Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 1 (June 2020), pp. 655 – 665

Numerical Simulation for Solving Fractional Riccati and Logistic Differential Equations as a Difference Equation

[1*]M. M. Khader, [2]N. H. Sweilam and [3]B. N. Kharrat

[1]Department of Mathematics and Statistics
College of Science
Imam Mohammad Ibn Saud Islamic
University (IMSIU)
Riyadh, Saudi Arabia

[1]Department of Mathematics
Faculty of Science
Benha University
Benha, Egypt

[2]Department of Mathematics
Faculty of Science
Cairo University
Giza, Egypt

[3]Department of Mathematics
Faculty of Science
Aleppo University
Aleppo, Syria

[1]mohamed.khader@fsc.bu.edu.eg; [2]nsweilam@sci.cu.edu.eg; [3]bachirkha@alepuniv.edu.sy

*Corresponding Author

Received: October 9, 2019; Accepted: January 30, 2020

Abstract

In this paper, we introduce a numerical treatment using the generalized Euler method (GEM) for the fractional (Caputo sense) Riccati and Logistic differential equations. In the proposed method, we invert the given model as a difference equation. We compare our numerical solutions with the exact solution and with those numerical solutions using the fourth-order Runge-Kutta method (RK4). The obtained numerical results of the two proposed problem models show the simplicity and efficiency of the proposed method.

Keywords: Fractional Riccati differential equation; Fractional logistic differential equation; Caputo fractional derivative; Generalized Euler method

MSC 2010 No.: 41A30, 465N20

1. Introduction

Fractional differential equations (FDEs) have recently been applied in various areas of engineering, science, finance, applied mathematics, bio-engineering, and others. However, many researchers remain unaware of this field. FDEs have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics, and engineering (Ajou et al. (2019), Sharma et al. (2019)). Consequently, considerable attention has been given to the solutions of FDEs of physical interest (Abro et al. (2019), Saad et al. (2019)). Most FDEs do not have exact solutions, so approximate and numerical techniques (Gòmez et al. (2016), Zaid and Momani (2008)) must be used. Recently, several numerical methods to solve fractional differential equations have been given, such as variational iteration method (Sweilam et al. (2007)), homotopy analysis method (Saad et al. (2017), Sweilam and Khader (2011)) and collocation method (Khader and Babatin (2013), Sweilam et al. (2012)).

The Riccati differential equation (RDE) is named after the Italian Nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid (Reid (1972)) contains the fundamental theories of Riccati equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that are considered classical today, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics (Lasiecka and Triggiani (1991)). The solution for this equation can be reached using classical numerical methods such as the forward Euler method and the Runge-Kutta method. Bahnasawi et al. (2004) presented the usage of the Adomian decomposition method to solve the non-linear RDE in an analytic form. Tan and Abbasbandy (2008) employed the analytic technique called the homotopy analysis method to solve the quadratic RDE.

The Logistic model can be obtained by applying the derivative operator on the Logistic equation. The model is initially published in 1838 (Cushing (1998)). The continuous logistic model is described by first-order ODE. The discrete logistic model is a simple iterative equation that reveals the chaotic property in certain regions (Alligood et al. (1996)). There are many variations in the population modeling (Ausloos (2006)). The Verhulst model is the classical example to illustrate the periodic doubling and chaotic behavior in dynamical system. The model that described the population growth may be limited by certain factors like population density (Ausloos (2006)). Typical applications of the Logistic equation are a common model of population growth and in medicine, where the logistic differential equation is used to model the growth of tumors. This application can be considered as an extension to the above-mentioned use in the framework of ecology. The solution for this equation to explain the constant population growth rate which doesn't include the limitation on food supply or the spread of diseases. The solution curve of the model increases exponentially from the multiplication factor up to the saturation limit which is the maximum carrying capacity (Pastijn (2006)), $\frac{dN}{dt} = \rho N(1 - \frac{N}{K})$ where N is the population with respect to time, ρ is the rate of maximum population growth and K is the carrying capacity. The solution of continuous Logistic equation is in the form of constant growth rate as in formula $N(t) = N_0 e^{\rho t}$ where N_0 is the initial population (Suansook and Paithoonwattanakij (2009)).

