Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 2 (December 2020), pp. 708 – 724

Nonparametric Estimation of Trend Function for Stochastic Differential Equations Driven by a Weighted Fractional Brownian Motion

[1]**Abdelmalik Keddi**, [2]**Fethi Madani, and** [3]**Amina Angelika Bouchentouf**

[1;2]Laboratory of Stochastic Models, Statistic and Applications
Tahar Moulay University of Saida
P.O. Box 138
En-Nasr, 20000 Saida, Algeria
[1]malokkeddi@yahoo.com; [2]fethi.madani@univ-saida.dz

[3]Department of Mathematics
Djillali Liabes University of Sidi Bel Abbes
B. P. 89
Sidi Bel Abbes 22000, Algeria
bouchentouf_amina@yahoo.fr

Received: August 5, 2019; Accepted: April 16, 2020

Abstract

In this paper, we consider the problem of nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion (weighted-fBm). Under some general conditions, the consistent uniform, the rate of convergence as well as the asymptotic normality of our estimator are established. In addition, a numerical example is provided to illustrate the validity of the considered estimator.

Keywords: Weighted fractional Brownian motion; Trend function; Kernel estimator; Stochastic differential equations; Nonparametric estimation

MSC 2010 No.: 62M09, 60G15

1. Introduction

A long/short-term memory stochastic processes with self-similarity have been extensively employed as models for various physical phenomena. They seemed to play an crucial role in analyzing network traffic, economy, and telecommunications. As a result, some effective mathematical models based on long-term/short term dependence processes with self-similarity have been suggested in these directions.

Fractional Brownian motion is a straightforward stochastic process with long/short term dependence and self-similarity, which is an appropriate generalization of standard Brownian motion. Interesting comprehensive surveys and literatures on fractional Brownian motion can be found in Norros et al. (1999), Hu (2005), Gradinaru et al. (2005), Mishura (2008), Biagini et al. (2008), and references therein.

On the other hand, in contrast to the extensive studies on fractional Brownian motion, only few systematic investigations have been conducted on other self-similar Gaussian processes. The principal cause is the complexity of the dependency structures for self-similar Gaussian processes that don't have stationary increments. It therefore seems interesting to study some extensions of the fractional Brownian motion, such as weighted fractional Brownian motion and the bi-fractional Brownian motion. Recently, weighted fractional Brownian motion has attracted considerable attention from many researchers, see for instance Shen et al. (2013), Guangjun et al. (2016), and Sun et al. (2017).

Nonparametric estimation of trend function for stochastic differential equations (SDEs) have been studied by several authors for their mathematical interest and their applications. The first paper on the subject is Kutoyants (1994) for the stochastic differential equation driven by a standard Brownian motion. Then, the problem was generalized; Mishra and Prakasa Rao (2011a) for the stochastic differential equation driven by a fractional Brownian motion; Mishra and Prakasa Rao (2011b) for nonparametric estimation of linear multiplier for fractional diffusion processes; Saussereau (2014) for nonparametric inference for fractional diffusion; and Prakasa Rao (2019) for nonparametric estimation of trend function for SDEs driven by mixed fractional Brownian motion.

In this investigation, we consider nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion (weighted-fBm). To the best of our knowledge, the problem of nonparametric estimation of trend function for stochastic differential equations driven by a weighted fractional Brownian motion (weighted-fBm) has not been attempted in the literature.

The rest of the paper is organized as follows. In Section 2, the basic properties of weighted fractional Brownian motion are stated. Section 3 is devoted to the preliminaries. Then, in Section 4, we give the main results; under some hypotheses, we establish the consistent uniform (Theorem 1), the rate of convergence (Theorem 2) as well as the asymptotic normality (Theorem 3) of the estimator. Further, Section 5 is devoted to some numerical examples. Section 6 is dedicated to the technical proofs. Finally, we conclude the paper in Section 7.

2. Weighted fractional Brownian motion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a stochastic basis satisfying the habitual hypotheses, i.e., a filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous and $\mathcal{F}_0$ contains every $\mathbb{P}$-null set. Let $\{B_t^{a,b}, t \geq 0\}$ be a weighted fractional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with parameters (a, b) such that $a > -1$, $\mid b \mid< 1$, and $\mid b \mid< a+1$ is a centered and self-similar Gaussian process with long/short-range dependence and the covariance function:

$$R^{a,b}(t,s) = \mathbb{E}\left(B_t^{a,b} B_s^{a,b}\right) = \int_0^{s\wedge t} u^a \left[(t-u)^b + (s-u)^b\right] du, \quad s,\, t \geq 0.$$

Clearly, for $a = b = 0$, $B^{a,b}$ coincides with the standard Brownian motion B. When $a = 0$, we have

$$\mathbb{E}\left(B_t^{a,b} B_s^{a,b}\right) = \frac{1}{b+1}\left[t^{b+1} + s^{b+1} - |s-t|^{b+1}\right],$$

which is the covariance function of the fBm with Hurst index $\dfrac{b+1}{2}$ when $|b| < 1$.

