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On a Multiserver Queueing System with Customers' Impatience Until the End of Service Under Single and Multiple Vacation Policies

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Abstract

This paper deals with a multiserver queueing system with Bernoulli feedback and impatient customers (balking and reneging) under synchronous multiple and single vacation policies. Reneged customers may be retained in the system. Using probability generating functions (PGFs) technique, we formally obtain the steady-state solution of the proposed queueing system. Further, important performance measures and cost model are derived. Finally, numerical examples are presented.

Keywords: Queueing models; Synchronous vacation; Impatient customers; Bernoulli feedback

MSC 2010 No.: 60K25, 68M20, 90B22

1. Introduction

Queueing models with server vacations have found a large applicability in many real time systems (computer and communication network, telecommunication, data/voice transmission, manufacturing system, etc). Since the last three decades these models have been extensively studied, superb surveys on the earlier works on vacation queues have been done in Doshi (1986), Takagi (1991), and Tian and Zhang (2006) and the references therein. Multi-server vacation systems were well studied over the past decade. The servers in these models may either take the same vacation together (synchronous vacation) or individual vacations (called asynchronous vacations) independently, and most of multi-server vacation models are based on synchronous vacations. Zhang and Tian (2003a), (2003b) first studied the Markovian multi-server queueing system with single and multiple synchronous vacations. Then, Tian and Zhang (2003) dealt with a $GI/M/c$ queueing model with phase-type vacations, where all servers take multiple vacations at the same time until a waiting customers exist at a vacation completion instant. Further, Gharbi and Ioualalen (2010) studied the finite-source multi-server queueing systems with single and multiple vacation policies.

Queueing systems with balking and reneging arise in many practical situations including telephone services, computer and communication systems, manufacturing systems, etc. For a detailed overview of main results on the subject, see Haghighi et al. (2010); Abou El-Ata and Hariri (1992); Al-Seedy (2009); Haghighi and Mishev (2014); Ammar (2014) and the reference therein.

Vacation queueing models with impatient customers played a powerful role in day-to-day as well as industrial congestion situations; computer systems, communication networks, call centers, systems operating in machining environment, manufacturing systems, transportation systems, etc. Over recent decades, a vast number of papers dealt with vacation queues with customers' impatience. Arumuganthan and Jeyakumar (2005) presented the steady state analysis of a bulk queue with multiple vacations, set up times with N-policy and close down times. Zhang et al. (2005) studied an $M/M/1/N$ queueing model with balking, reneging and server vacations. Both single server and multi-server vacation systems with impatient customers were studied by Altman and Yechiali (2006). Altman and Yechiali (2008) dealt with infinite-server queues with system's additional tasks and impatient customers. Further, Padmavathy et al. (2011) dealt with vacation queues, impatient customers and waiting server. The balking behavior in the single-server queue with general service and vacation times has been carried out by Antonis et al. (2011). Selvaraju and Goswami (2013) presented single and multiple operational vacations with customer's intolerance in an $M/M/1$ queue. Yue et al. (2014) considered an $M/M/c$ queueing system with impatient customers and synchronous vacation. Goswami (2014) analyzed the impatience in the queueing system with Bernoulli schedule working vacations and vacation interruption. Ammar (2015) presented the transient analysis of an $M/M/1$ queue with impatient behavior and multiple vacations. Later, in Ammar (2017), a transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers is given. Panda and Goswami (2016) presented the equilibrium balking strategies in renewal input queue with bernoulli-schedule controlled vacation and vacation interruption. Then, Optimal balking strategies in single-server Markovian queues with multiple vacations and N-policy was examined by Sun et al. (2016). A study of single server markovian queueing model with vacations and impatience times which depends of the state of the server was presented in Yue et al. (2016). Recently, vacation queues with impatience and retention of reneged customers have been well analyzed (Manoharan and Majid (2017); Majid and Manoharan (2018), Bouchentouf et al. (2018), Bouchentouf et al. (2019), Bouchentouf and Guendouzi (2018), (2019), (2020)).

In this investigation, we study a $M/M/c$ Bernoulli feedback queueing system with single and multiple synchronous vacations, impatient customers (balking and reneging) and retention of reneged customers. Our results have large applications. It can be employed to model many real life congestion situations, like manufacturing systems, production systems, communications, etc. And from the cost-economic point of view, it is beneficial to convince the reneged customers to do not leave the system and stay for their services.

The remaining part of the paper is structured as follows. The system description to formulate the mathematical model is presented in Section 2. In Section 3, the steady state probabilities of the system under single vacation policy are constructed and their steady-state solutions are established. Some explicit expressions of useful measures of effectiveness are derived. In Section 4, we carry out the stationary analysis of the model under multiple vacation policy using similar method given in Section 3, we obtain the closed-form expressions of some performance measures. In Section 5, a model for the costs incurred in the considered queueing system under multiple vacation policy is developed. Numerical illustration is carried out in Section 6, where performance and economic analysis of the model under multiple vacation policy are presented. Finally, in Section 7 we conclude the paper.

2. Model's mathematical formulation

Consider a $M/M/c$ queueing model with Bernoulli feedback, balking, reneging and retention of reneged customers. Customers arrive into the system according to a Poisson process with arrival rate λ, the service time is assumed to be exponentially distributed with rate μ. The service discipline is FCFS and there is a infinite space for customers to wait. The servers take vacation synchronously once the system becomes empty, and they also return to the system as one at the same time.

In this paper we consider two vacation type queueing models:

Model I: Single station vacation policy: servers take a single vacation when the system is empty. If they return from a vacation to an empty system, they wait dormant to the first arrival and thereafter they start a busy period. Otherwise, if there are customers waiting in the queue at the end of a vacation, the servers immediately start a busy period.

Model II: Multiple station vacation policy: if the servers return from a vacation to find an empty queue, they immediately leave all together for another vacation, otherwise, they return to serve the queue. If there are customers in the queue at the end of a vacation, the servers immediately start a busy period. Otherwise, they take all together another vacation.

For the two models, the vacation duration is exponentially distributed with mean $1/\phi$.

Whenever a customer arrives at the system and finds the servers on vacation (respectively, busy), he activates an impatience timer T_0 (respectively, T_1), which is exponentially distributed with parameter ξ_0 (respectively, ξ_1). If the customer's service has not been completed before the customer's timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers.

Each reneged customer may leave the system without getting service with probability α and may remain in the queue for his service with probability $\bar{\sigma} = (1 - \sigma)$.

A customer who on arrival finds at least one customer (resp. c customers) in the system, when the servers are on vacation period (resp. busy period) either decides to enter the queue with probability θ or balk with probability $\overline{\theta} = 1 - \theta$.

After completion of each service, the customer can either leave the system with probability β or come back to the system and join the end of the queue with probability β', where $\beta + \beta' = 1$.

The inter-arrival times, vacation periods, service times, and impatience times are mutually independent.

Let $L(t)$ be the number of customers in the system at time t, and $J(t)$ represents the status of the server at time t, such that

$$J(t) = \begin{cases} 0, \text{ all the servers are taking a vacation at time } t, \\ 1, \text{ the servers are busy at time } t. \end{cases}$$

3. Analysis of Model I: M/M/c/SV

In this part of paper we analyse the considered model under single station vacation, the state-transition diagram is presented in Figure 1.