The organization of this paper is as follows. In the next section, generalized Taylor's formula is introduced. In Section 3, generalized Euler's formula is presented. In Section 4, a numerical simulation is given to clarify the proposed method. Finally, in Section 5, the report ends with a brief conclusion and discussion.

2. Generalized Taylor's formula

In this section, we introduce a generalization of Taylor's formula that involves Caputo fractional derivatives (Zaid and Shawagfeh (2007)). Suppose that:

$$D^{k\alpha} f(t) \in C(0, a], \quad \text{for} \quad k = 0, 1, ..., n+1, \text{ where } \quad 0 < \alpha \leq 1,$$

where the Caputo fractional derivative operator D^{ν} of order ν is defined in the following form (Oldham and Spanier (1974)):

$$D^{\nu} f(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\nu-m+1}} d\tau, \quad \nu > 0, \quad t > 0, \quad m-1 < \nu \leq m, \ m \in \mathbb{N}.$$

Then, we have:

$$f(t) = \sum_{i=0}^{n} \frac{t^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{(D^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha}, \quad 0 \leq \xi \leq t, \quad \forall\, t \in (0, a]. \quad (1)$$

In case of $\alpha = 1$, the generalized Taylor's formula (1) reduces to the classical Taylor's formula (Arafa et al. (2012)).

3. Generalized Euler method

Zaid and Momani derived the generalized Euler's method that we have developed for the numerical solution of initial value problems with Caputo derivatives (Zaid and Shawagfeh (2007)). The method is a generalization of the classical Euler's method. Consider the following general form of IVP:

$$D^{\alpha} y(t) = f(t, y(t)), \qquad y(0) = y_0, \quad 0 < \alpha \leq 1, \ 0 < t < a. \quad (2)$$

In the proposed method we will not find a function $y(t)$ that satisfies IVP (2) but we will find a set of points $(t_j, y(t_j))$ and use it for our approximation. For convenience, we divide the interval $[0, a]$ into n subintervals $[t_j, t_{j+1}]$ of equal width $h = a/n$ by using the nodes $t_j = jh$, for $j = 0, 1, ..., n$. Assume that $y(t), D^{\alpha} y(t)$ and $D^{2\alpha} y(t)$ are continuous on $[0, a]$ and use the generalized Taylor's formula (1) to expand $y(t)$ about $t = t_0 = 0$. For each value t there is a value c_1 so that

$$y(t) = y(t_0) + \frac{D^{\alpha} y(t_0)}{\Gamma(\alpha+1)} t^{\alpha} + \frac{D^{2\alpha} y(c_1)}{\Gamma(2\alpha+1)} t^{2\alpha}. \quad (3)$$

Now, when $D^{\alpha}y(t_0) = f(t_0, y(t_0))$ and $h = t_1$ are substituted into Equation (3), the result is an expression for $y(t_1)$,

$$y(t_1) = y(t_0) + f(t_0, y(t_0))\frac{h^{\alpha}}{\Gamma(\alpha+1)} + D^{2\alpha}y(c_1)\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}.$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving $h^{2\alpha}$) and get:

$$y(t_1) = y(t_0) + f(t_0, y(t_0))\frac{h^{\alpha}}{\Gamma(\alpha+1)}.$$

The process is repeated and generates a sequence of points that approximates the solution $y(t)$. The general formula for generalized Euler's method (GEM) when $t_{j+1} = t_j + h$ is

$$y(t_{j+1}) = y(t_j) + f(t_j, y(t_j))\frac{h^{\alpha}}{\Gamma(\alpha+1)}, \qquad j = 0, 1, ..., n-1. \tag{4}$$

It is clear that if $\alpha = 1$, then the generalized Euler's formula (4) is reduced to the classical Euler's formula (Arafa et al. (2012)).

4. Numerical simulation

In this section, we illustrate the effectiveness of the proposed formula and validate the solution scheme for solving the fractional Riccati differential equation and the fractional Logistic differential equation. To achieve this propose, we consider the following two problems.