According to Bojdecki et al. (2007), Bojdecki et al. (2008), and Yan et al. (2014), the weighted-fBm has the following properties:

(1) $\mathbb{E}\left(B_t^{a,b}\right) = 0$ and $Var\left(B_t^{a,b}\right) = 2\int_0^t u^a (t-u)^b du.$

(2) $B_t^{a,b}$ is said to be self-similar with index $\dfrac{a+b+1}{2}$, that is, for every constant $\alpha > 0$,

$$\left\{B_{\alpha t}^{a,b},\, t \geq 0\right\} \triangleq \left\{\alpha^{\frac{a+b+1}{2}} B_t^{a,b},\, t \geq 0\right\}, \text{ for each } a > 0 \tag{1}$$

in the sense that the processes, on both the sides of the equality sign, have the same finite dimensional distributions.

(3) The process $B_t^{a,b}$ is not Markov and it is not a semi-martingale if $b \neq 0$.

(4) The process $B_t^{a,b}$ has independent increments for $b = 0$.

(5) The trajectories of the process $B^{a,b}$ are Hölder continuous of order δ for any $\delta < \dfrac{1}{2}(b+1)$.

(6) The trajectories of the process $B^{a,b}$ are continuous with the only exception of the case $a < 0$, $b < 0$, $a + b = -1$, where $B^{a,b}$ is discontinuous at 0.

(7) The weighted-fBm $B_t^{a,b}$ has not stationary increments in general, we have

$$c_{a,b}(t \vee s)^a |t-s|^{b+1} \leq \mathbb{E}\left[B_t^{a,b} - B_s^{a,b}\right]^2 \leq C_{a,b}(t \vee s)^a |t-s|^{b+1}.$$

(8) $B_t^{a,b}$ is long-range dependent for $b > 0$ and short-range dependent for $b < 0$.

The stochastic calculus with respect to the weighted-fBm has been recently developed by Kruk et al. (2007). More works on weighted-fBm can be found in Nualart (2006) and Alós et al. (2012).

Fix a time interval $[0, T]$. We denote by $\mathcal{E}$ the set of step function on $[0, T]$. Let $\mathcal{H}_{a,b}$ be the Hilbert space defined as the completion of the linear space $\mathcal{E}$ generated by the indicator functions $\left\{\mathbf{1}_{[0,t]};\ 0 \leq t \leq T\right\}$, with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}_{a,b}} = R^{a,b}(t, s),$$

where $R^{a,b}(t, s)$ is the covariance of $B_t^{a,b}$ and $B_s^{a,b}$. The application

$$\varphi \in \mathcal{E} \longrightarrow B^{a,b}(\varphi) = \int_0^T \varphi(u) dB_u^{a,b},$$

is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B^{a,b}$ and it can be extended to $\mathcal{H}_{a,b}$. Then, $B^{a,b}(\varphi)$ is a Gaussian process on $\mathcal{H}_{a,b}$ such that, for all $\varphi, \psi \in \mathcal{H}_{a,b}$ we have

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{a,b}} = \mathbb{E}\left(B^{a,b}(\varphi) B^{a,b}(\psi)\right) = \int_0^T \int_0^T \varphi(u)\psi(v)\phi_{a,b}(u, v) du dv, \tag{2}$$

with

$$\phi_{a,b}(u, v) := \frac{\partial^2 R^{a,b}(u, v)}{\partial u \partial v} = b(u \wedge v)^a (u \vee v - u \wedge v)^{b-1}.$$

When $0 < b < 1$, the Hilbert space $\mathcal{H}_{a,b}$ can be written as

$$\mathcal{H}_{a,b} = \left\{ \varphi : [0, T] \longrightarrow \mathbb{R};\ \|\varphi\|_{\mathcal{H}_{a,b}} < \infty \right\},$$

where

$$\|\varphi\|_{\mathcal{H}_{a,b}}^2 = \int_0^T \int_0^T \varphi(u)\varphi(v)\phi_{a,b}(u, v) du dv. \tag{3}$$

We can use the subspace $|\mathcal{H}_{a,b}|$ of $\mathcal{H}_{a,b}$ that is defined as the set of measurable function φ on $[0, T]$ such that

$$\|\varphi\|_{|\mathcal{H}_{a,b}|}^2 = \int_0^T \int_0^T |\varphi(u)|\, |\varphi(v)|\, \phi_{a,b}(u, v) du dv < \infty. \tag{4}$$

It was shown that $|\mathcal{H}_{a,b}|$ is a Banach space with the norm $\|\varphi\|_{|\mathcal{H}_{a,b}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}_{a,b}|$. Assume that $a > -1,\ 0 < b < 1,\ b < a + 1$ and $a + b > 0$. Then, the canonical Hilbert space $\mathcal{H}_{a,b}$ associated to $B^{a,b}$ satisfies

$$\mathbb{L}^2([0,T]) \subset \mathbb{L}^{\frac{2}{a+b+1}}([0,T]) \subset |\mathcal{H}_{a,b}| \subset \mathcal{H}_{a,b}. \tag{5}$$

