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The Traveling Wave Solution Of the Fuzzy Linear Partial Differential Equation

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Abstract

In this paper we are going to obtain fuzzy traveling wave solutions for fuzzy linear partial differential equations by considering the type of generalized Hukuhara differentiability. In particular, the fuzzy traveling wave solutions for fuzzy Advection equation, fuzzy linear Diffusion equation, fuzzy Convection-Diffusion-Reaction equation, and fuzzy Klein-Gordon equation are obtained.

Keywords: Fuzzy traveling wave solution; Generalized Hukuhara partial differentiability; The fuzzy linear convection-diffusion-reaction; The fuzzy linear Klein-Gordon equation

MSC 2010 No.: 35R13, 35F10

1. Introduction

This paper examines the fuzzy solution of the following generic form of the second-order fuzzy linear partial differential equation under generalized Hukuhara partial differentiability,

$$u_{t_{gH}} = F\Big(u, u_{x_{gH}}, u_{xx_{gH}}, u_{tt_{gH}}\Big). \tag{1}$$

We are interested in a particular class of solutions to the fuzzy linear partial differential equations, the traveling wave fuzzy solution.

Various complex phenomena in different branches of science and engineering such as plasma physics, fluid mechanic and optical fibers can be expressed in the form of Equation (1) because most of these phenomena have some uncertainty and ambiguity in their initial measurements, and a well-known way for modeling systems with uncertainties is fuzzy set theory.

The fuzzy partial differential equations are examined by Buckley and Feuring (Buckley and Feuring (1999)). They introduced the concepts of fuzzy partial differentiability based on generalized Hukurara difference for the fuzzy multivariable functions. In recent years, many methods have been utilized for finding an analytical or numerical fuzzy solution for fuzzy partial differential equations (Allahviranloo (2002); Allahviranloo and Taheri (2009); Bertone et al. (2013); Gouyandeha et al. (2017); Moghaddam and Allahviranloo (2018)) and fuzzy differential equations (Tapaswini and Chakraverty (2013); Gholami et al. (2019)).

The traveling wave solutions of the partial differential equations can provide physical aspects of the problems; therefore, they play an essential role in applied science fields (Wazwaz (2002); Ablowitz and Clarkson (1991); Griffiths and Schiesser (2011)). In this article, we will obtain the fuzzy traveling wave solution for particular cases of Equation (1): fuzzy Advection equation, fuzzy linear Diffusion equation, fuzzy Convection-Diffusion-Reaction equation, and fuzzy Klein-Gordon equation. We will discuss the fuzzy traveling wave solution of these equations by considering the type of $gH-$differentiability.

The organization of this paper is as follows. In Section 2, some concepts associated with fuzzy numbers and generalized Hukuhara differentiability are expressed. The method of obtaining fuzzy traveling wave solutions for second-order fuzzy linear partial differential equations by considering the type of generalized Hukuhara partial differentiability is described in Section 3. Next, in Section 4 the fuzzy traveling wave solution for fuzzy Advection equation, fuzzy linear Diffusion equation, fuzzy Convection-Diffusion-Reaction equation, and fuzzy Klein-Gordon equation are obtained and the corresponding formulas are shown. Conclusions are drawn in Section 5, and finally, in the Appendix the fuzzy solution of the eigenvalue problem for the second-order fuzzy differential equation is obtained.

2. Mathematical Preliminaries

In this section, the basic definitions and the necessary notation which will be used throughout the paper are introduced. The set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which defined over the real line is denoted by $\mathbb{E}$. The $\alpha-$cut of fuzzy number A for all $0 \leq \alpha \leq 1$ is defined as follows:

$$[A]^{\alpha} = \Big\{x \in \mathbb{R}^n \Big| A(x) \geq \alpha\Big\}, \qquad [A]^0 = cl\Big\{x \in \mathbb{R}^n \Big| A(x) > 0\Big\}.$$

Definition 2.1. (Kaufmann and Gupta (1985))

Triangular fuzzy numbers are particular numbers in $\mathbb{E}$ such that they define by an ordered triple $a = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$. Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ be two triangular fuzzy numbers, so

1. The triangular fuzzy number a is said to be non-negative if $a_1 \geq 0$;
2. $[a]^{\alpha} = [(a_1, a_2, a_3)]^{\alpha} = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha]$ for all $\alpha \in [0, 1]$;
3. Two triangle fuzzy number a and b are said equal if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$;
4. $a \oplus b = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$;
5. For all $\lambda \in \mathbb{R}$

$$\lambda a = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3), & \text{if } \lambda \geq 0, \\ (\lambda a_3, \lambda a_2, \lambda a_1), & \text{if } \lambda < 0. \end{cases}$$

Definition 2.2. (Alikhani and Bahrami)

The generalized Hukuhara difference of two fuzzy number $a, b \in \mathbb{E}$ is the fuzzy number c (if it exists), such that

$$a \ominus_{gH} b = c \Longleftrightarrow \begin{cases} (i)\ a = b \oplus c, & \text{or} \\ (ii)\ b = a \oplus (-1)c. \end{cases}$$

For two fuzzy triangular numbers $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, the generalized Hukuhara difference is defined as follows

$$a \ominus_{gH} b = c \Longleftrightarrow \begin{cases} (i)\ c = (a_1 - b_1, a_2 - b_2, a_3 - b_3), & \text{or} \\ (ii)\ c = (a_3 - b_3, a_2 - b_2, a_1 - b_1), \end{cases}$$

provided that c is a triangular fuzzy number. Bede has shown that if a and b are two triangular fuzzy number, then $a \ominus_{gH} b$ always exists in $\mathbb{E}$ (Bede (2013)).