Model 1: Fractional Riccati differential equation

Consider the following fractional Riccati differential equation:

$$D^{\alpha}u(t) + u^2(t) - 1 = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{5}$$

where α refers to the Caputo fractional derivative, we also assume an initial condition $u(0) = u^0$. The exact solution to this problem at $\alpha = 1$ and $u^0 = 0$ is:

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

Now, we solve numerically this model using the proposed method (GEM). In view of the GEM, the numerical scheme of the proposed model (5) is given in the following form:

$$u(t_{j+1}) = u(t_j) + f(t_j, u(t_j))\frac{h^{\alpha}}{\Gamma(\alpha+1)}, \tag{6}$$

where the quantity $f(t_j, u(t_j))$ is computed from the following function, at the points $t_j = jh,\ j = 0, 1, ..., n$,

$$f(t, u(t)) = 1 - u^2(t).$$

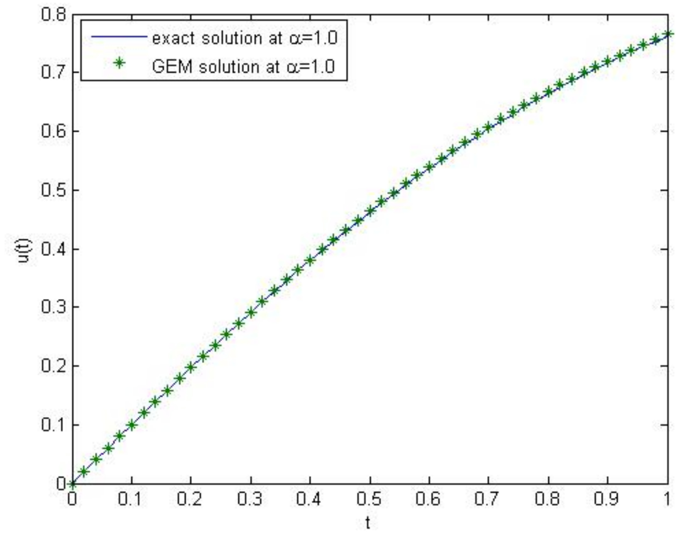

Figure 1. RDE model: A comparison between the exact solution and the numerical solution at $n = 50$.

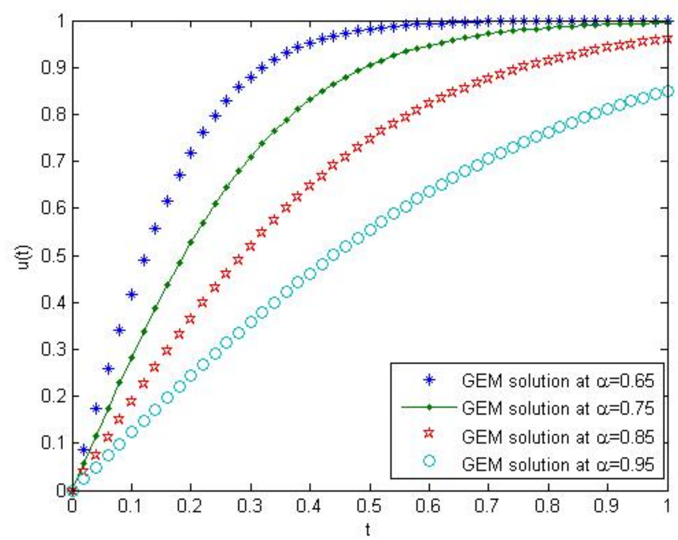

Figure 2. RDE model: The behavior of the numerical solution of FRDE with different values of α.

The numerical results of the proposed problem (5) are given in Figures 1 and 2. In Figure 1, we presented a comparison of the obtained numerical solution with the exact solution at $\alpha = 1$ in the interval $[0, 1]$ and $u^0 = 0$. From this figure, since the obtained numerical solutions are in excellent agreement with the exact solution, we can conclude that the proposed technique is well done for solving such a class of FDEs. In Figure 2, we presented the behavior of the numerical solution of the RDE with different values of α with $n = 50$. From this figure, we can see that the behavior of the obtained numerical solution follows the same behavior of the exact solution $\alpha = 1$. This conclusion ensures that the proposed method can solve the considered model effectively.