3. Preliminaries

Let $\{X_t, 0 \le t \le T\}$ be a process governed by the following equation:

$$dX_t = S(X_t)dt + \varepsilon dB_t^{a,b},\ X_0 = x_0,\ 0 \le t \le T, \tag{6}$$

where $\varepsilon > 0$, $B_t^{a,b}$ a weighted-fBm defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, and $S(.)$ is an unknown function. We suppose that x_t is a solution of the equation

$$\frac{dx_t}{dt} = S(x_t),\ x_0,\ 0 \le t \le T. \tag{7}$$

We suppose also that the function $S : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(**A1**) There exists $L > 0$ such that

$$|S(x) - S(y)| \le L\,|x - y|\,,\ 0 \le t \le T. \tag{8}$$

(**A2**) There exists $M > 0$ such that

$$|S(x)| \le M(1 + |x|),\ x \in \mathbb{R},\ 0 \le t \le T.$$

Then, the stochastic differential equation (6) has a unique solution $\{X_t, 0 \le t \le T\}$.

(**A3**) Assume that the function $S(x)$ is bounded by a constant C. Since the function x_t satisfies (7), it follows that

$$|S(x_t) - S(x_s)| \le L|x_t - x_s| = L\left|\int_s^t S(x_r)dr\right| \le LC|t - s|,\ t, s \in \mathbb{R}.$$

Let us define $\Sigma_0(L)$ as the class of all functions $S(x)$ satisfying the assumption (A1) and uniformly bounded by the same constant C. Let us also denote by $\Sigma_k(L)$ the class of all function $S(x)$ which are uniformly bounded by same constant C and which are k-times differentiable with respect to x satisfying the condition

$$\left|S^k(x) - S^k(y)\right| \le L\,|x - y|\,,\ x, y \in \mathbb{R}, \tag{9}$$

where $S^k(x)$ denote the k-th derivative of $S(x)$.

Lemma 3.1.

Assume that hypothesis $(A1)$ is verified. Let X_t and x_t be the solutions of the equations (6) and (7) respectively. Then, we have

$$|X_t - x_t| \leq e^{Lt}\varepsilon|B_t^{a,b}|, \tag{10}$$

and

$$\sup_{0\leq t\leq T} \mathbb{E}\left(X_t - x_t\right)^2 \leq e^{2LT}\varepsilon^2 2\int_0^T u^a\left(T-u\right)^b du. \tag{11}$$

Proof:

Proof of (10).

By (6) and (7), we have

$$X_t - x_t = \int_0^t \left(S(X_r) - S(x_r)\right) dr + \varepsilon B_t^{a,b}.$$

Thus,

$$|X_t - x_t| \ \leq L\int_0^t |X_r - x_r|\, dr + \varepsilon|B_t^{a,b}|.$$

Putting $u_t = |X_t - x_t|$, we have

$$u_t \leq \int_0^t u_r dr + \varepsilon|B_t^{a,b}|.$$

Finally, by using Gronwalls inequality, we obtain

$$|X_t - x_t| \leq e^{Lt}\varepsilon\left|B_t^{a,b}\right|. \tag{12}$$

Proof of (11).

From (12), we have

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt}\varepsilon^2\mathbb{E}\left|B_t^{a,b}\right|^2.$$

Then,

$$\mathbb{E}|X_t - x_t|^2 \leq e^{2Lt}\varepsilon^2 2\int_0^t u^a\,(t-u)^b\,du.$$

Finally, we obtain

$$\sup_{0\leq t\leq T}\mathbb{E}\,(X_t - x_t)^2 < e^{2LT}\varepsilon^2 2\int_0^T u^a\,(T-u)^b\,du.$$

■

4. Main results

The main goal of this work is to build a estimator of the trend function S_t in the model described by stochastic differential equation (6) using the method developed by Kutoyants (1994). Then, we study of asymptotic properties of the estimator as $\varepsilon \longrightarrow 0$.

For all $t \in [0, T]$, the kernel estimator $\hat{S}_t$ of S_t is given by

$$\hat{S}_t = \frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)dX_\tau, \tag{13}$$

where $G(u)$ is a bounded function with finite support $[A, B]$ satisfying the following hypotheses

(H1) $G(u) = 0$ for $u < A$ and $u > B$, and $\displaystyle\int_A^B G(u)du = 1$,

(H2) $\displaystyle\int_{-\infty}^{+\infty} G^2(u)du < \infty$,

(H3) $\displaystyle\int_{-\infty}^{+\infty} u^{2(k+1)}G^2(u)du < \infty$,

(H4) $\displaystyle\int_{-\infty}^{+\infty} |G(u)|^{\frac{2}{a+b+1}}\,du < \infty$.