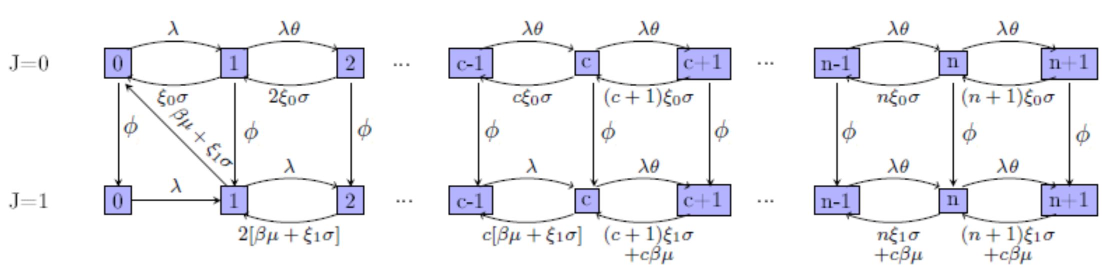

Figure 1. State-transition diagram for Model I

3.1. Stationary Analysis

The process $\{(J(t), L(t)),\ t \geq 0\}$ is defined as a continuous-time Markov process with a state space $\Omega = \{(j, n) :\ \ j = 0, 1,\ n = 0, 1, ...\}$. Let

$$P_{j,n} = \lim_{t\to\infty} \mathbb{P}\left[J(t) = j,\ L(t) = n\right],\ n \geq 0,\ j = 0, 1,$$

be the probability that there is n customers in the system, and servers are at state 0 or 1. The partial generating functions, $G_0(z)$ and $G_1(z)$, for $0 < z < 1$ are given as

$$G_0(z) = \sum_{n=0}^{\infty} P_{0,n} z^n, \qquad G_1(z) = \sum_{n=0}^{\infty} P_{1,n} z^n, \qquad 0 \leq z \leq 1. \tag{1}$$

The set of balance equations is given as follows:

$$(\lambda + \phi) P_{0,0} = \sigma\xi_0 P_{0,1} + (\beta\mu + \sigma\xi_1) P_{1,1}, \qquad n = 0, \tag{2}$$

$$(\theta\lambda + \sigma\xi_0 + \phi) P_{0,1} = \lambda P_{0,0} + 2\sigma\xi_0 P_{0,2}, \qquad n = 1, \tag{3}$$

$$(\theta\lambda + n\sigma\xi_0 + \phi) P_{0,n} = \theta\lambda P_{0,n-1} + (n+1)\sigma\xi_0 P_{0,n+1}, \qquad n \geq 2, \tag{4}$$

$$\lambda P_{1,0} = \phi P_{0,0}, \tag{5}$$

$$(\lambda + \beta\mu + \sigma\xi_1) P_{1,1} = \lambda P_{1,0} + 2(\beta\mu + \sigma\xi_1) P_{1,2} + \phi P_{0,1}, \qquad n = 1, \tag{6}$$

$$\begin{aligned}(\lambda + n(\beta\mu + \sigma\xi_1)) P_{1,n} = \lambda P_{1,n-1} + (n+1)(\beta\mu + \sigma\xi_1) P_{1,n+1} + \phi P_{0,n},\\ 2 \leq n \leq c-1,\end{aligned} \tag{7}$$

$$\begin{aligned}(\theta\lambda + n(\beta\mu + \sigma\xi_1)) P_{1,n} = \lambda P_{1,n-1} + \left[c\beta\mu + (n+1)\sigma\xi_1\right] P_{1,n+1} + \phi P_{0,n},\\ n = c,\end{aligned} \tag{8}$$

$$\begin{aligned}(\theta\lambda + c\beta\mu + n\sigma\xi_1)) P_{1,n} = \theta\lambda P_{1,n-1} + \left[c\beta\mu + (n+1)\sigma\xi_1\right] P_{1,n+1} + \phi P_{0,n},\\ n > c.\end{aligned} \tag{9}$$

The normalizing condition is as follows:

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} = 1. \tag{10}$$

Multiplying Equation (4) by z^n, summing all possible values of n, and using Equations (2)-(3), we get

$$\xi_0\sigma(1 - z)G_0'(z) = \left[\theta\lambda(1 - z) + \phi\right] G_0(z) - \lambda\bar{\theta}(z - 1)P_{0,0} - (\beta\mu + \sigma\xi_1)P_{1,1}. \tag{11}$$

Similarly, multiplying Equations (5)-(8) by z^n, then summing all possible values of n, we have

$$\begin{aligned}(1 - z)\left[(\theta\lambda z - c\beta\mu)G_1(z) - \xi_1\sigma z G_1'(z)\right] = z\phi G_0(z) - (\beta\mu + \sigma\xi_1)P_{1,1}z\\ + z\lambda\bar{\theta}(z - 1)Q_2(z) + \beta\mu(1 - z)Q_1(z) - c\beta\mu(1 - z)P_{1,0},\end{aligned} \tag{12}$$

with

$$Q_1(z) = \sum_{n=1}^{c-1}(n - c)P_{1,n} z^n, \qquad Q_2(z) = \sum_{n=0}^{c-1} P_{1,n} z^n. \tag{13}$$

Next, we solve the differential equation (11) and obtain $G_0(z)$ as follows,

$$G_0(z) = \exp\left\{\frac{\theta\lambda z - \phi ln(1-z)}{\sigma\xi_0}\right\}\left[G_0(0) - K_0(z)\right], \tag{14}$$

where

$$K_0(z) = \int_0^z \frac{\lambda\bar{\theta}(x-1)P_{0,0} + (\beta\mu + \sigma\xi_1)P_{1,1}}{\xi_0\sigma(1-x)} \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx.$$

In the same way, we solve differential equation (12) and obtain $G_1(z)$ as follows,

$$G_1(z) = \exp\left\{\frac{\theta\lambda z - c\beta\mu ln z}{\sigma\xi_1}\right\} K_1(z), \tag{15}$$

where

$$K_1(z) = \int_0^z \left[\frac{\phi G_0(x) - (\beta\mu + \sigma\xi_1)P_{1,1} + \lambda\bar{\theta}(x-1)Q_2(x)}{\xi_1\sigma(x-1)} - \frac{\beta\mu Q_1(x) + c\beta\mu P_{1,0}}{\xi_1\sigma x}\right] \times \exp - \left\{\frac{\theta\lambda x - c\beta\mu ln x}{\sigma\xi_1}\right\} dx.$$

Now, using Altman and Yechiali (2006), it yields

$$G_0(1) = \exp\left\{\frac{\theta\lambda}{\sigma\xi_1}\right\}\left[G_0(0) - K_0(1)\right] \times \lim_{z\to 1} \exp\left\{\frac{-\phi ln(1-z)}{\sigma\xi_1}\right\}. \tag{16}$$

Since $G_0(1) = \sum_{n=o}^{\infty} P_{0,n} > 0$ and $\lim_{z\to 1} \exp\left\{\frac{-\phi ln(1-z)}{\sigma\xi_1}\right\} = \infty$, we have

$$P_{0,0} = G_0(0) = \Gamma_1 P_{0,0} + \Gamma_2 P_{1,1}, \tag{17}$$

where

$$\Gamma_1 = \int_0^1 \frac{-\lambda\bar{\theta}}{\sigma\xi_0} \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx, \tag{18}$$

and

$$\Gamma_2 = \int_0^1 \frac{\beta\mu + \sigma\xi_1}{\sigma\xi_0(1-x)} \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx. \tag{19}$$

It is easily seen that from Equation (17) we get

$$P_{1,1} = T_1 P_{0,0}, \qquad \text{where} \qquad T_1 = (1-\Gamma_1)\,\Gamma_2^{-1}. \tag{20}$$

Now, substituting Equation (20) into Equation (14) and noting that $P_{0,0} = G_0(0)$, we get

$$G_0(z) = g_0(z) P_{0,0}, \tag{21}$$

where

$$g_0(z) = \exp\left\{\frac{\theta\lambda z - \phi ln(1-z)}{\sigma\xi_0}\right\} \times \left[1 - \int_0^z \left[\frac{-\lambda\bar{\theta}}{\xi_0\sigma} + \frac{(\beta\mu + \sigma\xi_1)T_1}{\xi_0\sigma(1-x)}\right] \times \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx\right]. \tag{22}$$