In addition, to validate our numerical solutions ($n = 60$) we make a comparison in Table 1 with the previous work of Khader (2013) by using the fractional Chebyshev finite difference method (FCheb-FDM) with distinct values of α. In this comparison, we compute the residual error function (REF) in the two methods via different values of $\alpha = 0.6, 0.8$ and 1.0. Also, we compute the allowed time $\bar{t}$ for obtaining these results by applying the two methods, where we used a computer with a processor (Intel(R) Core(TM) i5-2520M CPU-2.50GHz) and the amount of memory is 4.0GB and the code was written in MATLAB Program. This comparison shows the thoroughness of the proposed method in this article. For more details on the FCheb-FDM, see Khader (2013) and Khader (2016).

Table 1. A comparison of REF between the present method and FCheb-FDM via distinct values of α.

	Present Method–REF at:			Method (Khader (2013))-REF at:		
x	α=0.6	α=0.8	α=1.0	α=0.6	α=0.8	α=1.0
0.0	5.65214E-08	6.65214E-09	7.85214E-09	8.02134E-06	1.75120E-07	4.95122E-08
0.2	0.87541E-08	2.32541E-09	8.30214E-10	1.32014E-07	8.65421E-07	0.98541E-08
0.4	2.98542E-08	3.65210E-09	4.02145E-11	6.32145E-07	9.96521E-08	4.65201E-09
0.6	1.65487E-09	6.60040E-11	4.62140E-11	0.85017E-07	1.35004E-08	4.65214E-09
0.8	5.85582E-09	6.65217E-11	5.68520E-13	6.02541E-07	9.95200E-09	0.88241E-11
1.0	1.85214E-10	7.65410E-12	3.95124E-14	1.62541E-08	0.74120E-09	3.62104E-11
$\bar{t}$	40 sec	55 sec	50 sec	130 sec	125 sec	120 sec

Model 2: Logistic differential equation

Consider the following fractional Logistic differential equation:

$$D^{\alpha}u(t) = \rho u(t)(1 - u(t)), \quad t > 0, \;\; \rho > 0, \tag{7}$$

where α refers to the Caputo fractional derivative; we also assume an initial condition $u(0) = u^0,\; u^0 > 0$. The exact solution to this problem at $\alpha = 1$ is:

$$u(t) = \frac{u_0}{(1 - u_0)e^{-\rho t} + u_0}.$$

The existence and the uniqueness of the proposed problem (7) are introduced in details in El-Sayed et al. (2007).

Now, we solve numerically this model using the proposed method (GEM). In view of the GEM, the numerical scheme of the proposed model (7) is given in the following form:

$$u(t_{j+1}) = u(t_j) + g(t_j, u(t_j))\frac{h^{\alpha}}{\Gamma(\alpha + 1)}, \tag{8}$$

where the quantity $g(t_j, u(t_j))$ is computed from the function $g(t, u(t)) = \rho u(t)(1 - u(t))$, at the points $t_j = jh,\; j = 0, 1, ..., n$.

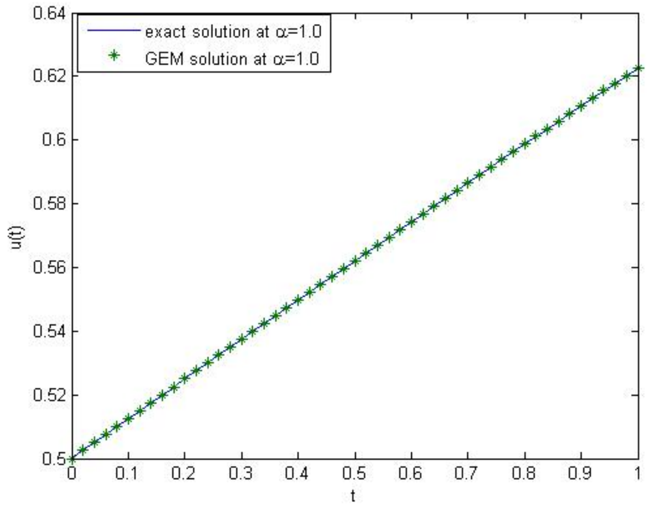

Figure 3. LDE model: A comparison between the exact solution and the numerical solution at $n = 50$.