Corollary 2.3.

Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ be triangular fuzzy numbers. Then, we easily see that

(i) $(-1)a = (-a_3, -a_2, -a_1)$,
(ii) $0 \ominus_{gH} a = (-a_3, -a_2, -a_1) = (-1)a$,
(iii) $0 \ominus_{gH} (-1)a = (a_1, a_2, a_3) = a$,

(iv) $a \ominus_{gH} (-1)b \neq a \oplus b$,
(v) $a \ominus_{gH} (b \ominus_{gH} c) \neq a \ominus_{gH} b \oplus c$,

where 0 is the singleton.

In this paper consider, $f : (a, b) \subseteq \mathbb{R} \to \mathbb{E}$ as a triangular fuzzy function such that $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$ where $f_i(t)$, $i = 1, 2, 3$ are real-valued functions. Moreover, a function $u : \mathbb{R} \times [0, \infty) \to \mathbb{E}$ is a fuzzy-valued function of two independent variables which consider $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ as a triangular fuzzy function.

Definition 2.4. (Bede (2013))

Let $f : (a, b) \to \mathbb{E}$ be a fuzzy valued function such that $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$. Consider $\lim_{t\to\infty} f_i(t) = L_i$ for $i = 1, 2, 3$, then we define

$$\lim_{t\to\infty} f(t) = (L_1, L_2, L_3).$$

Definition 2.5. (Bede (2013))

Let $f : (a, b) \to \mathbb{E}$ be a fuzzy valued function such that $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$, where $f_1(t), f_2(t)$ and $f_3(t)$ are real-valued differentiable functions on (a, b). Then f is a $[(i) - gH]-$differentiable function at $t_0 \in (a, b)$ if and only if

$$f'_{i.gH}(t_0) = \Big(f'_1(t_0), f'_2(t_0), f'_3(t_0)\Big),$$

defines a triangular fuzzy number. Similarly, f is a $[(ii) - gH]-$differentiable function at t_0 if and only if

$$f'_{ii.gH}(t_0) = \Big(f'_3(t_0), f'_2(t_0), f'_1(t_0)\Big),$$

is a triangular fuzzy number. In general, if $f(t)$ is a $[(i) - gH]-$ or $[(ii) - gH]-$ differentiable for all $t_0 \in (a, b)$, then f is generalized Hukuhara differentiable function on (a, b).

Definition 2.6. (Bede (2013))

Let $f : (a, b) \to \mathbb{E}$ is a fuzzy-valued function and $f(t) = \Big(f_1(t), f_2(t), f_3(t)\Big)$ and $t_0 \in (a, b)$ then

$$\int_a^b f(t)dt = \Big(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, \int_a^b f_3(t)dt\Big).$$

Theorem 2.7. (Bede and Stefanini (2013))

If f is gH-differentiable with no switching point in the interval $[a, b]$, then we have

$$\int_a^b f'_{gH}(t)dt = f(b) \ominus_{gH} f(a).$$

Proposition 2.8.

Let α and β be two real constants such that $\alpha, \beta \geq 0$ (or $\alpha, \beta \leq 0$). If $f(t)$ is a fuzzy-valued function, then

$$\alpha\, f(t) \ominus_{gH} \beta\, f(t) = (\alpha - \beta) \odot f(t). \tag{2}$$

Proof:

First consider α and β as positive constants. Then,

$$\alpha\, f(t) = \Big(\alpha f_1(t), \alpha f_2(t), \alpha f_3(t)\Big), \quad \beta\, f(t) = \Big(\beta f_1(t), \beta f_2(t), \beta f_3(t)\Big).$$

Using Definition 2.2 we observe that

$$\alpha\, f(t) \ominus_{gH} \beta\, f(t) = \Big(\min\Big\{\alpha f_1(t) - \beta f_1(t), \alpha f_3(t) - \beta f_3(t)\Big\}, \alpha f_2(t) - \beta f_2(t), \max\Big\{\alpha f_1(t) - \beta f_1(t), \alpha f_3(t) - \beta f_3(t)\Big\}\Big).$$

Now, let $(\alpha - \beta) > 0$. In this case,

$$\begin{aligned}\alpha\, f(t) \ominus_{gH} \beta\, f(t) &= \Big((\alpha-\beta) f_1(t), (\alpha-\beta) f_2(t), (\alpha-\beta) f_3(t)\Big)\\ &= (\alpha - \beta) \odot \Big(f_1(t), f_2(t), f_3(t)\Big),\end{aligned}$$

and, if $(\alpha - \beta) < 0$, therefore,

$$\begin{aligned}\alpha\, f(t) \ominus_{gH} \beta\, f(t) &= \Big((\alpha-\beta) f_3(t), (\alpha-\beta) f_2(t), (\alpha-\beta) f_1(t)\Big)\\ &= (\alpha - \beta) \odot \Big(f_1(t), f_2(t), f_3(t)\Big).\end{aligned}$$