Further, we suppose that the normalizing function ϕ_ε satisfies:

(H5) $\phi_\varepsilon \longrightarrow 0$ and $\varepsilon^2\phi_\varepsilon^{-1} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

The following Theorem gives uniform convergence of the estimator $\hat{S}_t$.

Theorem 4.1.

Suppose that the assumptions (A1)-(A3) and (H1)-(H5) hold true. Further, suppose that the trend function $S(x)$ belongs to $\Sigma_0(L)$. Then, for any $0 < c \leq d < T,\ a > -1,\ 0 < b < 1,\ b < a+1,$ and $a + b > 0$, the estimator $\hat{S}_t$ is uniformly consistent, that is,

$$\lim_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_0(L)} \sup_{c\leq t\leq d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2) = 0. \tag{14}$$

The following additional assumptions are useful for the rest of the theoretical study. Assume that

(H6) $\displaystyle\int_{-\infty}^{+\infty} u^j G(u)du = 0$ for $j = 1, 2, ..., k,$

(H7) $\displaystyle\int_{-\infty}^{+\infty} u^{k+1} G(u)du < \infty;$ and $\displaystyle\int_{-\infty}^{+\infty} u^{2(k+2)} G^2(u)du < \infty.$

The rate of convergence of the estimator $\hat{S}_t$ is established in the following Theorem.

Theorem 4.2.

Suppose that the function $S(x) \in \Sigma_k(L),\ a > -1,\ 0 < b < 1,\ b < a+1,\ a+b > 0$, and $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$. Then, under the hypotheses (A1)-(A3) and (H1)-(H7), we have

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\leq t\leq d} \mathbb{E}_S(|\hat{S}_t - S(x_t)|^2)\varepsilon^{\frac{-4(k+1)}{2k-a-b+3}} < \infty. \tag{15}$$

Finally, the following Theorem presents the asymptotic normality of the kernel type estimator $\hat{S}_t$ of $S(x_t)$.

Theorem 4.3.

Suppose that the function $S(x) \in \Sigma_{k+1}(L)$, $a > -1,\ 0 < b < 1,\ b < a+1,\ a+b > 0$, and $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$. Then, under the hypotheses (A1)-(A3) and (H1)-(H7), we have

$$\mathcal{L}\left\{\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right)\right\} \xrightarrow{\mathcal{D}} \mathcal{N}\left(m, \sigma^2_{a,b}\right), \quad \text{as} \ \ \varepsilon \to 0,$$

where

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G(u)u^{k+1}du,$$

and

$$\sigma^2_{a,b} = b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv.$$

5. Numerical example

In this section, we present a numerical analysis illustrating our theoretical result. We compare our kernel estimator for stochastic differential equations driven by a weighted fractional Brownian motion to the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra and Prakasa Rao (2011a)). The variances of the two kernel estimators are compared. $\sigma^2_{a,b}$ of our estimator is compared to that of the estimator σ^2_H.

Consider a function G verifying hypotheses (H1)-(H7):

$$G(t) = \frac{3}{8}\left(3 - 5t^2\right), \quad |t| \le 1.$$

1. The variance of the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in Mishra and Prakasa Rao (2011a) is given as:

For all $H \in (1/2, 1)$,

$$\sigma^2_H = H(2H-1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)\left|u-v\right|^{2H-2} dudv.$$

2. From Theorem 4.3, the variance of our estimator is as follows:

For all $a > -1,\ 0 < b < 1,\ b < a+1$ we have

$$\sigma^2_{a,b} = b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v - u\wedge v)^{b-1}dudv.$$

By developing a program in R software, we compute the numerical values of the variances. The obtained results are arranged in the following tables.

Table 1. The variance values σ^2_H

H	0.51	0.54	0.57	0.6	0.63	0.66	0.69	0.72	0.75	0.78	0.81	0.84
σ^2_H	0.0911	0.3240	0.5056	0.6461	0.7536	0.8351	0.8958	0.9403	0.9720	0.9935	1,0070	1.0143

Table 2. The variance values $\sigma^2_{0,b}$

b	0.02	0.08	0.14	0.2	0.26	0.32	0.38	0.44	0.5	0.56	0.62	0.68
$\sigma^2_{0,b}$	0.0911	0.3240	0.5056	0.6461	0.7536	0.8351	0.8958	0.9403	0.9720	0.9935	1.0070	1.0143

According to Tables 1-3 and Figure 1, we have

- For $a = 0$, the variance values $\sigma^2_{a,b}$ are equal to the variance values σ^2_H. This is due to the property of weighted fractional Brownian motion (wfBm); when a = 0 this later coincides with