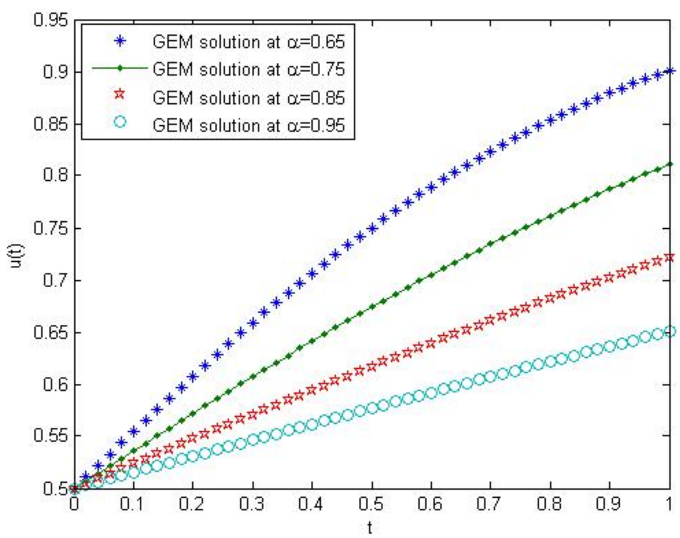

Figure 4. LDE model: The behavior of the numerical solution of FLDE with different values of α.

The numerical results of the proposed problem (7) are given in Figures 3 and 4. In Figure 3, we presented a comparison of the obtained numerical solution with the exact solution at $\alpha = 1$ in the interval $[0, 1]$ and $u^0 = 0.5,\ \rho = 0.5$. From this figure, since the obtained numerical solutions are in excellent agreement with the exact solution, so, we can conclude that the proposed technique is well done for solving such class of FDEs. In Figure 4, we presented the behavior of the numerical solution of the LDE with different values of α with $n = 50$. From this figure, we can see that the behavior of the obtained numerical solution follows the same behavior of the exact solution $\alpha = 1$. This conclusion ensures that the proposed method can be solved to the consider model effectively.

In addition, to validate our numerical solutions ($n = 60$) we make a comparison in Table 2 with the previous work of Khader (2016) by using the fractional Chebyshev finite difference method (FCheb-FDM) with distinct values of α. In this comparison, we compute the residual error function (REF) in the two methods via different values of $\alpha = 0.6, 0.8$ and 1.0. Also, we compute the allowed time $\bar{t}$ for obtaining these results by applying the two methods.

Table 2. A comparison of REF between the present method and FCheb-FDM via distinct values of α.

	Present Method–REF at:			Method (Khader (2016))-REF at:		
x	α=0.6	α=0.8	α=1.0	α=0.6	α=0.8	α=1.0
0.0	2.35785E-08	3.85214E-08	7.65412E-10	9.35752E-06	3.87520E-07	1.65201E-08
0.2	2.98521E-08	0.32541E-09	0.85214E-09	9.21450E-06	2.02541E-09	1.32145E-08
0.4	1.85210E-08	4.50040E-09	5.65210E-11	6.02145E-07	8.02145E-07	0.65420E-08
0.6	6.65420E-09	7.10654E-10	8.82410E-12	1.60215E-07	3.25414E-07	5.98541E-09
0.8	5.35241E-09	6.65214E-10	0.85214E-13	9.62541E-07	5.98720E-08	8.32541E-10
1.0	5.32145E-10	4.96521E-10	1.35204E-13	0.85214E-07	9.65412E-09	4.85210E-10
$\bar{t}$	50 sec	55 sec	60 sec	130 sec	125 sec	120 sec

5. Conclusion and discussion

This paper is devoted to implementing the generalized Euler method for studying the numerical solution for two of the well-known models, the fractional Riccati and Logistic differential equations. In this work, we are interested in studying the behavior of the numerical solution for the proposed problems for various fractional Brownian motions and also for standard motion $\alpha = 1$. In addition, we compared the obtained numerical solution with the exact solution. From this comparison, we can conclude that the obtained numerical solution using the suggested method is in excellent agreement with the exact solution and show that this approach can solve the problems effectively and illustrates the validity and the great potential of the proposed technique. All computations in this paper are done by using MATLAB 8.0. Finally, the recent appearance of FDEs as models in some fields of applied mathematics makes it necessary to investigate the analytical and numerical methods for such equations. In the future research, we will try to apply this method with different definitions of the new fractional derivative, such as the Atangana-Baleanu-Caputo operators.