Hence Equation (2) is obtained. The other case, when α and β are negative constants, can be proved in a similar manner. ■

Definition 2.9. (Allahviranloo et al. (2015))

A triangular fuzzy function $u(x, t)$, without any switching point on $\mathbb{D}$, is called

- $[(i) - p]-$differentiable with respect to t at (x_0, t_0) if and only if
$$u_{t_{i.gH}}(x,t) = \Big(\frac{\partial u_1(x,t)}{\partial t}, \frac{\partial u_2(x,t)}{\partial t}, \frac{\partial u_3(x,t)}{\partial t}\Big)\Big|_{x=x_0, t=t_0},$$
defines a triangular fuzzy number, and
- Its $[(ii) - p]-$differentiable if and only if
$$u_{t_{ii.gH}}(x,t) = \Big(\frac{\partial u_3(x,t)}{\partial t}, \frac{\partial u_2(x,t)}{\partial t}, \frac{\partial u_1(x,t)}{\partial t}\Big)\Big|_{x=x_0, t=t_0},$$
defines a triangular fuzzy number.

Moreover, if $u_x(x, t)$ is $[gH - p]-$differentiable at (x, t) with respect to x without any switching point on $\mathbb{D}$ and

- If the type of $[gH-p]-$differentiability of both $u(x,t)$ and $u_x(x,t)$ are the same, then $u_x(x,t)$ is $[i-p]$-differentiable with respect to x and

$$u_{xx_{i.gH}}(x,t)=\Big(\frac{\partial^2 u_1(x,t)}{\partial x^2},\frac{\partial^2 u_2(x,t)}{\partial x^2},\frac{\partial^2 u_3(x,t)}{\partial x^2}\Big)\Big|_{x=x_0,t=t_0}.$$

- If the type of $[gH-p]-$differentiability $u(x,t)$ and $u_x(x,t)$ are different, therefore, $u_x(x,t)$ is $[ii-p]-$differentiable with respect to x and

$$u_{xx_{ii.gH}}(x,t)=\Big(\frac{\partial^2 u_3(x,t)}{\partial x^2},\frac{\partial^2 u_2(x,t)}{\partial x^2},\frac{\partial^2 u_1(x,t)}{\partial x^2}\Big)\Big|_{x=x_0,t=t_0}.$$

Recently, Chalco-Canoa et al. have studied the gH-derivative of the product of a differentiable real-valued function and a gH-differentiable for interval functions (Chalco-Canoa et al. (2019)). In the following, we extend the results for triangular fuzzy functions.

Theorem 2.10.

Assume that $f:(a,b)\rightarrow\mathbb{E}$ is a fuzzy generalized Hukuhara differentiable function on (a,b) and type of $gH-$differentiability doesn't change in this interval and $h(t)$ is a monotonic real-valued continuous differentiable function. Then, the following cases are established:

1. If $f(t)$ is $[i-gH]-$differentiable and

1-1. $h(t)$ is a positive and increasing function then $h(t)\odot f(t)$ is $[i-gH]-$differentiable and

$$\Big(h(t)\odot f(t)\Big)'_{gH}=h'(t)\odot f(t)\oplus h(t)\odot f'_{i.gH}(t).$$

1-2. $h(t)$ is a positive and decreasing function then $h(t)\odot f(t)$ is $[i-gH]-$differentiable and

$$\Big(h(t)\odot f(t)\Big)'_{gH}=h(t)\odot f'_{gH}(t)\ominus_{gH}(-h'(t))\odot f(t).$$

1-3. $h(t)$ is a negative and increasing function then $h(t)\odot f(t)$ is $[ii-gH]-$differentiable and

$$\Big(h(t)\odot f(t)\Big)'_{gH}=h(t)\odot f'_{gH}(t)\ominus_{gH}(-h'(t))\odot f(t).$$

1-4. $h(t)$ is a negative and decreasing function then $h(t)\odot f(t)$ is $[ii-gH]-$differentiable and

$$\Big(h(t)\odot f(t)\Big)'_{gH}=h'(t)\odot f(t)\oplus h(t)\odot f'_{gH}(t).$$

2. If $f(t)$ is $[ii-gH]-$differentiable and

2-1. $h(t)$ is a positive and increasing function then $h(t)\odot f(t)$ is $[ii-gH]-$differentiable and

$$\Big(h(t)\odot f(t)\Big)'_{gH}=h(t)\odot f'_{gH}(t)\ominus_{gH}(-1)h'(t)\odot f(t).$$

2-2. $h(t)$ is a positive and decreasing function then $h(t)\odot f(t)$ is $[ii-gH]-$differentiable and

$$\Big(h(t)\odot f(t)\Big)'_{gH}=h'(t)\odot f(t)\oplus h(t)\odot f'_{gH}(t).$$

2-3. $h(t)$ is a negative and increasing function then $h(t) \odot f(t)$ is $[i - gH]-$differentiable and

$$\Big(h(t) \odot f(t)\Big)'_{gH} = h'(t) \odot f(t) \oplus h(t) \odot f'_{gH}(t).$$