Table 3. The variance values $\sigma^2_{a,b}$

$a \setminus b$	0.51	0.54	0.57	0.6	0.63	0.66	0.69	0.72	0.75	0.78	0.81	0.84
0.2	0.7990	0.7907	0.7810	0.7702	0.7584	0.7459	0.7329	0.7195	0.7058	0.6919	0.6778	0.6637
0.3	0.5643	0.5569	0.5486	0.5395	0.5298	0.5196	0.5091	0.4983	0.4874	0.4764	0.4654	0.4543
0.4	0.3549	0.3484	0.3413	0.3338	0.3260	0.3179	0.3097	0.3014	0.2931	0.2848	0.2766	0.2684
0.5	0.1807	0.1751	0.1693	0.1633	0.1573	0.1512	0.1450	0.1390	0.1330	0.1271	0.1213	0.1157
0.6	0.0470	0.0424	0.0378	0.0332	0.0287	0.0243	0.0201	0.0160	0.0120	0.0082	0.0046	0.0012

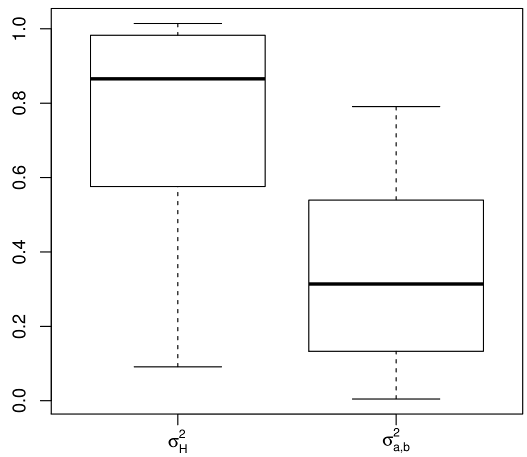

Figure 1. σ^2_H and $\sigma^2_{a,b}$, for $a \neq 0$

the fractional Brownian motion (fBm) with Hurst index $H = (b+1)/2$.

- As expected, for a fixed a (resp. b), the variance values of our estimator $\sigma^2_{a,b}$ decreases along the increasing of b (resp. a). While, the numerical values of the variance σ^2_H increases with the increases of the Hurst index H.

- For larger values of a, the values of $\sigma^2_{a,b}$ are smaller than σ^2_H, for instance, for $a = 0.5$ (resp. $a = 0.4$) the values of $\sigma^2_{a,b}$ become smaller than that of σ^2_H when $b \geq 0.54$ (resp. $b \geq 0.57$). While for $a = 0.6$, and any values of b, the variance values of our estimator are smaller than that of (Mishra and Prakasa Rao (2011a)). However, from Figure 1, we clearly observe that the variance values of our estimator is less than that of the kernel estimator for stochastic differential equations driven by fractional Brownian motion given in (Mishra and Prakasa Rao (2011a)). Thus, we can say that our kernel estimator for trend function for stochastic differential equations driven by a weighted fractional Brownian motion is better than that given in (Mishra and Prakasa Rao (2011a)).

6. Proof of Theorems

6.1. Proof of Theorem 4.1

From (6) and (13), we can see that

$$\begin{aligned}\mathbb{E}_S\left[\hat{S}_t - S(x_t)\right]^2 &\leq 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)(S(X_\tau) - S(x_\tau))\,d\tau\right]^2 \\ &\quad + 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\ &\quad + 3\mathbb{E}_S\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right]^2 \\ &\leq I_1 + I_2 + I_3.\end{aligned} \tag{16}$$

• **Concerning** I_1. Via the inequalities (8) and (11), and hypotheses (H1)-(H2), we get

$$\begin{aligned}I_1 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)(S(X_\tau) - S(x_\tau))\,d\tau\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right)du\right]^2 \\ &\leq 3(B-A)L^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G^2(u)\left(X_{t+\phi_\varepsilon u} - x_{t+\phi_\varepsilon u}\right)^2 du\right] \\ &\leq 3(B-A)L^2\left[\int_{-\infty}^{+\infty} G^2(u)\sup_{0\leq t+\phi_\varepsilon u\leq T}\mathbb{E}_S\left(X_{t+\phi_\varepsilon u} - x_{t+\phi_\varepsilon u}\right)^2 du\right] \\ &\leq 3(B-A)L^2 e^{2LT}\varepsilon^2 2\int_0^T u^a (T-u)^b\,du \\ &\leq C_1\varepsilon^2,\end{aligned} \tag{17}$$

where C_1 is a positive constant depending on T, L, a, b and $(B-A)$.

• **Concerning** I_2. Let

$$\begin{aligned}I_2 &= 3\mathbb{E}_S\left[\frac{1}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon}\right)S(x_\tau)d\tau - S(x_t)\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\,S(x_{t+\phi_\varepsilon u})du - S(x_t)\right]^2 \\ &= 3\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G(u)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right)du\right]^2.\end{aligned}$$

Next, by using hypotheses (A3), (H1), and (H3), we have

$$
\begin{aligned}
I_2 &\leq 3L^2C^2\mathbb{E}_S\left[\int_{-\infty}^{+\infty} G\left(u\right)\left(\phi_\varepsilon u\right)du\right]^2 \\
&\leq 3\left(B-A\right)L^2C^2\left[\int_{-\infty}^{+\infty} G^2\left(u\right)u^2du\right]\phi_\varepsilon^2 \\
&\leq C_2\phi_\varepsilon^2,
\end{aligned}
\tag{18}
$$

where C_2 is a positive constant depending on L and $(B-A)$.

