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Fuzzy Solutions to Second Order Three Point Boundary Value Problem

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Abstract

In this manuscript, the proposed work is to study the existence of second-order differential equations with three point boundary conditions. Existence is proved using fuzzy set valued mappings of a real variable whose values are normal, convex, upper semi continuous and compactly supported fuzzy sets. The sufficient conditions are also provided to establish the existence results of fuzzy solutions of second order differential equations for three point boundary value problem. By using Banach fixed point principle, a new existence theorem of solutions for these equations in the metric space of normal fuzzy convex sets with distance given by the maximum of the Hausdorff distance between level sets is obtained. Then to further establish the existence, fixed point theorem for absolute retracts is used by taking consideration that space of fuzzy sets can be embedded isometrically as a cone in Banach space. Finally, an example is provided to illustrate the result.

Keywords: Initial value problem; Hausdorff metric; Three point boundary value problem; Fuzzy solution; Contraction mapping; Fixed point; Absolute retracts

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1. Introduction

Multi-point problems plays a significant role in the field of applied mathematics and physics. In the recent years, many researchers have shown their interest regarding the existence of solutions when subjected to boundary conditions involving three or more points. Many problems in the theory of elastic stability can be treated using the method of multi-point problems (Timoshenko (1961)). These problems arise in the study of the equilibrium states of a heated bar. Il'in and Moiseev (1987) started the study of multi-point boundary value problem for linear second order ordinary differential equations. Boucheriff and Bouguima (1998) discussed the nonlinear second order ordinary differential equations with nonlocal conditions. Benchohra and Ntouyas (2001) analyzed an extension of three and four-point boundary value problems for second order differential equations to the multivalued case by the means of fixed point theorem for condensing multivalued map due to Martelli. Henderson (2005) employed shooting methods to obtain solutions of the three-point boundary value problem for the second order equation. (Xie et al. (2016)) developed a new algorithm based on differential transform method for solving multi point boundary value problems. Cui (2016) discussed the boundary value problems for fractional differential equations. Liu et al. (2017) investigated the upper and lower solutions for mixed fractional four point boundary value problem.

The study of second order fuzzy differential equations has gained much attention in the recent years. Agarwal et al. (2005) discussed the fuzzy solutions for multi point boundary value problems. Georgiou et al. (2005) observed the initial value problems for higher-order fuzzy differential equations. Chalishajar et al. (2017) discussed the fuzzy solutions for nonlocal impulsive neutral functions differential equations. Ramesh and Vengataasalam (2015a) investigated the existence of fuzzy solutions for impulsive differential equations. Khastan and Nieto (2010) studied the boundary value problem for second order fuzzy differential equations. Allahviranloo et al. (2009) proved the existence and uniqueness of solutions of second-order fuzzy differential equations. Chalishajar et al. (2019) studied the existence of solutions in detail for fractional differential equations. Ali et al. (2016) proved the existence of positive solutions for fractional differential equations. Ramesh and Vengataasalam (2015b) analysed the fuzzy solutions for second order boundary value problems.

The highlights of this manuscript is as follows:

- Researchers have studied the second order systems/inclusions and/or three point boundary value problems extensively. Also, fuzzy solution of the dynamical systems have been studied, but the fuzzy solution of the second order three point boundary value problem has not been studied in detail in the literature.
- We have studied the fuzzy solution of the second order three point boundary value problem with an application for absolute retracts and Banach fixed point theorem.

The rest of this work is organized as follows: In Section 2, we will recall briefly some basic definitions and preliminary facts which will serve as a motivation to get existence results. In Section

3, we prove the existence of fuzzy solutions for three point boundary value problems for second order differential equations and in Section 4, we provide an example to illustrate our result. Our approach here depends on Banach fixed point theorem and a fixed point theorem in absolute retract spaces.