Equation (21) shows that $G_0(z)$ can be expressed in terms of $P_{0,0}$. Equation (15) shows that $G_1(z)$ may be expressed in terms of $G_0(x), \Gamma_1, \Gamma_2, Q_1(x)$ and $Q_2(x)$. Thus, once $P_{0,0}$ and $P_{1,j}(j = 1, 2, ..., c-1)$ are obtained, $G_0(z)$ and $G_1(z)$ are completely determined. The steady-state probabilities $P_{0,n}$ $(1 \leq n \leq c-1)$, $P_{1,n}$ $(0 \leq n \leq c-1)$ are computed recursively by solving the system (2)-(8). Therefore, solving Equations (2)-(4) recursively, it yields

$$P_{0,n} = \omega_n P_{0,0}, \qquad n \geq 1, \tag{23}$$

where

$$\begin{aligned}
\omega_1 &= \left[\lambda + \phi - (\beta\mu + \sigma\xi_1)T_1\right](\sigma\xi_0)^{-1},\\
\omega_2 &= \left[(\theta\lambda + \sigma\xi_0 + \phi)\,\omega_1 - \lambda\right](2\sigma\xi_0)^{-1},\\
\omega_n &= \left[(\theta\lambda + (n-1)\sigma\xi_0 + \phi)\,\omega_{n-1} - \theta\lambda\omega_{n-2}\right](n\sigma\xi_0)^{-1}, \;\; n \geq 3.
\end{aligned}$$

Next, substituting Equation (23) in Equations (6)-(7), we get easily

$$P_{1,n} = T_n P_{0,0}, \qquad 0 \leq n \leq c-1, \tag{24}$$

where

$$\begin{aligned}
T_0 &= \frac{\phi}{\lambda},\\
T_2 &= \frac{(\lambda + \beta\mu + \sigma\xi_1)T_1 - (\omega_1 + 1)\phi}{2(\beta\mu + \sigma\xi_1)},\\
T_n &= \frac{(\lambda + (n-1)(\beta\mu + \sigma\xi_1))T_{n-1} - \lambda T_{n-2} - \phi\omega_n}{n(\beta\mu + \sigma\xi_1)}, \;\; 3 \leq n \leq c-1.
\end{aligned}$$

By substituting Equation (24) into Equation (13), we have

$$Q_1(z) = P_{0,0}\sum_{n=1}^{c-1}(n-c)T_n z^n, \qquad Q_2(z) = P_{0,0}\sum_{n=0}^{c-1} T_n z^n. \tag{25}$$

Putting $z = 1$ in Equation (11), we get

$$G_0(1) = \frac{(\beta\mu + \sigma\xi_1)T_1}{\phi}P_{0,0}, \tag{26}$$

or, equivalently,

$$g_0(1) = \frac{(\beta\mu + \sigma\xi_1)T_1}{\phi}. \tag{27}$$

Substituting Equations (21), (20) and (25) into Equation (15), it yields

$$G_1(z) = g_1(z)P_{0,0}, \tag{28}$$

where

$$g_1(z) = \exp\left\{\frac{\theta\lambda z - c\beta\mu lnz}{\sigma\xi_1}\right\} \times \int_0^z \left[\frac{\phi g_0(x) - (\beta\mu + \sigma\xi_1)T_1 + \lambda\bar{\theta}(x-1)(\sum_{n=1}^{c-1} T_n x^n + (\phi/\lambda))}{\xi_1\sigma(x-1)} - \frac{\beta\mu(\sum_{n=1}^{c-1}(n-c)T_n x^n - (c\phi/\lambda))}{\xi_1\sigma x}\right] \times \exp - \left\{\frac{\theta\lambda x - c\beta\mu lnx}{\sigma\xi_1}\right\} dx.$$

Finally, the only unknown $P_{0,0}$ is obtained from the normalization condition (10) and is given by

$$P_{0,0} = [g_0(1) + g_1(1)]^{-1}. \tag{29}$$

This completes the evaluation of steady-state probabilities.

3.2. Performance Measures

3.2.1. *Mean number of customers in the system.*

Let $E[L_0]$ be the mean system size when servers are taking vacation, and let $E[L_1]$ denote the mean system size when servers are busy. We first derive the mean system sizes $E[L_0]$ and $E[L_1]$. Deriving Equation (11) and taking $z = 1$, we obtain,

$$\begin{aligned} E[L_0] &= G_0'(1) \\ &= \frac{\theta\lambda G_0(1) + \lambda\bar{\theta}P_{0,0}}{\xi_0\sigma + \phi}, \end{aligned} \tag{30}$$

where $G_0(1)$ is given by Equation (26) and $P_{0,0}$ by Equation (29). From Equation (28), we have

$$\begin{aligned} E[L_1] &= G_1'(1) \\ &= \lim_{z\to 1} G_1'(z) \\ &= \frac{1}{\sigma\xi_1}\left[(\theta\lambda - c\beta\mu)G_1(1) + \phi E[L_0] + \lambda\bar{\theta}Q_2(1)\right. \\ &\quad \left. -\beta\mu Q_1(1) + c\beta\mu(\phi/\lambda)P_{0,0}\right]. \end{aligned} \tag{31}$$

3.2.2. *Probability that the servers are in vacation.*

$$
\begin{aligned}
P_{vac} &= \mathbb{P}(J=0) \\
&= \sum_{n=0}^{\infty} P_{0,n} \\
&= G_0(1),
\end{aligned} \tag{32}
$$

where $G_0(1)$ is given by Equation (26).

3.2.3. *Probability that the servers are in busy period.*

$$
\begin{aligned}
P_{ser} &= \sum_{n=0}^{\infty} P_{1,n} \\
&= 1 - P_{vac} \\
&= 1 - G_0(1).
\end{aligned} \tag{33}
$$

3.2.4. *Mean number of served customers.*

When the system is in state $(1, n)$, service rates of the servers are $n\mu$ for $n = 1, 2, \ldots, c-1$ and $c\mu$ for $n = c, c+1, \ldots$, respectively. Thus, the expected number of customers served per unit of time is given by

$$
\begin{aligned}
N_s &= \beta\mu \sum_{n=1}^{c-1} nP_{1,n} + c\beta\mu \sum_{n=c}^{\infty} P_{1,n} \\
&= \beta\mu \left[cG_1(1) + Q_1(1) - cP_{1,0} \right].
\end{aligned} \tag{34}
$$

3.2.5. *Average rate of abandonment.*

Let R_a the average rate of abandonment of a customer due to impatience,

$$
R_a = R_r + R_b, \tag{35}
$$

where R_r is the average reneging rate and R_b denotes the average balking rate of a customer.

When the system is in state $(0, n)$, $n = 0, 1, \ldots$, the instantaneous reneging rate is $n\sigma\xi_0$. When the system is in state $(1, n)$, $n = 1, 2, \ldots$, the instantaneous reneging rate of a customer is $n\sigma\xi_1$. Thus, the average rate of reneging of a customer is given by

$$
\begin{aligned}
R_r &= \sigma \left[\xi_0 \sum_{n=0}^{\infty} nP_{0,n} + \xi_1 \sum_{n=1}^{\infty} nP_{1,n} \right] \\
&= \sigma \left(\xi_0 E\left[L_0\right] + \xi_1 E\left[L_1\right] \right).
\end{aligned} \tag{36}
$$

And when the system is in state $(0, n)$, $n = 1, 2, \ldots$, or in state $(1, n)$, $n = c, c+1, \ldots$, the instantaneous balking rate of a customer is $\lambda(1-\theta)$. Thus, the balking average rate of a customer is given by

$$\begin{aligned} R_b &= \lambda(1-\theta)\left[\sum_{n=1}^{\infty} P_{0,n} + \sum_{n=c}^{\infty} P_{1,n}\right] \\ &= \lambda(1-\theta)\left(1 - P_{0,0} - \sum_{n=0}^{c-1} P_{1,n}\right) \\ &= \lambda(1-\theta)\left(1 - P_{0,0} - \sum_{n=0}^{c-1} T_n P_{0,0}\right). \end{aligned} \tag{37}$$

3.2.6. *Average rate of retention.*

The average retention rate is given as

$$\begin{aligned} R_{ret} &= (1-\sigma)\left[\xi_0 \sum_{n=0}^{\infty} nP_{0,n} + \xi_1 \sum_{n=1}^{\infty} nP_{1,n}\right] \\ &= (1-\sigma)\left(\xi_0 E\left[L_0\right] + \xi_1 E\left[L_1\right]\right). \end{aligned} \tag{38}$$

4. Analysis of type II model: M/M/c/MV

In this section we study the model defined in Section 2 under multiple station vacation, the state-transition diagram is presented in Figure 1.