• **Concerning** I_3. Since $a>-1,\ 0<b<1,\ b<a+1,$ and $0<a+b<1$, we have

$$
\begin{aligned}
I_3 &= 3\mathbb{E}_S\left[\frac{\varepsilon}{\phi_\varepsilon}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right]^2 \\
&= 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\mathbb{E}_S\left[\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right]^2 \\
&\leq 3\frac{\varepsilon^2}{\phi_\varepsilon^2}\left[C(a,b)\left(\int_0^T\left|G\left(\frac{\tau-t}{\phi_\varepsilon}\right)\right|^{\frac{2}{a+b+1}}d\tau\right)^{a+b+1}\right] \\
&= 3C(a,b)\frac{\varepsilon^2}{\phi_\varepsilon^2}\left[\phi_\varepsilon^{a+b+1}\left(\int_{-\infty}^{+\infty}\left|G\left(u\right)\right|^{\frac{2}{a+b+1}}du\right)^{a+b+1}\right] \\
&\leq C_3\frac{\varepsilon^2}{\phi_\varepsilon}\phi_\varepsilon^{a+b}\ \text{(using hypothesis (H4))},
\end{aligned}
\tag{19}
$$

where C_3 is a positive constant depending on a and b.

Combining (16), (17), (18), and (19), we have

$$
\sup_{S(x)\in\Sigma_0(L)}\sup_{c\leq t\leq d}\mathbb{E}_S\left[\hat{S}_t-S(x_t)\right]^2\leq C_4\left(\varepsilon^2+\phi_\varepsilon^2+\frac{\varepsilon^2}{\phi_\varepsilon}\phi_\varepsilon^{a+b}\right).
$$

Finally, under the assumption (H5), we obtain

$$
\lim_{\varepsilon\longrightarrow 0}\sup_{S(x)\in\Sigma_0(L)}\sup_{c\leq t\leq d}\mathbb{E}_S\left[\hat{S}_t-S(x_t)\right]^2=0.
$$

6.2. Proof of Theorem 4.2

Using the Taylor formula, we easily get

$$
S\left(x_{t+\phi_\varepsilon u}\right)=S\left(x_t\right)+\sum_{j=1}^{k}S^j\left(x_t\right)\frac{\left(\phi_\varepsilon u\right)^j}{j!}+\left(S^k(x_{t+\lambda(\phi_\varepsilon u)})-S^k(x_t)\right)\frac{\left(\phi_\varepsilon u\right)^k}{k!},\ \lambda\in(0,1).
$$

Then, by substituting this expression in I_2, under the condition $S(x) \in \Sigma_{k+1}(L)$, and the assumptions (H6)-(H7), we obtain

$$\begin{aligned}
I_2 &= 3\mathbb{E}_S \left[\frac{1}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon} \right) S(x_\tau) d\tau - S(x_t) \right]^2 \\
&= 3\mathbb{E}_S \left[\int_{-\infty}^{+\infty} G(u) \left(S(x_{t+\phi_\varepsilon u}) - S(x_t) \right) du \right]^2 \\
&= 3\mathbb{E}_S \left[\frac{\phi_\varepsilon^k}{k!} \int_{-\infty}^{+\infty} G(u)\, u^k \left(S^k(x_{t+\lambda(\phi_\varepsilon u)}) - S^k(x_t) \right) du \right]^2 \\
&\leq 3C^2 L^2 \left[\frac{\phi_\varepsilon^{k+1}}{k!} \int_{-\infty}^{+\infty} G(u)\, u^{k+1} du \right]^2 \\
&\leq 3C^2 L^2 (B-A) \frac{\phi_\varepsilon^{2(k+1)}}{(k!)^2} \left[\int_{-\infty}^{+\infty} G^2(u)\, u^{2(k+1)} du \right] \\
&\leq C_5 \phi_\varepsilon^{2(k+1)},
\end{aligned} \tag{20}$$

where C_5 is a positive constant depending on L and $(B-A)$.

Next, from (17), (19), and (20), we find

$$\sup_{S(x)\in\Sigma_k(L)} \sup_{c\leq t\leq d} \mathbb{E}_S \left(\hat{S}_t - S(x_t) \right)^2 \leq C_6 \left(\varepsilon^2 \phi_\varepsilon^{a+b-1} + \phi_\varepsilon^{2(k+1)} + \varepsilon^2 \right).$$

Putting $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$, it yields

$$\limsup_{\varepsilon\longrightarrow 0} \sup_{S(x)\in\Sigma_k(L)} \sup_{c\leq t\leq d} \mathbb{E}_S \left(\hat{S}_t - S(x_t) \right)^2 \varepsilon^{\frac{-4(k+1)}{2k-a-b+3}} < \infty.$$

This completes the proof of Theorem (4.2).