2. Preliminaries

In this section, we discuss the basics of fuzzy differential equations which are used through out this paper. (For more details, refer to Puri and Ralescu (1986); Arara and Benchohra (2006); Qui et al. (2016); Farahrooz et al. (2017); Khastan and Rodríguez-López (2019)). If one intends to analyze a real world phenomenon, uncertain factors must also be addressed. The theory of fuzzy sets may be one of the best non-probabilistic solutions for modelling uncertain cases, which leads us to investigate the theory of fuzzy differential equations. A fuzzy set may be viewed as a generalization of the concept of an ordinary set whose membership function only takes two values $\{0, 1\}$. For fuzzy sets, product operations, sum, and complement are also established in the same way as for classic sets.

In a dynamic environment, fuzzy differential equations appear as a natural way to model the propagation of epistemic uncertainty. Many concepts such as fuzzy numbers and its various properties, alpha cuts, Hausdorff metric, Measurability, Integrability and Differentiability are employed in this work to establish our existence result for second order three point boundary value problem. Fuzzy numbers play a particularly important role in this regard as a special type of fuzzy sets. In many respects, fuzzy numbers more realistically depict the physical world than single-valued numbers.

Also, Hausdorff metric is used to determine measurements of the distance between two subsets of a metric space. The Hausdorff distance is the largest of all the distances between one point and the nearest point in the other set. It plays a pivotal role in many Engineering applications. It is used in computer graphics to measure the difference between two different representations of the same 3D object, especially when generating level of detail to efficiently display complex 3D models.

The notion of alpha cut method is a standard method used to perform various arithmetic operations such as addition, multiplication, division, subtraction. Using the resolution identity principle, alpha cuts are used to decompose a fuzzy set into a weighted combination of classical sets. In fuzzy set theory the concept of alpha cut is important because it creates a bridge between fuzzy sets and crisp sets. In the next section, we will discuss the existence results using the above mentioned ideas.

3. Existence Results

Here, the purpose is to study the existence of fuzzy solutions for three point boundary value problems for second order differential equations. So, we consider the following three point boundary value problem,

$$x^{''}(t) - a(t)f(t, x(t)) = 0, \quad t \in J = [0, 1], \tag{1}$$

$$x(0) = \hat{0} \in E^n, \quad \alpha x(\eta) = x(1), \tag{2}$$

where E^n be the set of all upper semi-continuous, convex, normal fuzzy numbers with bounded α-level, $f : J \times E^n \to E^n$ a continuous function and $\eta \in [0, 1]$ and $0 < \alpha < \frac{1}{\eta}$.

Definition 3.1.

A function $x \in C(J, E^n)$ is said to be a solution of the boundary value problem $(1) - (2)$ if x satisfies the equation $x^{''}(t) = a(t)f(t, x(t))$ on $[0, 1]$ and the condition (2).

3.1. Application of Banach fixed point principle

Our first result for the problem $(1) - (2)$ is based on Banach fixed point principle.

Theorem 3.1.

Assume:

(H1) There exists a constant $d_2 > 0$ such that

$$d_\infty(f(t, x), f(t, y)) \leq d_\infty d_2(x(t), y(t)) \text{ for } t \in J, \quad x \in E^n.$$

(H2) If $(d_1 d_2[2 + \alpha(1 - \eta)]) < 1$, then the three point nonlinear second order boundary value problem $(1) - (2)$ has a unique fuzzy solution, where

$$d_1 = \frac{\int_0^1 (1 - s)a(s)ds}{1 - \alpha\eta}.$$

Proof:

By transforming the second order boundary value problem $(1) - (2)$ into a fixed point problem, it is easy to understand that the solutions of the three point boundary value problem $(1) - (2)$ are fixed points of the operator Φ.