2-4. $h(t)$ is a negative and decreasing function then $h(t) \odot f(t)$ is $[i - gH]-$differentiable and

$$\Big(h(t) \odot f(t)\Big)'_{gH} = h(t) \odot f'_{gH}(t) \ominus_{gH} (-1)h'(t) \odot f(t).$$

Proof:

Let $f(t) = (f_1(t), f_2(t), f_3(t))$, the product of $f(t)$ and $h(t)$ is equal to

$$\Big(h(t) \odot f(t)\Big) = \begin{cases} \Big(h(t)f_1(t), h(t)f_2(t), h(t)f_3(t)\Big), \ \text{if } h(t) > 0, \\ \\ \Big(h(t)f_3(t), h(t)f_2(t), h(t)f_1(t)\Big), \text{if } h(t) < 0. \end{cases}$$

Then $\Big(h(t) \odot f(t)\Big)'_{gH} = \Big(\mathcal{H}_1(t), \mathcal{H}_2(t), \mathcal{H}_3(t)\Big)$ where

$$\begin{aligned} \mathcal{H}_1(t) &= \min\Big\{h'(t)f_1(t) + h(t)f_1'(t), h'(t)f_3(t) + h(t)f_3'(t)\Big\}, \\ \mathcal{H}_2(t) &= h'(t)f_2(t) + h(t)f_2'(t), \\ \mathcal{H}_3(t) &= \max\Big\{h'(t)f_1(t) + h(t)f_1'(t), h'(t)f_3(t) + h(t)f_3'(t)\Big\}. \end{aligned}$$

If $\mathcal{H}_1(t) = h'(t)f_1(t)+h(t)f_1'(t)$ and $\mathcal{H}_3(t) = h'(t)f_3(t)+h(t)f_3'(t)$, then we say that $\Big(h(t)\odot f(t)\Big)$ is a $[(i) - gH]-$differentiable function. Otherwise, it is $[(ii) - gH]-$differentiable.

Now consider the Case 1-4. According to the assumptions, $f(t)$ is a $[(i) - gH]-$differentiable function and $h(t)$ is a negative and decreasing function. Then $h(t) < 0$ and $h'(t) < 0$,

$$\begin{aligned} h(t)f'_{i.gH}(t) \oplus h'(t)f(t) &= h(t)\Big(f_1'(t), f_2'(t), f_3'(t)\Big) \oplus h'(t)\Big(f_1(t), f_2(t), f_3(t)\Big) \\ &= \Big(h(t)f_3'(t), h(t)f_2'(t), h(t)f_1'(t)\Big) \oplus \Big(h'(t)f_3(t), h'(t)f_2(t), h'(t)f_1(t)\Big) \\ &= \Big(h(t)f_3'(t) + h'(t)f_3(t), h(t)f_2'(t) + h'(t)f_2(t), h(t)f_1'(t) + h'(t)f_1(t)\Big) \\ &= \Big(h(t) \odot f(t)\Big)'_{ii.gH}. \end{aligned}$$

So $h(t) \odot f(t)$ is $[ii - gH]-$differentiable. Similarly, the rest of the cases can be proved. ∎

Corollary 2.11.

In the particular case of Theorem 2.10, suppose that $h'(t) = 0$. It can be easily shown that

1. If $f(t)$ is $[(i) - gH]-$differentiable, then $h(t) \odot f(t)$ is $[(i) - gH]-$differentiable,
2. If $f(t)$ is $[(ii) - gH]-$differentiable, then $h(t) \odot f(t)$ is $[(ii) - gH]-$differentiable,

and

$$\Big(h(t) \odot f(t)\Big)'_{gH} = h(t) \odot f'_{gH}(t).$$

Theorem 2.12. (Moghaddam and Allahviranloo (2018))

Let $Z := F(\xi(t), \eta(t))$ is a fuzzy valued function, where $\xi(t)$ and $\eta(t)$ are differentiable real valued functions of t. Then, F is gH-differentiable function of t and we have:

$$Z_{t_{gH}} = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t} \oplus \frac{d_{gH}F}{d\eta} \odot \frac{\partial \eta}{\partial t}.$$

Corollary 2.13.

Let $Z(x,t) = F(\xi)$ is a fuzzy valued function, where $\xi(x,t)$ is differentiable real valued function of x and t. Then, by Theorem 2.12,

$$Z_{t_{gH}}(x,t) = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t},$$
$$Z_{x_{gH}}(x,t) = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial x}.$$

Theorem 2.14.