6.3. Proof of Theorem 4.3

From (6) and (13), we can see that

$$\begin{aligned}
\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \left(\hat{S}_t - S(x_t) \right) &= \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \left[\int_{-\infty}^{+\infty} G(u) \left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u}) \right) du \right. \\
&\quad \left. + \int_{-\infty}^{+\infty} G(u) \left(S(x_{t+\phi_\varepsilon u}) - S(x_t) \right) du + \frac{\varepsilon}{\phi_\varepsilon} \int_0^T G\left(\frac{\tau - t}{\phi_\varepsilon} \right) dB_\tau^{a,b} \right].
\end{aligned}$$

Thus,

$$\varepsilon^{\frac{-2(k+1)}{2k-a-b+3}}\left(\hat{S}_t - S(x_t)\right) = r_1(t) + r_2(t) + \eta_\varepsilon(t).$$

Hence, by the Slutsky's Theorem, it suffices to show the following three claims:

$$r_1(t) \xrightarrow{\mathbb{P}} 0, \ as\ \varepsilon \to 0, \tag{21}$$

$$r_2(t) \xrightarrow{\mathbb{P}} m, \ as\ \varepsilon \to 0, \tag{22}$$

and

$$\eta_\varepsilon(t) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{a,b}\right), \ as\ \varepsilon \to 0. \tag{23}$$

Proof of (21).

Let

$$r_1(t) = \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \int_{-\infty}^{+\infty} G\left(u\right)\left(S(X_{t+\phi_\varepsilon u}) - S(x_{t+\phi_\varepsilon u})\right) du.$$

By applying the inequality (17), we have

$$0 \le \mathbb{E}\left[r_1^2(t)\right] \le C_1 \varepsilon^{\frac{2(1-a-b)}{2k-a-b+3}} \longrightarrow 0.$$

Then,

$$r_1(t) \xrightarrow{\mathbb{P}} 0, \ as\ \varepsilon \to 0.$$

Proof of (22).

Let

$$r_2(t) = \varepsilon^{\frac{-2(k+1)}{2k-a-b+3}} \int_{-\infty}^{+\infty} G\left(u\right)\left(S(x_{t+\phi_\varepsilon u}) - S(x_t)\right) du.$$

Take any $t, u \in [0, T]$, via the Taylor expansion

$$S\left(x_{t+\phi_\varepsilon u}\right) = S\left(x_t\right) + \sum_{j=1}^{k+1} S^j\left(x_t\right)\frac{\left(\phi_\varepsilon u\right)^j}{j!} + \left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{\left(\phi_\varepsilon u\right)^{k+1}}{(k+1)!}, \ \lambda \in (0,1),$$

using the conditions (H6), (H7), and choosing $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$, we get

$$\begin{aligned}
\mathbb{E}\left[r_2(t)-m\right]^2 &= \mathbb{E}\left[\int_{-\infty}^{+\infty} G\left(u\right)\left(S^{k+1}(x_{t+\lambda(\phi_\varepsilon u)}) - S^{k+1}(x_t)\right)\frac{u^{k+1}}{(k+1)!}du\right]^2 \\
&\leq L^2C^2\left(\int_{-\infty}^{+\infty} G(u)u^{k+2}\frac{\phi_\varepsilon}{(k+1)!}du\right)^2 \\
&\leq L^2C^2(B-A)[(k+1)!]^{-2}\left(\int_{-\infty}^{+\infty} G^2(u)u^{2(k+2)}du\right)\phi_\varepsilon^2 \\
&\leq C_7\phi_\varepsilon^2,
\end{aligned}$$

where C_7 is a positive constant depending on L, $B-A$ and k; and

$$m = \frac{S^{k+1}(x_t)}{(k+1)!}\int_{-\infty}^{+\infty} G\left(u\right)u^{k+1}du.$$

Therefore,

$$\mathbb{E}\left[r_2(t)-m\right]^2 \longrightarrow 0 \ \text{ as } \ \varepsilon \longrightarrow 0.$$

Then,

$$r_2(t) \xrightarrow{\mathbb{P}} m, \ as\ \varepsilon \rightarrow 0.$$

Proof of (23).

Let

$$\eta_\varepsilon(t) = \varepsilon^{\frac{-(a+b-1)}{2k-a-b+3}}\phi_\varepsilon^{-1}\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}. \tag{24}$$

In fact, we have to evaluate the variance of (24). To this end, let

$$\mathbb{E}\left(\eta_\varepsilon(t)\right)^2 = \varepsilon^{\frac{-2(a+b-1)}{2k-a-b+3}}\phi_\varepsilon^{-2}\mathbb{E}\left(\int_0^T G\left(\frac{\tau-t}{\phi_\varepsilon}\right)dB_\tau^{a,b}\right)^2.$$

Moreover, using (2), $a>-1,\ 0<b<1,\ b<a+1,$ and $a+b>0$, we have

$$\mathbb{E}\left(\eta_\varepsilon(t)\right)^2 = \varepsilon^{\frac{-2(a+b-1)}{2k-a-b+3}}\phi_\varepsilon^{a+b-1}\left[b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(u)G(v)(u\wedge v)^a(u\vee v-u\wedge v)^{b-1}dudv\right].$$

Then, by taking $\phi_\varepsilon = \varepsilon^{\frac{2}{2k-a-b+3}}$, we get

$$\mathbb{E}\left(\eta_{\varepsilon}(t)\right)^{2}=b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}G(u)G(v)(u\wedge v)^{a}(u\vee v-u\wedge v)^{b-1}dudv.$$

Finally, this last equation allows us to achieve the proof of Theorem (4.3).