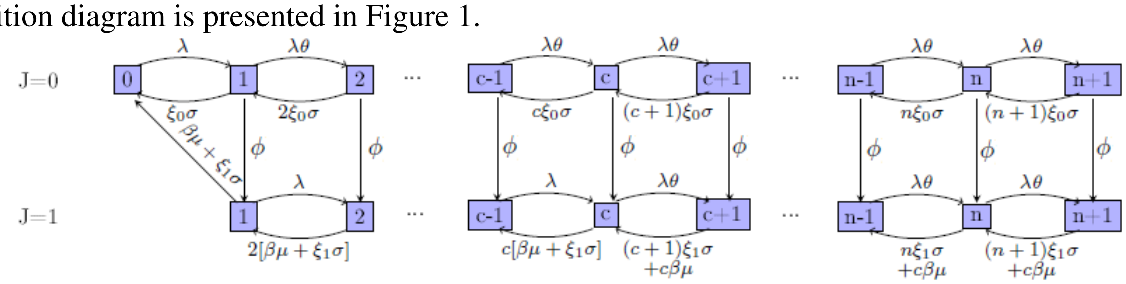

Figure 2. State-transition diagram for Model II

4.1. Stationary Analysis.

The process $\{(J(t), L(t)),\ t \geq 0\}$ is defined as before, a continuous-time Markov process with a state space $\Delta = \{(0, n):\ n = 0, 1, \ldots\} \cup \{(1, n):\ n = 1, 2, \ldots\}$. Let

$$P_{j,n} = \lim_{t\to\infty} \mathbb{P}(J(t) = j, L(t) = n),\ \ j = 0, 1,\ \ n = 0, 1, \ldots,$$

be the system state (steady-state) probabilities. The partial generating functions, $G_0(z)$ and $G_1(z)$, for $0 < z < 1$ are given as

$$G_0(z) = \sum_{n=0}^{\infty} P_{0,n} z^n, \qquad G_1(z) = \sum_{n=1}^{\infty} P_{1,n} z^n, \qquad 0 \le z \le 1. \tag{39}$$

The set of balance equations is given as follows

$$\lambda P_{0,0} = \sigma\xi_0 P_{0,1} + (\beta\mu + \sigma\xi_1) P_{1,1}, \qquad n = 0, \tag{40}$$

$$(\theta\lambda + \sigma\xi_0 + \phi) P_{0,1} = \lambda P_{0,0} + 2\sigma\xi_0 P_{0,2}, \qquad n = 1, \tag{41}$$

$$(\theta\lambda + n\sigma\xi_0 + \phi) P_{0,n} = \theta\lambda P_{0,n-1} + (n+1)\sigma\xi_0 P_{0,n+1}, \quad n \ge 2, \tag{42}$$

$$(\lambda + \beta\mu + \sigma\xi_1) P_{1,1} = 2(\beta\mu + \sigma\xi_1) P_{1,2} + \phi P_{0,1}, \qquad n = 1, \tag{43}$$

$$(\lambda + n(\beta\mu + \sigma\xi_1)) P_{1,n} = \lambda P_{1,n-1} + (n+1)(\beta\mu + \sigma\xi_1) P_{1,n+1} + \phi P_{0,n}, \quad 2 \le n \le c-1, \tag{44}$$

$$(\theta\lambda + n(\beta\mu + \sigma\xi_1)) P_{1,n} = \lambda P_{1,n-1} + [c\beta\mu + (n+1)\sigma\xi_1] P_{1,n+1} + \phi P_{0,n}, \quad n = c, \tag{45}$$

$$(\theta\lambda + c\beta\mu + n\sigma\xi_1)) P_{1,n} = \theta\lambda P_{1,n-1} + [c\beta\mu + (n+1)\sigma\xi_1] P_{1,n+1} + \phi P_{0,n}, \quad n > c. \tag{46}$$

The normalizing condition is as follows:

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=1}^{\infty} P_{1,n} = 1. \tag{47}$$

Then, multiplying Equation (42) by z^n, summing all possible values of n, and using Equations (40) and (41), we get

$$\xi_0\sigma(1-z)G_0'(z) = [\theta\lambda(1-z) + \phi] G_0(z) - \left[\lambda\bar{\theta}(z-1) + \phi\right] P_{0,0} - (\beta\mu + \sigma\xi_1)P_{1,1}. \tag{48}$$

Similarly, multiplying Equations (43)-(46) by z^n, then summing all possible values of n, we have

$$\begin{aligned}(1-z)\left[(\theta\lambda z - c\beta\mu)G_1(z) - \xi_1\sigma z G_1'(z)\right] &= z\phi G_0(z) - (\phi P_{0,0} + (\beta\mu + \sigma\xi_1)P_{1,1})\, z \\ &\quad + z\lambda\bar{\theta}(z-1)R_2(z) + \beta\mu(1-z)R_1(z),\end{aligned} \tag{49}$$

with

$$R_1(z) = \sum_{n=1}^{c-1} (n-c) P_{1,n} z^n, \qquad R_2(z) = \sum_{n=1}^{c-1} P_{1,n} z^n. \tag{50}$$

Next, we solve Equation (48) and obtain $G_0(z)$ as

$$G_0(z) = \exp\left\{\frac{\theta\lambda z - \phi ln(1-z)}{\sigma\xi_0}\right\} \left[G_0(0) - K_0'(z)\right], \tag{51}$$

where

$$K_0'(z) = \int_0^z \frac{\left[\lambda\bar{\theta}(x-1) + \phi\right] P_{0,0} + (\beta\mu + \sigma\xi_1)P_{1,1}}{\xi_0\sigma(1-x)} \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx.$$

By solving the differential equation (49), we obtain $G_1(z)$ as follows:

$$G_1(z) = \exp\left\{\frac{\theta\lambda z - c\beta\mu lnz}{\sigma\xi_1}\right\} K_1'(z), \tag{52}$$

where

$$K_1'(z) = \int_0^z \left[\frac{\phi G_0(x) - (\phi P_{0,0} + (\beta\mu + \sigma\xi_1)P_{1,1}) + \lambda\bar{\theta}(x-1)R_2(x)}{\xi_1\sigma(x-1)} - \frac{\beta\mu R_1(x)}{\xi_1\sigma x}\right] \times \exp - \left\{\frac{\theta\lambda x - c\beta\mu lnx}{\sigma\xi_1}\right\} dx.$$

Via Altman and Yechiali (2006), we have

$$G_0(1) = \exp\left\{\frac{\theta\lambda}{\sigma\xi_1}\right\}\left[G_0(0) - K_0'(1)\right] \times \lim_{z\to 1} \exp\left\{\frac{-\phi ln(1-z)}{\sigma\xi_1}\right\}. \tag{53}$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $\lim_{z\to 1} \exp\left\{\frac{-\phi ln(1-z)}{\sigma\xi_1}\right\} = \infty$, it yields

$$P_{0,0} = G_0(0) = \Gamma_2 P_{1,1} + \Gamma_3 P_{0,0}, \tag{54}$$

where Γ_2 has been defined in Equation (19), and

$$\Gamma_3 = \int_0^1 \frac{\left[\lambda\bar{\theta}(x-1) + \phi\right]}{\sigma\xi_0(1-x)} \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx. \tag{55}$$