Let $Z(x,t) = F(\xi)$ be a triangular fuzzy function such that $F(\xi) = \Big(F_1(\xi), F_2(\xi), F_3(\xi)\Big)$, where $\xi(x,t)$ is differentiable real-valued function of x and t. Then, F is gH-differentiable function of ξ. Then,

1. If $F(\xi)$ is $[(i) - gH]-$differentiable and $\frac{\partial \xi}{\partial t} > 0$, then $Z(x,t)$ is $[(i) - p]-$differentiable with respect to t;
2. If $F(\xi)$ is $[(i) - gH]-$differentiable and $\frac{\partial \xi}{\partial t} < 0$, then $Z(x,t)$ is $[(ii) - p]-$differentiable with respect to t;
3. If $F(\xi)$ is $[(ii) - gH]-$differentiable and $\frac{\partial \xi}{\partial t} > 0$, then $Z(x,t)$ is $[(ii) - p]-$differentiable with respect to t;
4. If $F(\xi)$ is $[(ii) - gH]-$differentiable and $\frac{\partial \xi}{\partial t} < 0$, then $Z(x,t)$ is $[(i) - p]-$differentiable with respect to t.

Proof:

Consider $F(\xi) = \Big(F_1(\xi), F_2(\xi), F_3(\xi)\Big)$ is $[(i) - gH]$-differentiable for every $t \in J$ and $\frac{\partial \xi}{\partial t} > 0$. Using Corollary 2.13,

$$Z_{t_{gH}}(x,t) = \frac{d_{gH}F}{d\xi} \odot \frac{\partial \xi}{\partial t}.$$

Now, $F(\xi)$ is $[(i)-gH]$-differentiable, then

$$\begin{aligned}\frac{d_{i.gH}F}{d\xi}\odot\frac{\partial\xi}{\partial t}&=\Big(\frac{dF_1}{d\xi},\frac{dF_2}{d\xi},\frac{dF_3}{d\xi}\Big)\odot\frac{\partial\xi}{\partial t}\\&=\Big(\frac{dF_1}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_2}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_3}{d\xi}\frac{\partial\xi}{\partial t}\Big)\\&=Z_{t_{i.gH}}(x,t).\end{aligned}$$

Now, if $\frac{\partial\xi}{\partial t}<0$ we obtain

$$\begin{aligned}\frac{d_{i.gH}F}{d\xi}\odot\frac{\partial\xi}{\partial t}&=\Big(\frac{dF_1}{d\xi},\frac{dF_2}{d\xi},\frac{dF_3}{d\xi}\Big)\odot\frac{\partial\xi}{\partial t}\\&=\Big(\frac{dF_3}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_2}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_1}{d\xi}\frac{\partial\xi}{\partial t}\Big)\\&=Z_{t_{ii.gH}}(x,t).\end{aligned}$$

This concludes the proof for $[(i)-gH]$-differentiability.

Now, consider $F(\xi)$ is $[(ii)-gH]$-differentiable and $\frac{\partial\xi}{\partial t}>0$

$$\begin{aligned}\frac{d_{ii.gH}F}{d\xi}\odot\frac{\partial\xi}{\partial t}&=\Big(\frac{dF_3}{d\xi},\frac{dF_2}{d\xi},\frac{dF_1}{d\xi}\Big)\odot\frac{\partial\xi}{\partial t}\\&=\Big(\frac{dF_3}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_2}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_1}{d\xi}\frac{\partial\xi}{\partial t}\Big)\\&=Z_{t_{ii.gH}}(x,t).\end{aligned}$$

Now, if $\frac{\partial\xi}{\partial t}<0$, then

$$\begin{aligned}\frac{d_{ii.gH}F}{d\xi}\odot\frac{\partial\xi}{\partial t}&=\Big(\frac{dF_3}{d\xi},\frac{dF_2}{d\xi},\frac{dF_1}{d\xi}\Big)\odot\frac{\partial\xi}{\partial t}\\&=\Big(\frac{dF_1}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_2}{d\xi}\frac{\partial\xi}{\partial t},\frac{dF_3}{d\xi}\frac{\partial\xi}{\partial t}\Big)\\&=Z_{t_{i.gH}}(x,t).\end{aligned}$$

■

3. The Traveling Wave Fuzzy Solution

Consider the following generic form of second-order fuzzy linear partial differential equation in two independent variables,

$$u_{t_{gH}}=F\Big(u,u_{x_{gH}},u_{xx_{gH}},u_{tt_{gH}}\Big). \tag{3}$$

We are interested in a particular class of solutions to the fuzzy linear partial differential equations, the traveling waves. For a fuzzy traveling wave solution with profile U and velocity $c\in\mathbb{R}$, we want to find a solution $u:\mathbb{R}\times[0,\infty)\to\mathbb{E}$ of the form

$$u(x,t)=U(x-ct),\quad \xi(x,t)=x-ct, \tag{4}$$

where profile U is a continuous function and $gH-$differentiable in ξ and c is arbitrary constant generally termed the wave velocity. This solution corresponds to a linear translation along the x axis with respect to t. If $c > 0$, profile $U(x-ct)$ at a later time t is moving to the positive x direction by a amount ct with speed c. Similarly, $u(x,t) = U(x-ct)$ with $c < 0$ shows a moving to the left of x axis with speed $|c|$. In this paper we consider $c > 0$, it means we are just interested such solutions which are moving to the positive x direction.

To find a traveling wave solution for Equation (3), consider $u(x,t) = U(\xi)$ where $\xi = x - ct$. The partial derivatives of $\xi(x,t) = x - ct$ are as follows

$$\frac{\partial \xi}{\partial t} = -c, \quad \frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2} = 0. \tag{5}$$

By using the Theorems 2.14, 2.10, Corollary 2.11 and Equation (5), and by considering the type of $gH-$differentiability for U, we have different cases as follows.

