducting. The distinction between good and poor conductors, or insulators, is relative and arbitrary. All substances exhibit conductivity to some degree, but the range of observed values of σ is tremendous. The conductivity of copper, for example, is some 10^7 times as great as that of such a "good" conductor as sea water, and 10^{19} times that of ordinary glass.

In the present paper the explicit formulas for elements of the fundamental solutions of the time-dependent Maxwell's equations in conducting media with arbitrary values of σ have been obtained.

These formulas contain singular and regular generalized functions (distributions). The singular generalized functions appearing in explicit formulas characterize the electromagnetic wave propagation from point pulse sources on the fronts of waves, and regular generalized functions describe the electric and magnetic fields in regions bounded by fronts. Moreover, elements of the fundamental solution related with magnetic fields are divergenceless, and elements related with electric fields are linear combination of point sources of the form $\theta(t)e^k\delta(x)$ and divergenceless terms. We note that for the case of nonconducting media ($\sigma = 0$) the elements of the fundamental solutions contain the singular generalized functions with the supports on the fronts of electromagnetic waves (see formulas (22), (23)).

The generalized initial value problem for Maxwell's equations with arbitrary density of current in conducting media was stated. The components of the density of current are generalized functions from $\mathcal{D}'(\mathbf{R}^4)$. The generalized solution of this problem was obtained as the convolution of the fundamental matrix and components of the density of current. If components of the density of current are classical smooth functions from the class $C(\mathbf{R}^3 \times [0, +\infty))$, then this convolution can be calculated explicitly. As a result of this calculation the new Kirchhoff's formulas for the components of the classical solution of the initial value problem for Maxwell's equations with zero initial data were obtained. To the best of the author's knowledge, these formulas were not obtained before.

These explicit formulas can be used in the boundary integral method, Green's functions method and for computation of electric and magnetic fields in conducting media and materials.

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Appendix A

In this section we describe briefly operations of the direct product and convolution of generalized functions follow notations from Vladimirov (1971). Let us consider the class of test functions $\mathcal{D}(\mathbf{R}^n)$ and class of generalized functions $\mathcal{D}'(\mathbf{R}^n)$. We will say that a function is locally integrable in $\mathbf{R}^n$, if this function is integrable on every compact subset of $\mathbf{R}^n$.

Direct product. Let $f(x)$ and $g(y)$ be locally integrable functions in the spaces $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. Then, the function $f(x)g(y)$ will be locally integrable in the space $\mathbf{R}^{n+m}$. It defines the regular generalized function acting on the test function $\varphi(x, y) \in \mathcal{D}(\mathbf{R}^{n+m})$ according to the formulas

$$\langle f(x)g(y),\varphi\rangle=\int_{\mathbf{R}^n} f(x)\int_{\mathbf{R}^m} g(y)\varphi(x,y)\,dy\,dx=\langle f(x),\langle g(y),\varphi\rangle\rangle\,,$$

$$\langle g(y)f(x),\varphi\rangle=\int_{\mathbf{R}^m} g(y)\int_{\mathbf{R}^n} f(x)\varphi(x,y)\,dx\,dy=\langle g(y),\langle f(x),\varphi\rangle\rangle\,.$$

We shall take above equations as the definition of the direct product $f(x)\bullet g(y)$ of generalized functions $f\in\mathcal{D}'(\mathbf{R}^n)$ and $g\in\mathcal{D}'(\mathbf{R}^m)$:

$$\langle f(x)\bullet g(y),\varphi\rangle=\langle f(x),\langle g(y),\varphi\rangle\rangle\,,$$

$$\langle g(y)\bullet f(x),\varphi\rangle=\langle g(y),\langle f(x),\varphi\rangle\rangle\,,$$

for every $\varphi(x,y)\in\mathcal{D}(\mathbf{R}^{n+m})$.

The following properties are satisfied:

1) $f(x)\bullet g(y)=g(y)\bullet f(x)$;

2) If $f_k\to f$ as $k\to+\infty$ in $\mathcal{D}'(\mathbf{R}^n)$, then $f_k\bullet g\to f\bullet g$ as $k\to+\infty$ in $\mathcal{D}'(\mathbf{R}^n)$;

3) If $f\in\mathcal{D}'(\mathbf{R}^n)$, $g\in\mathcal{D}'(\mathbf{R}^m)$ and $h\in\mathcal{D}'(\mathbf{R}^k)$, then $f\bullet(g\bullet h)=(f\bullet g)\bullet h$;

4) $D^\alpha(f\bullet g)=(D^\alpha f)\bullet g$.

Convolution. Let $f(x)$ and $g(x)$ be locally integrable functions in the space $\mathbf{R}^n$ and $\int_{\mathbf{R}^n}|g(y)f(x-y)|\,dy$ be locally integrable in $\mathbf{R}^n$. Then,

$$(f*g)(x)=\int_{\mathbf{R}^n} f(y)g(x-y)\,dy=\int_{\mathbf{R}^n} g(y)f(x-y)\,dy=(g*f)(x)$$

is known as the convolution of f and g. The function $(f*g)(x)$ is locally integrable in $\mathbf{R}^n$ and therefore, defines a regular generalized function, acting on the test function $\varphi(x)\in\mathcal{D}(\mathbf{R}^n)$ according to the rule:

$$\begin{aligned}\langle (f*g)(x),\varphi(x)\rangle&=\int_{\mathbf{R}^n}(f*g)(\xi)\varphi(\xi)\,d\xi=\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} g(y)f(\xi-y)\,dy\right)\varphi(\xi)\,d\xi\\&=\int_{\mathbf{R}^n} g(y)\left(\int_{\mathbf{R}^n} f(x)\varphi(x+y)\,dx\right)\,dy\\&=\int_{\mathbf{R}^{2n}} f(x)g(y)\varphi(x+y)\,dx\,dy\\&=\langle f(x)\bullet g(y),\varphi(x+y)\rangle\end{aligned}$$

for every $\varphi(x)\in\mathcal{D}(\mathbf{R}^n)$.