Then, $\Phi : C(J, E^n) \to C(J, E^n)$ is defined by

$$\Phi(x)(t) = \int_0^t (t - s)a(s)f(s, x(s))ds + \frac{\alpha t}{1 - \alpha\eta} \int_0^\eta (\eta - s)a(s)f(s, x(s))ds - \frac{t}{1 - \alpha\eta} \left[\int_0^1 (1 - s)a(s)f(s, x(s))ds \right].$$

Now for $t \in [0,1]$,

$$\begin{aligned}
d_\infty(\Phi x(t), \Phi y(t)) = & \Bigg(\int_0^t (t-s)a(s)f(s,x(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(s,x(s))ds \\
& - \frac{t}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,x(s))ds\right], \\
& \int_0^t (t-s)a(s)f(s,y(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(s,y(s))ds \\
& - \frac{t}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,y(s))ds\right]\Bigg) \\
\leq & \Bigg(\int_0^1 (1-s)a(s)f(s,x(s))ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(s,x(s))ds \\
& - \frac{1}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,x(s))ds\right], \int_0^1 (1-s)a(s)f(s,y(s))ds \\
& + \frac{\alpha}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(s,y(s))ds \\
& - \frac{1}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,y(s))ds\right]\Bigg) \\
\leq & \int_0^1 (1-s)a(s)d_\infty(f(s,x(s)), f(s,y(s))ds \\
& + \frac{\alpha}{1-\alpha\eta}\int_0^1 (1-s)a(s)d_\infty(f(s,x(s)), f(s,y(s))ds \\
& - \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)d_\infty(f(s,x(s)), f(s,y(s))ds \\
\leq & [(1-\alpha\eta)d_1 + \alpha d_1 + d_1]d_\infty(f(s,x(s)), f(s,y(s))) \\
\leq & d_1[2+\alpha(1-\eta)] \sup_{t\in[0,1]} d_\infty d_2(x(s), y(s)).
\end{aligned}$$

Thus,

$$H_1(\Phi x(t), \Phi y(t)) \leq (d_1 d_2[2+\alpha(1-\eta)])H_1(x,y).$$

Thus, Φ is a contraction. Hence, by Banach fixed point theorem, Φ has a unique fixed point. ■

3.2. Application of fixed point theorem for Absolute retracts

Now, we shall prove the existence of fuzzy solutions for the problem $(1)-(2)$ by a fixed point theorem in absolute retract spaces. There are several definitions available on the concepts of absolute retract spaces (see Dugundji and Granas (1982); Ma (1997)).

Theorem 3.2.

Assume the following hypotheses:
(H3) Let $f : [0,1] \times E^n \to E^n$ be continuous and there exists a continuous non-decreasing function

$\psi : [0, \infty) \to (0, \infty)$ and $m \in L^1(J, \mathbb{R}_+)$ such that

$$d_\infty(a(t)f(t,x), \hat{0}) \leq m(t)a(t)\psi(d_\infty(x, \hat{0})) \text{ for } t \in J, \ x \in E^n.$$

(H4) There exists $\gamma > 0$ with

$$\frac{\gamma}{\psi(\gamma)\left(1 + \dfrac{\alpha+1}{1-\alpha\eta}\right)\displaystyle\int_0^1 m(s)a(s)ds} \geq 1,$$

such that for each $t \in J$, the set

$$\begin{cases} \displaystyle\int_0^t (t-s)a(s)f(s,x(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(s,x(s))ds \\ \displaystyle -\frac{t}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,x(s))ds\right], \quad x \in \mathcal{A}, \end{cases}$$

is a totally bounded subset of E^n, where

$$\mathcal{A} = \{x \in C(J, E^n) : d_\infty(x(t), \hat{0}) \leq \gamma, \ \ t \in J\}.$$

Then, the problem $(1) - (2)$ has atleast one fuzzy solution on J.

Proof:

Let $\mathcal{A} \cong \mathcal{B} \equiv \{\bar{J}_x \in C(J, E^n) : y \in C(J, E^n), d_\infty((x(t), \hat{0}) \leq \gamma), \text{ and } t \in J\}$. So, $\mathcal{B}$ is a convex subset of the Banach space $C(J, X)$. Therefore, $\mathcal{B}$ is an absolute retract. This implies that $\mathcal{A}$ is an absolute retract. Now, it remains to show that Φ maps $\mathcal{A}$ into $\mathcal{A}$, Φ is continuous and Φ is an equicontinuous set.