Case 1. If $U(\xi)$ is a $[(i)-gH]-$differentiable fuzzy function, then

1-1. $u(x,t)$ is $[(ii)-p]-$differentiable with respect to t and

$$u_{t_{ii.gH}} = \frac{d_{i.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial t} = (-1)c \odot \frac{d_{i.gH}U}{d\xi}.$$

1-2. $u(x,t)$ is $[(i)-p]-$differentiable with respect to x and

$$u_{x_{i.gH}} = \frac{d_{i.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{i.gH}U}{d\xi}.$$

1-3. $u_{t_{gH}}(x,t)$ is $[(i)-p]-$differentiable with respect to t and

$$u_{tt_{i.gH}} = \frac{d^2_{i.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial t}\right)^2 = c^2 \odot \frac{d^2_{i.gH}U}{d\xi^2}.$$

1-4. $u_{x_{gH}}(x,t)$ is $[(i)-p]-$differentiable with respect to x and

$$u_{xx_{i.gH}} = \frac{d^2_{i.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial x}\right)^2 = \frac{d^2_{i.gH}U}{d\xi^2}.$$

Case 2. Consider $U(\xi)$ is a $[(ii)-gH]-$differentiable fuzzy function. Hence, we have

2-1. $u(x,t)$ is $[(i)-p]-$differentiable with respect to t and

$$u_{t_{i.gH}} = \frac{d_{ii.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial t} = (-1)c \odot \frac{d_{ii.gH}U}{d\xi}.$$

2-2. $u(x,t)$ is $[(ii)-p]-$differentiable with respect to x and

$$u_{x_{ii.gH}} = \frac{d_{ii.gH}U}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{ii.gH}U}{d\xi}.$$

2-3. $u_{t_{gH}}(x,t)$ is $[(ii)-p]-$differentiable with respect to t and

$$u_{tt_{ii.gH}} = \frac{d^2_{ii.gH}U}{d\xi^2} \odot \left(\frac{\partial \xi}{\partial t}\right)^2 = c^2 \odot \frac{d^2_{ii.gH}U}{d\xi^2}.$$

2-4. $u_{x_{gH}}(x,t)$ is $[(ii)-p]-$differentiable with respect to x and

$$u_{xx_{ii.gH}} = \frac{d^2_{ii.gH}U}{d\xi^2} \odot \Big(\frac{\partial \xi}{\partial x}\Big)^2 = \frac{d^2_{ii.gH}U}{d\xi^2}.$$

So Equation (3) reduces to

$$(-1)c \odot \frac{d_{i.gH}U}{d\xi} = F\Big(U, \frac{d_{i.gH}U}{d\xi}, \frac{d^2_{i.gH}U}{d\xi^2}, c^2\frac{d^2_{i.gH}U}{d\xi^2}\Big), \tag{6}$$

and

$$(-1)c \odot \frac{d_{ii.gH}U}{d\xi} = F\Big(U, \frac{d_{ii.gH}U}{d\xi}, \frac{d^2_{ii.gH}U}{d\xi^2}, c^2\frac{d^2_{ii.gH}U}{d\xi^2}\Big). \tag{7}$$

Equations (6) and (7) are fuzzy ordinary differential equations in ξ. Solutions of such equations usually depend upon some arbitrary constants that we can find them by initial conditions and some auxiliary conditions are often used to obtain a fuzzy solution for this equation. Generally, the boundary conditions

$$\lim_{\xi\to-\infty} U(\xi) = u_l, \quad \lim_{\xi\to+\infty} U(\xi) = u_r, \tag{8}$$

are usually imposed. Then, U is called wave front if $u_l \neq u_r$. However, if $u_l = u_r$, the corresponding wave is known as a pulse wave.

In this paper we consider the following auxiliary boundary conditions

$$\lim_{\xi\to\pm\infty} U(\xi) = 0, \quad \lim_{\xi\to\pm\infty} \frac{dU}{d\xi} = 0, \quad \lim_{\xi\to\pm\infty} \frac{d^2U}{d\xi^2} = 0. \tag{9}$$

Moreover, the traveling wave solution must be valid in the initial condition $u(x,0) = f(x)$. Then, by using auxiliary conditions (9), the fuzzy initial condition $f(x)$ has to satisfies in the following conditions:

$$\lim_{x\to\pm\infty} f(x) = 0, \quad \lim_{x\to\pm\infty} f'_{gH}(x) = 0, \quad \lim_{x\to\pm\infty} f''_{gH}(x) = 0. \tag{10}$$

For this reason, in the following sections, the initial value is selected in such a way that conditions (10) are applied.

4. The Traveling Wave Solution for Fuzzy Linear Partial Differential Equations

In the following, in order to illustrate the efficiency of the proposed method, several important equations of linear fuzzy partial differential equations are investigated and their traveling wave solutions are presented.