The functional which is defined by

$$\langle f*g,\varphi\rangle=\langle f(x)\bullet g(y),\varphi(x+y)\rangle\,,$$

for every $\varphi(x) \in \mathcal{D}(\mathbf{R}^n)$ is known as the convolution of f and g. We note that since $\varphi(x+y)$ does not belong to $\mathcal{D}(\mathbf{R}^{2n})$ (it does not have a compact support in $\mathbf{R}^{2n}$), then, the right side of the above equality does not exist for all pairs of generalized functions f and g, and in this way the convolution does not always exist.

Properties of the convolution

1) If the convolution $f * g$ exists, then there exists also a convolution $g * f$ and $f * g = g * f$.

2) Let f be an arbitrary generalized function, g be a generalized function with compact support, then $f * g$ exists in $\mathcal{D}'(\mathbf{R}^n)$.

3) The convolution of any generalized function $f \in \mathcal{D}'(\mathbf{R}^n)$ with the Dirac delta function $\delta(x)$ exists and

$$f * \delta = \delta * f.$$

4) If the convolution $f * g$ exists, then convolutions $D^\alpha f * g$ and $f * D^\alpha g$ exist and

$$D^\alpha f * g = f * D^\alpha g = D^\alpha (f * g).$$

5) Let $c > 0$ be a constant, $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, $t \in \mathbf{R}$,

$$\Sigma^+ = \left\{ (x,t) \in \mathbf{R}^3 \times [0,\infty) : \qquad t - \frac{|x|}{c} \geq 0 \right\},$$

be a closure of the light cone; $f(x,t)$, $g(x,t)$ be generalized functions from $\mathcal{D}'(\mathbf{R}^4)$ such that $f(x,t) = 0$ for $t < 0$ and $\operatorname{supp} \mathrm{g(x,t)} \subset \Sigma^+$. Then (see, for example, Vladimirov (1971, pp. 159 – 163, Section 11.2)), the convolution $f * g$ exists in $\mathcal{D}'(\mathbf{R}^4)$. Here $\operatorname{supp} \mathrm{g(x,t)}$ is the support of the generalized function $g(x,t)$.

Property of the fundamental solution

Let $\mathbf{L}$ be a differential operator with constant coefficients. Then, the general function $T \in \mathcal{D}'(\mathbf{R}^n)$ which satisfies the equation $\mathbf{L}\,T = \delta(x)$ in $\mathbf{R}^n$ is said to be the fundamental solution of $\mathbf{L}$.

Let $f \in \mathcal{D}'(\mathbf{R}^n)$ be a generalized function, such that the convolution $T * f$ exists in $\mathcal{D}'(\mathbf{R}^n)$. Then (see, for example, Vladimirov (1971, pp. 143 – 144, Section 10.3)), the solution of $\mathbf{L}\,u = f$ exists in $\mathcal{D}'(\mathbf{R}^n)$, and it is given by the formula $u = T * f$. This solution is unique in the class of generalized functions belonging to $\mathcal{D}'(\mathbf{R}^n)$ for which the convolution with T exists.

Appendix B

Let $\mathbf{E}(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t),)$, $\mathbf{H}(x,t) = (H_1(x,t), H_2(x,t), H_3(x,t),)$ be electric and magnetic fields, $\tilde{J}(x,t) = (J_1(x,t), J_2(x,t), J_3(x,t),)$ be the density of electric currents and

$\rho(x,t)$ be the density of the electric charges. The wave propagation of electromagnetic waves in the conducting medium with the dielectric permittivity $\varepsilon > 0$, magnetic permeability $\mu > 0$ and conductivity σ arising from sources $\tilde{J}(x,t)$, $\rho(x,t)$ is described by the following Maxwell's equations (see, for example, Stratton (2007)),

$$-\varepsilon\frac{\partial \mathbf{E}}{\partial t} - \sigma\mathbf{E} + \mathrm{curl}\,\mathbf{H} = \tilde{J}, \tag{40}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \mathrm{curl}\,\mathbf{E} = \mathbf{0}, \tag{41}$$

$$\mathrm{div}\,(\varepsilon\mathbf{E}) = \rho, \tag{42}$$

$$\mathrm{div}\,(\mu\mathbf{H}) = 0. \tag{43}$$

We note that Equations (40) – (43) are consistent if $\tilde{J}(x,t)$, $\rho(x,t)$ satisfy the following equality

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon}\rho + \mathrm{div}\,\tilde{J} = 0. \tag{44}$$

Equality (44) is well known as the conservation law of electric charges and currents, and it is always assumed to be satisfied, when Maxwell's equations are studied.

The electric currents and electric charges are sources of electromagnetic fields. We have never found in the real physical phenomena their magnetic equivalents – magnetic currents and magnetic charges, i.e., they are always equal to zero. From mathematical point of view, equations (41) and (43) do not contain inhomogeneous terms. These inhomogeneous terms of (41) and (43) are always equal to zero.

Equations (40), (41) and (44) with conditions

$$\mathbf{E}|_{t<0} = 0, \quad \mathbf{H}|_{t<0} = 0, \quad \rho|_{t<0} = 0, \quad \tilde{J}|_{t<0} = 0, \tag{45}$$

imply (42), (43). Therefore, if $\tilde{J}$, ρ satisfy (44), (45) and vector functions $\mathbf{E}$, $\mathbf{H}$ are a solution of (40), (41), (45), then $\mathbf{E}$, $\mathbf{H}$ satisfy Maxwell's equations (40), (41), (42) and (43).