Claim 1: To show $\Phi : \mathcal{A} \to \mathcal{A}$.

Let $x \in \mathcal{A}$ and $t \in [0, 1]$.

From $(H3)$, we have

$$\begin{aligned} & d_\infty(\Phi x(t), \hat{0}) \\ & = d_\infty\Bigg(\int_0^t (t-s)a(s)f(s,x(s))ds \\ & \quad + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(s,x(s))ds \\ & \quad - \frac{t}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,x(s))ds\right], \hat{0}\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t (t-s) d_\infty(a(s)f(s,x(s)),\hat{0})ds \\
&\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s) d_\infty(a(s)f(s,x(s)),\hat{0})ds \\
&\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s) d_\infty(a(s)f(s,x(s)),\hat{0})ds \\
&\leq \int_0^1 m(s)a(s)\psi(d_\infty(x(s),\hat{0}))ds \\
&\quad + \frac{\alpha}{1-\alpha\eta}\int_0^1 m(s)a(s)(d_\infty(x(s),\hat{0}))ds \\
&\quad + \frac{1}{1-\alpha\eta}\int_0^1 m(s)a(s)(d_\infty(x(s),\hat{0}))ds \\
&\leq \psi(\gamma)\left(1+\frac{\alpha+1}{1-\alpha\eta}\right)\int_0^1 m(s)a(s)ds \\
&\leq \gamma.
\end{aligned}
$$

Thus $\Phi(\mathcal{A}) \subset A$.

Claim 2: To show Φ is continuous.

Let $\{x_n\} \in \mathcal{A}$ be a sequence such that $x_n \to x \in A$ in $C([0,1], E^n)$.

$$
\begin{aligned}
&H_1(\Phi x_n(t), \Phi x(t)) \\
&= H_1\Bigg(\int_0^t (t-s)a(s)f(s,x_n(s))ds \\
&\quad + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(s,x_n(s))ds \\
&\quad - \frac{t}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,x_n(s))ds\right], \\
&\quad \int_0^t (t-s)a(s)f(s,x(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(s,x(s))ds \\
&- \frac{t}{1-\alpha\eta}\left[\int_0^1 (1-s)a(s)f(s,x(s))ds\right]\Bigg) \\
&\leq \int_0^1 H_1(a(s)f(s,x_n(s)), a(s)f(s,x(s))ds \\
&+ \frac{\alpha}{1-\alpha\eta}\int_0^1 H_1(a(s)f(s,x_n(s)), a(s)f(s,x(s)))ds \\
&+ \frac{1}{1-\alpha\eta}\int_0^1 H_1(a(s)f(s,x_n(s)), a(s)f(s,x(s)))ds.
\end{aligned}
$$

Thus,

$$H_1(\Phi x_n, \Phi x) \leq \left(1 + \frac{\alpha+1}{1-\alpha\eta}\right) \int_0^1 H_1(a(s)f(s, x_n(s)), a(s)f(s, x(s)))ds.$$

Consider

$$\rho_n(s) = d_\infty(a(s)f(s, x_n(s)), a(s)f(s, x(s))).$$

Since f is continuous, $\rho_n(t) \to 0$ as $n \to \infty$ for $t \in [0, 1]$.

Using the assumption $(H3)$, we have that

$$\begin{aligned}\rho_n(t) &\leq d_\infty(a(t)f(t, x_n(t)), \hat{0}) + d_\infty(\hat{0}, a(t)f(t, x(t)))\\ &\leq m(t)a(t)[\psi(d_\infty(x_n(t), \hat{0})) + \psi(d_\infty(x(t), \hat{0}))]\\ &\leq 2m(t)a(t)\psi(\gamma).\end{aligned}$$

This results in

$$\lim_{n\to\infty} \int_0^1 \rho_n(s)ds = \int_0^1 \lim_{n\to\infty} \rho_n(s)ds = 0.$$

Hence,

$$H_1(\Phi x_n, \Phi x) \to 0 \text{ as } n \to \infty.$$

So, $\Phi : \mathcal{A} \to \mathcal{A}$ is continuous.