4.1. Fuzzy Linear Advection Equation

One of the simplest fuzzy partial differential equations is the fuzzy linear advection equation (or fuzzy linear transport equation). Consider u is a quantity to be transported and the positive constant a is the velocity. Consider the fuzzy advection

$$u_{t_{gH}} = (-1)au_{x_{gH}}, \tag{11}$$

with fuzzy initial condition

$$u(x,0) = f(x). \tag{12}$$

To obtain a traveling fuzzy solution for Equation (11), consider

$$u(x,t) = U(\xi), \quad \xi = x - ct.$$

First, we consider $U(\xi)$ is $[(i) - gH]-$differentiable. Then,

$$(-1)c\frac{d_{i.gH}U}{d\xi} = (-1)a\frac{d_{i.gH}U}{d\xi}, \quad \Rightarrow \quad c\frac{d_{i.gH}U}{d\xi} \ominus_{gH} a\frac{d_{i.gH}U}{d\xi} = 0.$$

Two constants a and c are positive. Therefore, Proposition 2.8 implies

$$(c-a)\frac{d_{i.gH}U}{d\xi} = 0.$$

For non-constant U, we have $\frac{d_{i.gH}U}{d\xi} \neq 0$ which implies that $c = a$. So any function $U(x - at)$ with sufficiently smooth U which satisfies in the initial fuzzy value (12), $\lim_{\xi\to\infty} U(\xi) = 0$ and $[(i) - gH]$-differentiable, is a traveling wave solution. In fact, the traveling fuzzy solution of Equation (11) is $u(x,t) = f(x - at)$, such that $u(x,t)$ is $[(ii) - p]-$differentiable with respect to t and $[(i) - p]-$differentiable with respect to x.

Similarly, consider $U(\xi)$ is $[(ii) - gH]-$differentiable. Then, in this case, the initial fuzzy value $f(x)$ has to be $[(ii) - gH]-$differentiable at $\xi = x - ct$ and we obtain the traveling wave solution $u(x,t) = f(x-at)$ which is $[(i)-p]-$differentiable with respect to t and $[(ii)-p]-$differentiable with respect to x. For instant, let $a = 1$ in Equation (11) and initial fuzzy function (12) is equal $f(x) = (3.8, 7.7, 9.3)(0.2)^x$. This initial value satisfies the conditions in (10). Hence, the traveling wave fuzzy solution is $u(x,t) = (3.8, 7.7, 9.3)(0.2)^{x-t}$. We plot these functions in Figure 1, and as you can see this solution is $[(ii) - p]-$differentiable with respect to t and $[(i) - p]-$differentiable with respect to x.

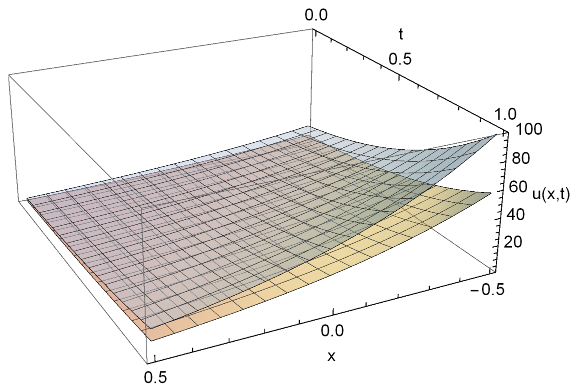
4.2. Fuzzy Linear Diffusion Equation

Consider the following fuzzy linear diffusion equation

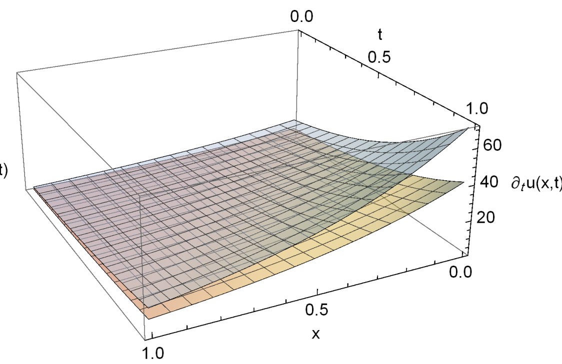
$$u_{t_{gH}} = Du_{xx_{gH}}, \tag{13}$$

with the initial condition

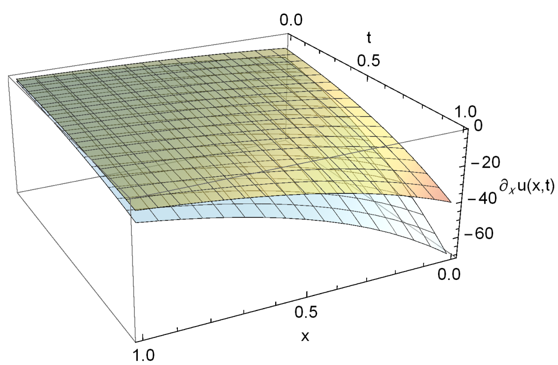
$$u(x,0) = f(x), \tag{14}$$



(a) Graph of $u(x,t)$.



(b) Graph of $u_{t_{gH}}(x,t)$.



(c) Graph of $u_{x_{gH}}(x,t)$.