Acknowledgment:

The authors are very grateful to the editor and the referees for carefully reading the paper and for their comments and suggestions, which have improved the paper.

REFERENCES

Abro, K.A., Khan, I. and Gòmez-Aguilar, J.F. (2019). Thermal effects of magnetohydro-dynamic micropolar fluid embedded in porous medium with Fourier sine transform technique, J. of the

Brazilian Society of Mechanical Sciences and Engineering, Vol. 41, No. 4, pp. 12–19.

Abro, K.A., Mirbhar, M.N. and Gòmez-Aguilar, J.F. (2019). Functional application of Fourier sine transform in radiating gas flow with non-singular and non-local kernel, Journal of the Brazilian Society of Mechanical Sciences and Engineering, Vol. 41, No. 10, pp. 1–8.

Ajou, A.E.L., Oqielat, M.N., Zhour, Z.A., Kumar, S. and Momani, S. (2019). Solitary solutions for time-fractional nonlinear dispersive PDEs in the sense of conformable fractional derivative, Chaos, Vol. 29, pp. 1–15.

Alligood, K.T., Sauer, T.D. and Yorke, J.A. (1996). *An Introduction to Dynamical Systems*, Springer, Verlag New York, Inc.

Arafa, A.A.M., Rida, S.Z. and Khalil, M. (2012). Fractional modeling dynamics of HIV and $CD4^+$ T-cells during primary infection, Nonlinear Biomedical Physics, Vol. 6, No. 1, pp. 1–7.

Ausloos, M. (2006). The Logistic map and the route to chaos, From the Beginnings to Modern Applications XVI, 411.

Bahnasawi, A.A., El-Tawil, M.A. and Abdel-Naby, A. (2004). Solving Riccati differential equation using ADM, Applied Mathematics Computation, Vol. 157, pp. 503–514.

Cushing, J.M. (1998). An Introduction to Structured Population Dynamics, Society for Industrial and Applied Mathematics, Vol. 71, pp. 1–4.

El-Sayed, A.M.A., El-Mesiry, A.E.M., and El-Saka, H.A.A. (2007). On the fractional-order Logistic equation, Applied Mathematics Letters, Vol. 20, No. 7, pp. 817–823.

Gòmez-Aguilar, J.F. (2017). Space-time fractional diffusion equation using a derivative with non-singular and regular kernel, Physica A: Statistical Mechanics and its Applications, Vol. 465, pp. 562–572.

Gòmez-Aguilar, J.F., Hernàndez, M.M., Lopez-Lopez, G. and Alvarado, V. (2016). Modeling and simulation of the fractional space-time diffusion equation, Communications in Nonlinear Science and Numerical Simulation, Vol. 30, No. 1, pp. 115–127.

Goufoa, E.F.D., Kumar, S. and Mugisha, S.B. (2020). Similarities in a fifth-order evolution equation with and with no singular kernel, Chaos, Solitons & Fractals, Vol. 130, pp. 1–7.

Lasiecka, I. and Triggiani, R. (1991). Differential and algebraic Riccati equations with application to boundary/point control problems, continuous theory and approximation theory, Lecture notes in control and information sciences, Berlin, Springer.

Khader, M.M. (2011). On the numerical solutions for the fractional diffusion equation, Communications in Nonlinear Science and Numerical Simulations, Vol. 16, pp. 2535– 2542.

Khader, M.M. (2013a). An efficient approximate method for solving linear fractional Klein- Gordon equation based on the generalized Laguerre polynomials, International Journal of Computer Mathematics, Vol. 90, No. 9, pp. 1853–1864.

Khader, M.M. (2013b). Numerical treatment for solving fractional Riccati differential equation, Journal of the Egyptian Mathematical Society, Vol. 21, pp. 32–37.

Khader, M.M. (2013c). The use of generalized Laguerre polynomials in spectral methods for fractional-order delay differential equations, J. of Computational and Nonlinear Dynamics, Vol. 8, pp. 1-5.

Khader, M.M. (2016). Numerical treatment for solving fractional Logistic differential equation, Differential Equations and Dynamical Systems, Vol. 24, No. 1, pp. 99–107.

Khader, M.M. and Babatin, M.M. (2013). Numerical treatment for solving fractional SIRC model and influenza A, Computational and Applied Mathematics, Vol. 4, pp. 1–10.