From Equation (54) we get

$$P_{1,1} = T_1' P_{0,0}, \qquad \text{where} \qquad T_1' = (1-\Gamma_3)\,\Gamma_2^{-1}. \tag{56}$$

Substituting Equation (56) into Equation (51) and noting that $P_{0,0} = G_0(0)$, we have

$$G_0(z) = g_0(z) P_{0,0}, \tag{57}$$

where

$$g_0(z) = \exp\left\{\frac{\theta\lambda z - \phi ln(1-z)}{\sigma\xi_0}\right\} \times \left[1 - \int_0^z \left[\frac{\left[\lambda\bar{\theta}(x-1) + \phi\right]}{\xi_0\sigma(1-x)} + \frac{(\beta\mu + \sigma\xi_1)T_1'}{\xi_0\sigma(1-x)}\right] \times \exp - \left\{\frac{\theta\lambda x - \phi ln(1-x)}{\sigma\xi_0}\right\} dx\right]. \tag{58}$$

Equation (57) shows that $G_0(z)$ can be expressed in terms of $P_{0,0}$. Equation (52) shows that $G_1(z)$ can be expressed in terms of $G_0(x), \Gamma_3, \Gamma_2, R_1(x)$ and $R_2(x)$. Then, once $P_{0,0}$ and $P_{1,j} (j = 1, 2, ..., c-1)$ are obtained, $G_0(z)$ and $G_1(z)$ are completely determined. The steady-state probabilities $P_{0,n}$, $P_{1,n}$ $(1 \leq n \leq c-1)$ are computed recursively by solving Equations (40)-(46). Thus, via Equations (40)-(42) we get

$$P_{0,n} = \omega_n' P_{0,0}, \qquad n \geq 1, \tag{59}$$

where

$$\omega_1' = \left[\lambda - (\beta\mu + \sigma\xi_1)T_1'\right](\sigma\xi_0)^{-1},$$
$$\omega_2' = \left[(\theta\lambda + \sigma\xi_0 + \phi)\,\omega_1' - \lambda\right](2\sigma\xi_0)^{-1},$$
$$\omega_n' = \left[(\theta\lambda + (n-1)\sigma\xi_0 + \phi)\,\omega_{n-1}' - \theta\lambda\omega_{n-2}'\right](n\sigma\xi_0)^{-1}, \;\; n \geq 3.$$

Using Equation (59) in Equations (43)-(45), it yields

$$P_{1,n} = T'_n P_{0,0}, \qquad 1 \leq n \leq c-1, \tag{60}$$

where

$$T'_2 = \frac{(\lambda + \beta\mu + \sigma\xi_1)T'_1 - \phi\omega'_1}{2(\beta\mu + \sigma\xi_1)},$$
$$T'_n = \frac{(\lambda + (n-1)(\beta\mu + \sigma\xi_1))T'_{n-1} - \lambda T'_{n-2} - \phi\omega'_n}{n(\beta\mu + \sigma\xi_1)}, \;\; 3 \leq n \leq c-1.$$

Substituting Equation (60) into Equation (50), we have

$$R_1(z) = P_{0,0}\sum_{n=1}^{c-1}(n-c)T'_n z^n, \qquad R_2(z) = P_{0,0}\sum_{n=1}^{c-1} T'_n z^n. \tag{61}$$

Letting $z = 1$ in Equation (48), we get

$$G_0(1) = \frac{\phi + (\beta\mu + \sigma\xi_1)T'_1}{\phi} P_{0,0}, \tag{62}$$

or, equivalently,

$$g_0(1) = \frac{\phi + (\beta\mu + \sigma\xi_1)T'_1}{\phi}. \tag{63}$$

Substituting Equations (56)-(57) and (61) into Equation (52), we have

$$G_1(z) = g_1(z) P_{0,0}, \tag{64}$$

where

$$g_1(z) = \int_0^z \left[\frac{\phi g_0(x) - (\phi + (\beta\mu + \sigma\xi_1)T'_1) + \bar{\theta}\lambda(x-1)\sum_{n=1}^{c-1} T'_n x^n}{\xi_1\sigma(x-1)} - \frac{\beta\mu\sum_{n=1}^{c-1}(n-c)T'_n x^n}{\xi_1\sigma x} \right] \times \exp - \left\{\frac{\theta\lambda x - c\beta\mu ln x}{\sigma\xi_1}\right\} dx \exp\left\{\frac{\theta\lambda z - c\beta\mu ln z}{\sigma\xi_1}\right\}.$$

Finally, the only unknown $P_{0,0}$ is obtained from the normalization condition (47). Thus

$$P_{0,0} = \left[g_0(1) + g_1(1)\right]^{-1}. \tag{65}$$

This completes the evaluation of steady-state probabilities.

4.2. Performance Measures

4.2.1. *Mean number of customers in the system.*

As for the first model, deriving Equation (48) and taking $z = 1$, we obtain,

$$\begin{aligned} E\left[L_0\right] &= G_0'(1) \\ &= \frac{\theta\lambda G_0(1) + \lambda\bar{\theta}P_{0,0}}{\xi_0\sigma + \phi}, \end{aligned} \tag{66}$$

where $G_0(1)$ is given by Equation (62) and $P_{0,0}$ by Equation (65). From Equation (64), we get

$$\begin{aligned} E\left[L_1\right] &= G_1'(1) \\ &= \frac{1}{\sigma\xi_1}\left[(\theta\lambda - c\beta\mu)G_1(1) + \phi E\left[L_0\right] + \bar{\theta}\lambda R_2(1) - \beta\mu R_1(1)\right]. \end{aligned} \tag{67}$$

4.2.2. *Probability that the servers are in vacation period.*

$$P_{vac} = G_0(1), \tag{68}$$

where $G_0(1)$ is given by Equation (62).

4.2.3. *Probability that the servers are in busy period serving customers.*

$$P_{ser} = 1 - G_0(1). \tag{69}$$

4.2.4. *Mean number of customers served.*

The expected number of customers served per unit of time is given by

$$N_s = \beta\mu\left[cG_1(1) + R_2(1)\right]. \tag{70}$$

4.2.5. *Average rate of abandonment.*

The average rate of abandonment of a customer due to impatience is as follows

$$R_{abd} = R_r + R_b, \tag{71}$$

where

$$R_r = \sigma\left(\xi_0 E\left[L_0\right] + \xi_1 E\left[L_1\right]\right), \tag{72}$$

and the average rate of balking is defined as

$$R_b = \lambda(1-\theta)\left(1 - P_{0,0} - \sum_{n=1}^{c-1} T_n' P_{0,0}\right). \tag{73}$$

4.2.6. *Average rate of retention.*

The average rate of retention is presented as

$$R_{ret} = (1-\sigma)\left(\xi_0 E\left[L_0\right] + \xi_1 E\left[L_1\right]\right). \tag{74}$$

5. Cost model

In this section, we develop a model for the costs incurred in the considered queueing system under multiple vacation policy. To this end, let's consider the following symbols and notations.

- C_{busy} : Cost per unit time when the servers are busy.
- C_{vac} : Cost per unit time when the servers are on vacation.
- C_q : Cost per unit time when a customer joins the queue and waits for service.
- C_b : Cost per unit time when a customer balks.
- C_s : Cost per service per unit time.
- C_r : Cost per unit time when a customer reneges, either during busy or vacation period.
- C_{ret} : Cost per unit time when a customer is retained, either during busy or vacation period.
- C_F : Fixed server purchase cost per unit.
- C_{s-f} : Cost per unit time when a customer returns to the system as a feedback customer.