Claim 3: To show $\Phi(\mathcal{A})$ is an equicontinuous set of $C([0, 1], E^n)$.

Consider $\tau_1, \tau_2 \in [0, 1]$, where $\tau_1 < \tau_2$, $x \in \mathcal{A}$.

Then, we have,

$$\begin{aligned}d_\infty(\Phi x(\tau_2), \Phi x(\tau_1)) = d_\infty\Bigg(&\int_0^{\tau_2} (\tau_2 - s)a(s)f(s, x(s))ds\\ &+ \frac{\alpha\tau_2}{1-\alpha\eta} \int_0^\eta (\eta - s)a(s)f(s, x(s))ds\\ &- \frac{\tau_2}{1-\alpha\eta} \left[\int_0^1 (1-s)a(s)f(s, x(s))ds\right],\\ &\int_0^{\tau_1} (\tau_1 - s)a(s)f(s, x(s))ds\\ &+ \frac{\alpha\tau_1}{1-\alpha\eta} \int_0^\eta (\eta - s)a(s)f(s, x(s))ds\\ &- \frac{\tau_1}{1-\alpha\eta} \left[\int_0^1 (1-s)a(s)f(s, x(s))ds\right]\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= d_\infty\Bigg(\int_0^{\tau_1} (\tau_1 - s)a(s)f(s,x(s))ds + \int_0^{\tau_1} (\tau_2 - \tau_1)a(s)f(s,x(s))ds \\
&\quad + \int_{\tau_1}^{\tau_2} (\tau_2 - s)a(s)f(s,x(s))ds \\
&\quad + \frac{\alpha\tau_2 - \alpha\tau_1}{1-\alpha\eta} \int_0^{\eta} (\eta - s)a(s)f(s,x(s))ds \\
&\quad + \frac{\alpha\tau_1}{1-\alpha\eta} \int_0^{\eta} (\eta - s)a(s)f(s,x(s))ds \\
&\quad - \frac{\tau_2 - \tau_1}{1-\alpha\eta} \int_0^{1} (1 - s)a(s)f(s,x(s))ds \\
&\quad - \frac{\tau_1}{1-\alpha\eta} \int_0^{1} (1 - s)a(s)f(s,x(s))ds, \\
&\quad \int_0^{\tau_1} (\tau_1 - s)a(s)f(s,x(s))ds \\
&\quad + \frac{\alpha\tau_1}{1-\alpha\eta} \int_0^{\eta} (\eta - s)a(s)f(s,x(s))ds \\
&\quad - \frac{\tau_1}{1-\alpha\eta} \left[\int_0^{1} (1 - s)a(s)f(s,x(s))ds \right] \Bigg) \\
&= d_\infty\Bigg(\int_0^{\tau_1} (\tau_2 - \tau_1)a(s)f(s,x(s))ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)a(s)f(s,x(s))ds \\
&\quad + \frac{\alpha\tau_2 - \alpha\tau_1}{1-\alpha\eta} \int_0^{\eta} (\eta - s)a(s)f(s,x(s))ds \\
&\quad - \frac{\tau_2 - \tau_1}{1-\alpha\eta} \int_0^{1} (1 - s)a(s)f(s,x(s))ds, \hat{0} \Bigg) \\
&\leq \tau_2 \int_{\tau_1}^{\tau_2} d_\infty(a(s)f(s,x(s)),\hat{0})ds + \int_0^{\tau_1} (\tau_2 - \tau_1) d_\infty(a(s)f(s,x(s)),\hat{0})ds \\
&\quad + (\alpha+1)\frac{\tau_2 - \tau_1}{1-\alpha\eta} \int_0^{1} d_\infty(a(s)f(s,x(s)),\hat{0})ds \\
&\leq \tau_2 \int_{\tau_1}^{\tau_2} m(s)a(s)\psi(x(s),\hat{0})ds + \int_0^{\tau_1} (\tau_2 - \tau_1) m(s)a(s)\psi(x(s),\hat{0})ds \\
&\quad + (\alpha+1)\frac{\tau_2 - \tau_1}{1-\alpha\eta} \int_0^{1} m(s)a(s)\psi(x(s),\hat{0})ds \\
&\leq \int_{\tau_1}^{\tau_2} \tau_2 m(s)a(s)\psi(\gamma)ds + \int_0^{\tau_1} (\tau_2 - \tau_1)\psi(\gamma)ds \\
&\quad + (\alpha+1)\frac{\tau_2 - \tau_1}{1-\alpha\eta} \int_0^{1} m(s)a(s)\psi(\gamma)ds.
\end{aligned}
$$