7. Conclusion

In this paper, we studied the problem of nonparametric estimation of the trend function for stochastic differential equations driven by a weighted fractional Brownian motion. We presented the kernel estimator of the trend function based on the continuous observation and obtained the uniform convergence (with rate) and the asymptotic normality of the proposed estimator. For further work, it will be interesting to investigate the problem of estimating the trend function for SDEs driven by a generalized fractional Brownian motion such as, sub-fractional Brownian motion, mixed sub-fractional Brownian motion, and Lévy fractional Brownian motion.

Acknowledgement:

The authors are garteful to anonymous referees and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for valuable comments and suggestions towards the improvement of this paper.

REFERENCES

Alós, E., Mazet, O. and Nualart, D. (2001). Stochastic calculus with respect to Gaussian processes, The Annals of Probability, Vol. 29, No. 2, pp. 766-801.

Biagini, F., Hu, Y., Øksendal, B. and Zhang, T. (2008). *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer Science & Business Media.

Bojdecki, T., Gorostiza, L. and Talarczyk, A. (2007). Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems, Electronic Communications in Probability, Vol. 12, pp. 161-172.

Bojdecki, T., Gorostiza, L. G. and Talarczyk, A. (2008). Occupation time limits of inhomogeneous Poisson systems of independent particles, Stochastic Processes and their Applications, Vol. 118, No. 1, pp. 28-52.

Gradinaru, M., Nourdin, I., Russo, F. and Vallois, P. (2005). m-order integrals and generalized Itô's formula; the case of a fractional Brownian motion with any Hurst index, In Annales de l'Institut Henri Poincare (B) Probability and Statistics, (Vol. 41, No. 4, pp. 781-806). No longer published by Elsevier.

Guangjun, S. H. E. N., Xiuwei, Y. I. N. and Litan, Y. A. N. (2016), Least squares estimation for Ornstein-Uhlenbeck processes driven by the weighted fractional Brownian motion. Acta Mathematica Scientia, Vol. 36, No. 2, pp. 394-408.

Hu, Y. (2005). *Integral Transformations and Anticipative Calculus for Fractional Brownian Motions*, American Mathematical Soc.

Kruk, I., Russo, F. and Tudor, C. A. (2007). Wiener integrals, Malliavin calculus and covariance measure structure, Journal of Functional Analysis, Vol. 249, No. 1, pp. 92-142.

Kutoyants, Y. A. (1994). *Identification of Dynamical Systems with Small Noise* (Vol. 300). Springer Science & Business Media.

Mishra, M. N. and Prakasa Rao, B. L. (2011a). Nonparametric estimation of trend for stochastic differential equations driven by fractional Brownian motion, Statistical inference for stochastic processes, Vol. 14, No. 2, pp. 101-109.

Mishra, M. N. and Prakasa Rao, B. L. S. (2011b). Nonparametric estimation of linear multiplier for fractional diffusion processes, Stochastic Analysis and Applications, Vol. 29, No. 4, pp. 706-712.

Mishura, I. S. (2008). *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Vol. 1929. Springer Science & Business Media.

Norros, I., Valkeila, E. and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, Bernoulli, Vol. 5, No. 4, pp. 571-587.

Nualart. D, 2006. *The Malliavin Calculus and Related Topics*, Berlin: Springer, 1995 (2006).

Prakasa Rao, B. L. S. (2019). Nonparametric estimation of trend for stochastic differential equations driven by mixed fractional Brownian motion, Stochastic Analysis and Applications, Vol. 37, No. 2, pp. 271-280.

Saussereau, B. (2014). Nonparametric inference for fractional diffusion. Bernoulli, Vol. 20, No. 2, pp. 878-918.

Shen, G., Yan, L. and Cui, J. (2013). Berry-Esséen bounds and almost sure CLT for quadratic variation of weighted fractional Brownian motion, Journal of Inequalities and Applications, Vol. 2013, No. 1, 275.

Sun, X., Yan, L. and Zhang, Q. (2017). The quadratic covariation for a weighted fractional Brownian motion, Stochastics and Dynamics, Vol. 17, No. 4, 1750029.

Yan, L., Wang, Z. and Jing, H. (2014). Some path properties of weighted-fractional Brownian motion, Stochastics An International Journal of Probability and Stochastic Processes, Vol. 86, No. 5, pp. 721-758.