Let R be the revenue earned by providing service to a customer.
$\mathcal{T}_{e.cost}$ be the total expected cost per unit time of the system.
$\mathcal{T}_{e.rev}$ be the total expected revenue per unit time of the system.
$\mathcal{T}_{e.pro}$ be the total expected profit per unit time of the system.

Thus

$$\mathcal{T}_{e.cost} = C_{busy}P_{busy} + C_{vac}P_{vac} + C_q E(L_q) + C_b R_b + C_r R_r$$

$$+C_{ret}R_{ret} + c\mu(C_s + \bar{\beta}C_{s-f}) + cC_F,$$

with $E(L_q)$ denotes the average number of customers in the queue, such that

$$E(L_q) = \sum_{n=0}^{+\infty} nP_{0,n} + \sum_{n=c}^{+\infty}(n-c)P_{1,n} \tag{75}$$

$$= E(L_0) + E(L_1) - (c \times G_1(1)) - R_1(1). \tag{76}$$

The total expected revenue per unit time of the system is given by:

$$\mathcal{T}_{e.rev} = R \times N_s.$$

Now, the total expected profit is presented as

$$\mathcal{T}_{e.pro} = T_{e.rev} - \mathcal{T}_{e.cost}.$$

After getting the expected cost per unit time $\mathcal{T}_{e.cost}$, the total expected revenue Δ and the total expected profit Θ functions in terms of various parameters involved. The economic analysis of the model will be performed numerically by using these functions and the results are discussed accordingly.

6. Numerical analysis

This section is devoted to the numerical study of different performance measures and cost profit aspects associated with the model under multiple vacation policy. More precisely, we present the variation in important performance measures and different types of costs involved with the change in diverse parameters of the system. Indeed, using a program implemented under R environment, we present some numerical examples to illustrate the effect of various parameters on the performance measures of the system. For the whole analysis we fix the different costs as follows: $C_{busy} = 5$, $C_{vac} = 6$, $C_q = 5$, $C_b = 5$, $C_s = 4$, $C_{ren} = 5$, $C_{ret} = 5$, $C_{s-f} = 5$, $C_F = 4$, and $R = 50$.

Case 1: Variation of arrival rate λ.

We check the behavior of some system characteristics and the profit function for various values of λ and β by keeping all other variables fixed. For the analysis we fix the parameters $c = 3$, $\theta = 0.8$, $\mu = 5$, $\phi = 0.1$, $\sigma = 0.8$, $\xi_0 = 0.5$, and $\xi_1 = 0.6$.

Table 1. Total expected profit variation vs. λ

	β / λ	7	7.5	8	8.5
$\mathcal{T}_{e.cost}$		726.3430	795.0661	867.3303	943.2274
$\mathcal{T}_{e.rev}$	0.4	803.9488	901.8072	1005.4203	1114.9089
$\mathcal{T}_{e.pro}$		77.6058	106.7411	138.0901	171.6815
$\mathcal{T}_{e.cost}$		660.4133	718.789	779.7721	843.3918
$\mathcal{T}_{e.rev}$	0.6	670.8670	745.6624	823.6789	905.0529
$\mathcal{T}_{e.pro}$		10.4538	26.8734	43.9068	61.6611
$\mathcal{T}_{e.cost}$		624.4847	678.4826	734.9177	793.9311
$\mathcal{T}_{e.rev}$	0.8	576.4310	635.4586	695.5587	756.5159
$\mathcal{T}_{e.pro}$		-48.0536	-43.0242	-39.3590	-37.4151

According to Figures 3-5 and Table 1, we constat that, for fixed β, the mean number of customers in the system $E(L)$ increases with the increase of the arrival rate λ. Consequently, the mean number of customers served N_s is significant, this implies a significant augmentation in the total expected profit $\mathcal{T}_{e.pro}$. On the other hand, for fixed ϕ, along the increase in the non-feedback β, the mean number of customers in the system $E(L)$ decreases, this implies a decrease in the mean number of customers served N_s. Thus, the total expected profit $\mathcal{T}_{e.pro}$ decreases significantly.

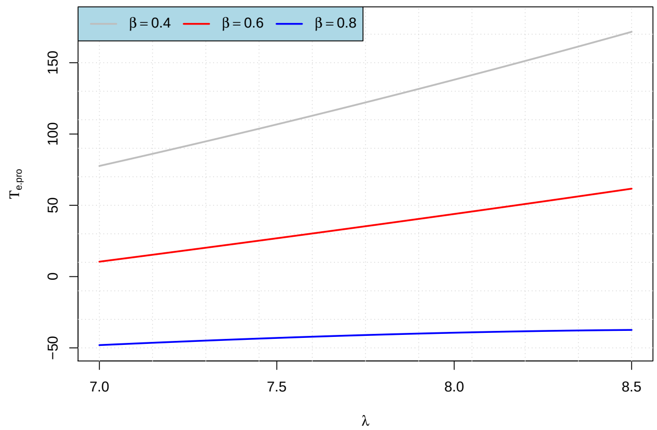

Figure 3. Total expected profit variation curves vs.λ

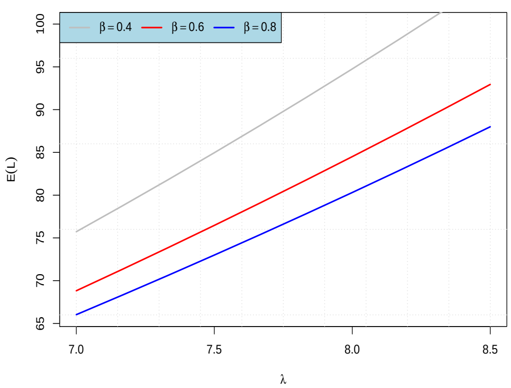

Figure 4. E(L) vs. λ

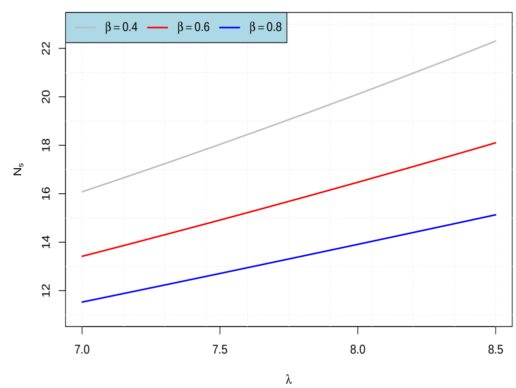

Figure 5. N_s vs. λ

Case 2: Variation of service rate μ.

We examine the behavior of some system indices and the profit function for various values of μ and ϕ by keeping all other variables fixed. For this case we fix the parameters $c = 3$, $\beta = 0.5$, $\lambda = 4$, $\theta = 0.8$, $\sigma = 0.8$, $\xi_0 = 0.5$ and $\xi_1 = 0.85$. Via Figures 6-8 and Table 2, it is clearly seen that along the increases of the service and vacation rates μ and ϕ, the number of customers served N_s grows, consequently system size $E(L)$ decreases. Therefore, the total expected profit $\mathcal{T}_{e.pro}$ increases significantly.

Case 3: Variation of impatience rate during busy period ξ_1.