Khader, M.M. and Hendy, A.S. (2013). A numerical technique for solving fractional variational problems, Mathematical Methods in Applied Sciences, Vol. 36, No. 10, pp. 1281–1289.

Khader, M.M. and Saad, K.M. (2018). A numerical study using Chebyshev collocation method for a problem of biological invasion, pp. fractional Fisher equation, International Journal of Biomathematics, Vol. 11, No. 8, pp. 1–15.

Khader, M.M. and Sweilam, N.H. (2013). On the approximate solutions for system of fractional integro-differential equations using Chebyshev pseudo-spectral method, Applied Mathematical Modelling, Vol. 37, pp. 9819–9828.

Kumar, S., Kumar, A., Momani, S., Aldhaifalla, M. and Nisar, K.S. (2019). Numerical solutions of nonlinear fractional model arising in the appearance of the strip patterns in two-dimensional systems, Advances in Difference Equations, Vol. 413, pp. 120–130.

Oldham, K.B. and Spanier, J. (1974). *The Fractional Calculus*, Academic Press, New York.

Pastijn, H. (2006). Chaotic Growth with the Logistic Model of P.-F. Verhulst, Understanding Complex Systems, The Logistic Map and the Route to Chaos, Vol. 10, pp. 3–11.

Podlubny, I. (1999). *Fractional Differential Equations,* Academic Press, New York.

Reid, W.T. (1972). *Riccati Differential Equations Mathematics in Science and Engineering*, New York, Academic Press.

Saad, K.M. (2018). A reliable analytical algorithm for space-time fractional cubic isothermal autocatalytic chemical system, Pramana-J. Phys., Vol. 91, No. 51, pp. 20–25.

Saad, K.M., Al-Shareef, E.H., Mohamed, M.S and Xiao-Jun, Y. (2017). Optimal q-homotopy analysis method for time-space fractional gas dynamics equation, European Physical Journal Plus, Vol. 132, No. 1, pp. 1–13.

Saad, K.M., Khader, M.M., Gòmez-Aguilar, J.F. and Baleanu, D. (2019). Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, Chaos, Vol. 29, pp. 125–132.

Sharma, B., Kumar, S., Cattani, C. and Baleanu, D. (2019). Nonlinear dynamics of Cattaneo-Christov heat flux model for third-grade power-law fluid, J. Comput. Nonlinear Dynam., Vol. 3, pp. 345–354.

Suansook, Y. and Paithoonwattanakij, K. (2009). Dynamic of Logistic model at fractional order, IEEE International Symposium on Industrial Electronics, Vol. 14, pp. 15–25.

Sweilam, N.H. and Khader, M.M. (2010). A Chebyshev pseudo-spectral method for solving fractional integro-differential equations, ANZIAM, Vol. 51, pp. 464–475.

Sweilam, N.H. and Khader, M.M. (2011). Semi exact solutions for the bi-harmonic equation using homotopy analysis method, World Applied Sciences Journal, Vol. 13, pp. 1–7.

Sweilam, N.H., Khader, M.M. and Al-Bar, R.F. (2007). Numerical studies for a multi-order fractional differential equation, Physics Letters A, Vol. 371, pp. 26–33.

Sweilam, N.H., Khader, M.M. and Mahdy, A.M.S. (2012). Numerical studies for fractional-order Logistic differential equation with two different delays, Journal of Applied Mathematics, Vol. 2012, pp. 1-14.

Sweilam, N.H., Khader, M.M. and Nagy, A.M. (2011). Numerical solution of two-sided space-fractional wave equation using finite difference method, J. of Computational and Applied Mathematics, Vol. 235, pp. 2832–2841.

Tan, Y. and Abbasbandy, S. (2008). Homotopy analysis method for quadratic Riccati differential equation, Communications in Nonlinear Science and Numerical Simulations, Vol. 13, No. 3, pp. 539–546.

Zaid, M.O. and Momani, S. (2008). An algorithm for the numerical solution of differential equations of fractional order, Journal of Applied Mathematics and Informatics, Vol. 26, pp. 15–27.

Zaid, M.O. and Shawagfeh, N.T. (2007). Generalized Taylor's formula, Applied Mathematics Computation, Vol. 186, pp. 286–293.