So, by Theorems (3.4) and (3.5), we obtain that Φ has a fixed point x which is a solution of the problem $(1)-(2)$. ■

4. Example

In this section, we consider an example to illustrate our result.

$$x^{''}(t) - (1+t)^7(x^2 + x^{\frac{1}{3}}) = 0, \quad t \in J = [0,1], \tag{3}$$

$$x(0) = \hat{0} \in E^n, \quad 3x\left(\frac{1}{5}\right) = x(1). \tag{4}$$

The problem (3) -(4) can be regarded as a boundary value problem of the form (1)-(2), where $\eta = \frac{1}{5}, \alpha = 3, a(t) = (1+t)^7, f(t,x) = (x^2 + x^{\frac{1}{3}})$. It is easy to see that the conditions $(H1) - (H4)$ holds. Hence, Theorem (3.4) and Theorem (3.5) are satisfied.

5. Conclusion and future scope

In this paper, well known fixed point theorems are employed to get the existence results. The main idea behind this work is to get the existence result for solutions of second-order three-point boundary value problem by applying the concept of fuzzy numbers and the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semi-continuous and compactly supported fuzzy sets in R^n. Also, the ideas of alpha cuts are employed to obtain the existence theorem of solutions in the metric space of normal fuzzy convex sets with distance given by the maximum of the Hausdorff distance between level sets. Sufficient conditions are also determined to get the desired result. To establish the existence result, an example is also provided. Similarly, the m-point boundary value problem can be studied using Banach fixed point theorem, and fixed point theorem for absolute retracts. Using other fixed point theorems in the literature, one can attempt to check the existence and uniqueness of solutions of three-point boundary value problems. Further, it can be extended to study the positive solutions of the differential system.

Impulsive differential equations have been studied extensively because of its various applications in the field of dynamical systems in recent years, for it helps in the description of the processes which experience a sudden change of their state at certain moments. It can also be added to the second-order differential system to get the existence and uniqueness of the three-point boundary value problem. Then, further, it can be extended to study the m-point boundary value problem.

The fractional-order models are more realistic and practical than the traditional integer-order models. Several researchers have also addressed boundary value problems for nonlinear fractional differential equations, as fractional derivatives provide an excellent method for explaining memory and hereditary properties of various materials and processes. But literature work on fuzzy fractional differential equations for problems with three-point boundaries is less. Consequently, one can get impressive results by using the method described in this paper. This treatment can also be extended to a multi-point problem. Also, by employing many well-known fixed point theorems, one can able to existence and uniqueness for the fractional differential system.

The study of nonlinearity, which may be positive or negative in a multi-point boundary value problem, has many engineering applications. The treatment of fuzzy techniques for the nonlinear problem, which takes a negative value, also called as semi-tone problem is untreated in the literature. By employing various fixed point theorems, one may test the existence and uniqueness of fuzzy solutions. The integro differential equations play a significant role in mathematical modelling. One might derive a remarkable result by using fuzzy techniques for multi-point boundary value problems for second-order integro differential systems combined with impulsive effects. In the subsequent studies, we shall extend the ideas mentioned above using the fuzzy set-valued mappings of a real variable whose values are normal, convex, upper semi-continuous, and compactly supported fuzzy sets along with several fixed point theorems to check the existence of the solutions for the second-order differential system.

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