We analyze the behavior of the system characteristics by varying ξ_1 and c, and fixing the parameters $\phi = 0.2$, $\theta = 0.8$, $\lambda = 4$, $\beta = 0.8$, $\mu = 4$, $\sigma = 0.8$, and $\xi_0 = 0.85$. From Figures 9-11 and Table

Table 2. Total cost variation vs. μ

	ϕ / μ	1.3	1.6	1.9	2.2
$\mathcal{T}_{e.cost}$		254.5262	246.4486	241.6383	238.9954
$\mathcal{T}_{e.rev}$	0.07	241.2788	245.2543	246.7307	246.8657
$\mathcal{T}_{e.pro}$		-13.2474	-1.1943	5.0924	7.8703
$\mathcal{T}_{e.cost}$		259.5818	251.1900	246.2299	243.5346
$\mathcal{T}_{e.rev}$	0.1	257.2323	261.0472	262.0694	261.5931
$\mathcal{T}_{e.pro}$		-2.3495	9.8572	15.8395	18.05844
$\mathcal{T}_{e.cost}$		262.3111	253.8694	248.9873	246.4428
$\mathcal{T}_{e.rev}$	0.13	267.5969	270.9617	271.2408	269.8559
$\mathcal{T}_{e.pro}$		5.2858	17.0923	22.2535	23.4131

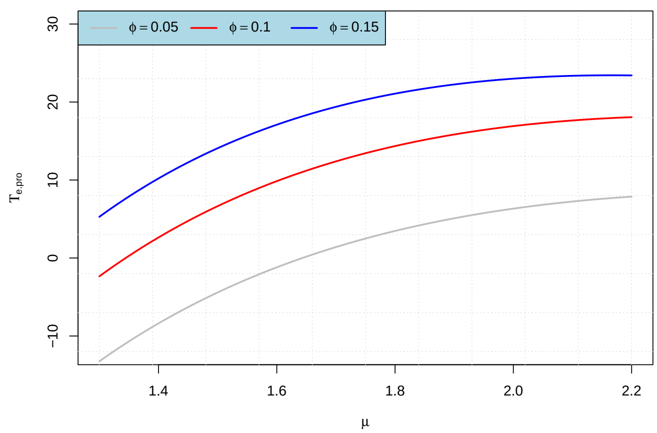

Figure 6. Total expected profit variation curves vs. μ

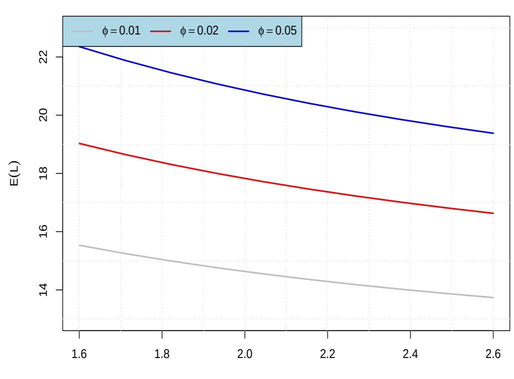

Figure 7. E(L) vs. μ

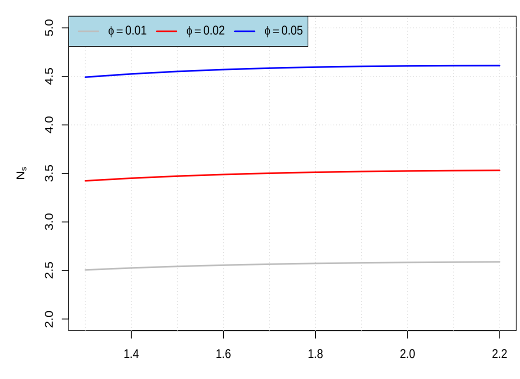

Figure 8. N_s vs. μ

Table 3. Total cost variation vs. ξ_1

	c / ξ_1	1	1.4	1.6	1.8
$\mathcal{T}_{e.cost}$		69.7	69.8	69.9555	70.0636
$\mathcal{T}_{e.rev}$	1	20	19	17.524	16.3741
$\mathcal{T}_{e.pro}$		-49	-51	-52.4315	-53.6895
$\mathcal{T}_{e.cost}$		122	119	117.3160	115.7033
$\mathcal{T}_{e.rev}$	2	129	125	120.6422	116.7341
$\mathcal{T}_{e.pro}$		7.08	5.41	3.3262	1.0308
$\mathcal{T}_{e.cost}$		196	188	182.7164	177.9936
$\mathcal{T}_{e.rev}$	3	230	221	213.0694	205.1455
$\mathcal{T}_{e.pro}$		34	33	30.3530	27.1518

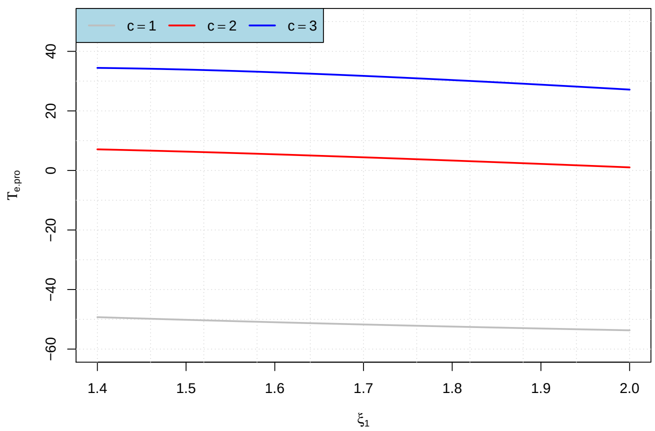

Figure 9. Total expected profit variation curves vs. ξ_1

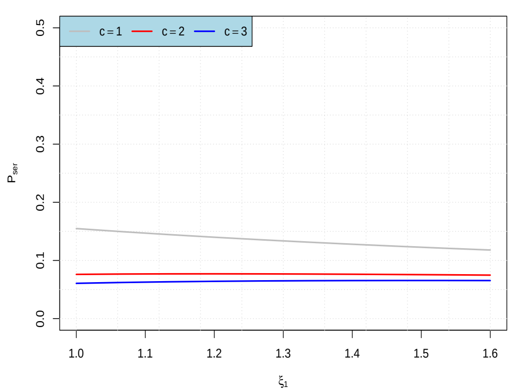

Figure 10. P_{ser} vs. ξ_1

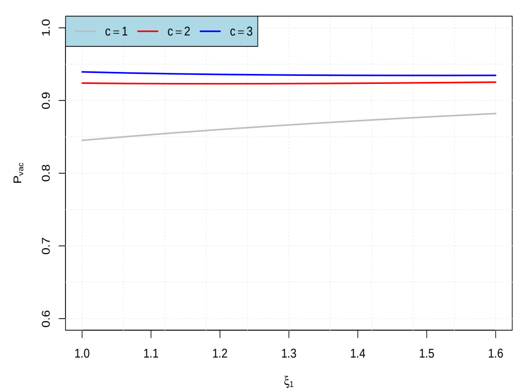

Figure 11. P_{vac} vs. ξ_1

3, we see that the augmentation of ξ_1 implies a decrease in the probability of serving customers, as it should be. Then, the probability of vacation period increases. This yield to a diminution in the size of the system $\mathbb{E}(L)$. Therefore, the mean number of customers served N_s is reduced. The increase in the number of servers has positive impact on the behavior of the system which results in an augmentation of the total expected profit $\mathcal{T}_{e.pro}$.

Case 4: Variation of non-balking rate θ.

In this subpart, we study the behavior of the parameters by varying θ and ξ_0. Let's fix the parameters $c = 3$, $\phi = 0.1$, $\lambda = 4$, $\beta = 0.8$, $\mu = 4$, $\sigma = 0.8$, $\xi_1 = 0.85$. From Figures 12-14 and

Table 4. Variation of system performance measures vs. (ξ_0, θ)

ξ_0 / θ	0.2	0.4	0.6	0.8	
$\mathcal{T}_{e.cost}$		207.3673	257.6433	277.46	282.7103
$\mathcal{T}_{e.rev}$	0.4	157.5653	248.9523	295.1911	319.2383
$\mathcal{T}_{e.pro}$		-49.8021	-8.6911	17.7311	36.528
$\mathcal{T}_{e.cost}$		183.5943	232.355	259.8643	269.1332
$\mathcal{T}_{e.rev}$	0.5	127.5822	215.5849	275.5405	308.1107
$\mathcal{T}_{e.pro}$		-56.0121	-16.7701	15.6761	38.9775
$\mathcal{T}_{e.cost}$		166.8161	209.3882	241.4381	261.4863
$\mathcal{T}_{e.rev}$	0.6	105.6353	182.2394	249.3144	275.0214
$\mathcal{T}_{e.pro}$		-61.1807	-27.1488	7.8763	13.5350

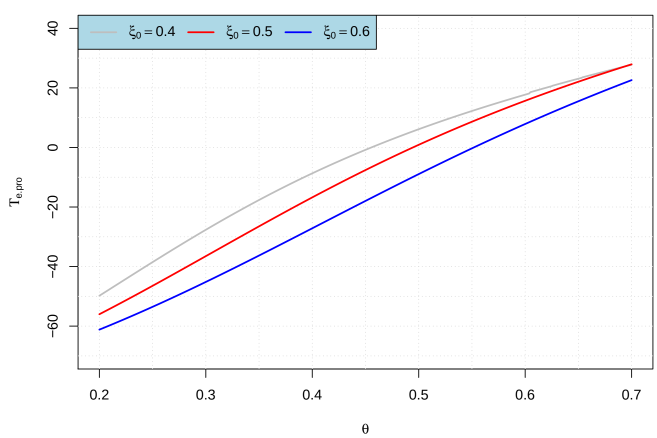

Figure 12. Total expected profit variation curves vs. θ

Table 4 we constat that for fixed ξ_0, the mean number of customers in the system $E(L)$ increases with the increase of non-balking rate θ. Consequently, the number of customers served N_s grows significantly. This implies an increase in the total expected profit $\mathcal{T}_{e.pro}$. In addition, when the impatience rate ξ_0 grows, for fixed θ, the system size decreases, which results in a decrease in the

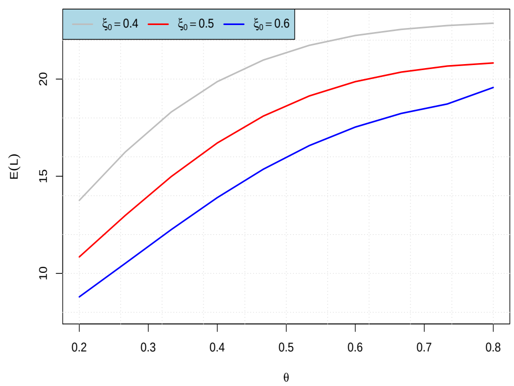

Figure 13. E(L) vs. θ

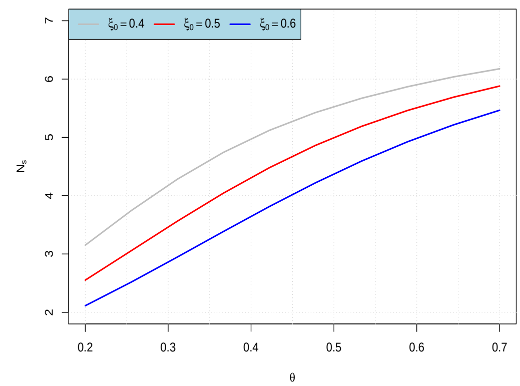

Figure 14. N_s vs. θ

mean number of customers served as well as in the total expected profit.

Case 5: Variation of non-retention rate σ.

We vary σ and ϕ and fix the parameters $c = 3$, $\theta = 0.8$, $\lambda = 4$, $\beta = 0.5$, $\mu = 4$, $\xi_0 = 1$, $\xi_0 = 2$.

Table 5. Total cost variation vs. σ

	ϕ / σ	0.6	0.7	0.8	0.9
$\mathcal{T}_{e.cost}$		231	194.7085	171	155.4570
$\mathcal{T}_{e.rev}$	0.05	182	135.7	100	73.9758
$\mathcal{T}_{e.pro}$		-49	-59.0087	-71	-81.4813
$\mathcal{T}_{e.cost}$		236	202.3086	179	162.3447
$\mathcal{T}_{e.rev}$	0.1	211	170.7674	137	108.9651
$\mathcal{T}_{e.pro}$		-25	-31.5412	-42	-53.3796
$\mathcal{T}_{e.cost}$		237	204.7325	182	165.6776
$\mathcal{T}_{e.rev}$	0.15	225	188.8468	157	129.8103
$\mathcal{T}_{e.pro}$		-11	-15.8857	-25	-35.8673

From Figures 15-17 and Table 5, we clearly see that for fixed ϕ, the increasing of the non-retention probability σ implies a significant decease in the system size $E(L)$ and consequently in the mean number of customers served N_s. Therefore, a lost in the total expected profit is considerable. Then, we conclude that the retention probability plays an important role in the economy of any firm.

7. Conclusion

In this investigation, we analyzed the synchronous vacation policy in a $M/M/c$ model. Two different types of vacation policies are discussed, multiple vacation (MV) and single vacation (SV)

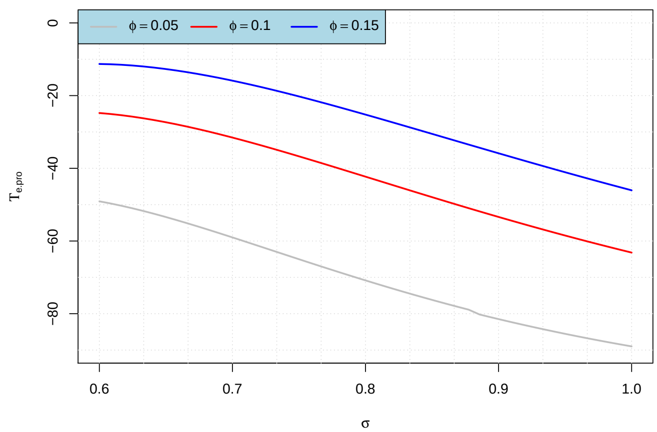

Figure 15. Total expected profit variation curves vs. σ

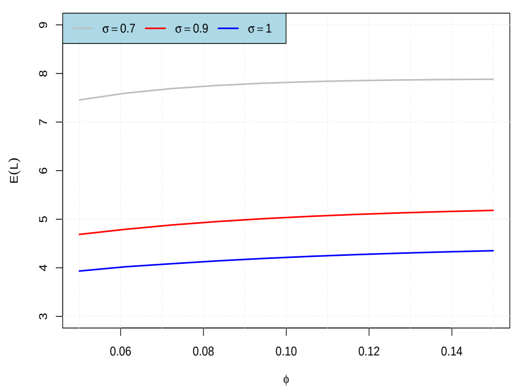

Figure 16. E(L) vs. σ

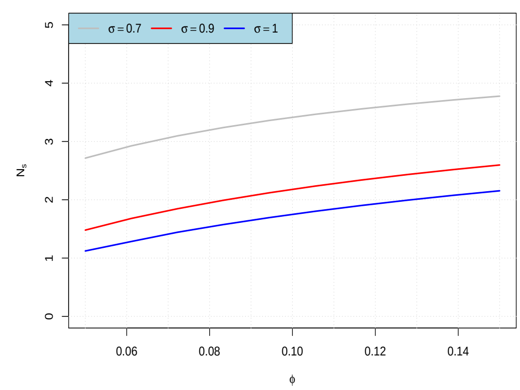

Figure 17. N_s vs. σ

policies. We obtained useful performance measures of the two models. We carried out some numerical examples regarding the multiple vacation queueing model. The theoretical results presented in this work may have potential applications in many real life systems such as call centers, communications, manufacturing and production-inventory systems, and many other related areas. From the presented study, we conclude that the analysis of multi-server vacation models is far more complex compared to single server vacation models. For future work, it will be interesting to extend the study of our model to non-Markovian models with general impatient times and vacation times.

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