Figure 1. Graph of $u(x,t) = (3.8, 7.7, 9.3)(0.2)^{x-t}$ and its partial derivatives for $\alpha = \frac{1}{3}$ and $0 \leq x, t \leq 1$.

where $f(x) \in \mathbb{E}$. For a traveling wave fuzzy solution, we consider

$$u(x,t) = U(\xi), \quad \text{with} \quad \xi = x - Dt. \tag{15}$$

We want to find a fuzzy solution for Equation (13) such that $u(x,t)$ will be $[(i)-p]-$differentiable with respect to t and $[(ii)-p]-$differentiable with respect to x. For these reasons, by Case 2 in Section 3, $U(\xi)$ has to be $[(ii)-gH]-$differentiable, and therefore,

$$(-1)D\frac{d_{ii.gH}U}{d\xi} = D \odot \frac{d^2_{ii.gH}U}{d\xi^2}. \tag{16}$$

After cancellation of D and rearrange this equation and using Corollary 2.3, we have the following second order differential equation

$$\frac{d^2_{ii.gH}U}{d\xi^2} \oplus \frac{d_{ii.gH}U}{d\xi} = 0. \tag{17}$$

By integrating Equation (17) once, we obtain

$$\frac{d_{ii.gH}U}{d\xi} \oplus U = C_1, \tag{18}$$

where C_1 is the integration constant. By using auxiliary conditions (9) when $\xi \to \pm\infty$, we have

$$U(\xi) = \frac{d_{ii.gH}U}{d\xi} = 0.$$

So the integration constant C_1 is equal to zero and we have the following first order fuzzy differential equation

$$\frac{d_{ii.gH}U}{d\xi} \oplus U = 0,$$

that has the following fuzzy solution (Armand and Gouyandeh (2017))

$$U(\xi) = Ce^{-\xi},$$

which satisfies the condition $U(\xi) = 0$ when $\xi \to \infty$. Therefore

$$u(x,t) = Ce^{-(x-Dt)}.$$

Using initial condition (14), we can write

$$C = f(x)e^{x},$$

so, we obtain the following fuzzy solution for the fuzzy linear diffusion equation

$$u(x,t) = f(x)e^{Dt}. \tag{19}$$

For example, for the fuzzy linear diffusion equation (13), consider $D = 3$ and the fuzzy initial condition $f(x) = (2.1, 5.7, 8.3)e^{-x^2}$. Then by Equation (19), we have the following traveling wave fuzzy solution

$$u(x,t) = (2.1, 5.7, 8.3)e^{-x^2+3t}. \tag{20}$$

This function and its first partial derivative with respect to t and x are shown in Figure 2. These figures show that $u(x,t)$ is $[(i)-p]-$differentiable with respect to t and $[(ii)-p]-$differentiable with respect to x.

4.3. The fuzzy linear Convection-Diffusion-Reaction Equation

Consider the following fuzzy linear convection-diffusion-reaction equation

$$\begin{cases} u_{t_{gH}} = (-1)a \odot u_{x_{gH}} \oplus D \odot u_{xx_{gH}} \ominus_{gH} r \odot u, \; (x,t) \in \mathbb{R} \times (0,\infty), \\ \\ u(x,0) = f(x), \end{cases} \tag{21}$$

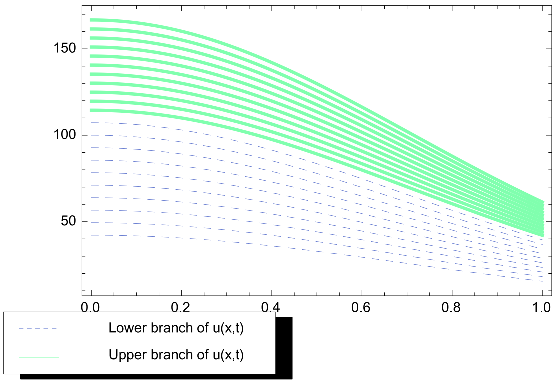
where a, D and r are real positive constants. For a traveling wave solution to (21), we consider

$$u(x,t) = U(\xi), \qquad \xi = (x - at), \tag{22}$$

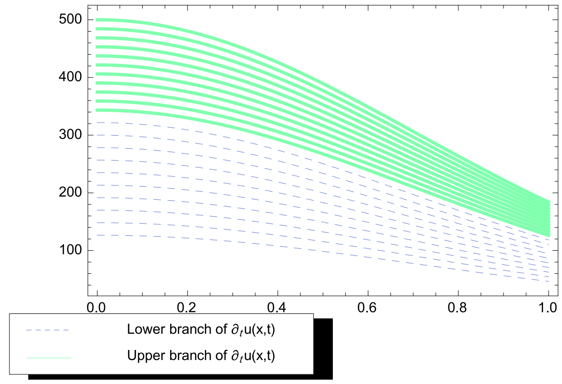
$$(-1)a \odot \frac{d_{ii.gH}U}{d\xi} = (-1)a \odot \frac{d_{ii.gH}U}{d\xi} \oplus D \odot \frac{d^2_{ii.gH}U}{d\xi^2} \ominus_{gH} r \odot U. \tag{23}$$

Then we have the following second order fuzzy differential equation

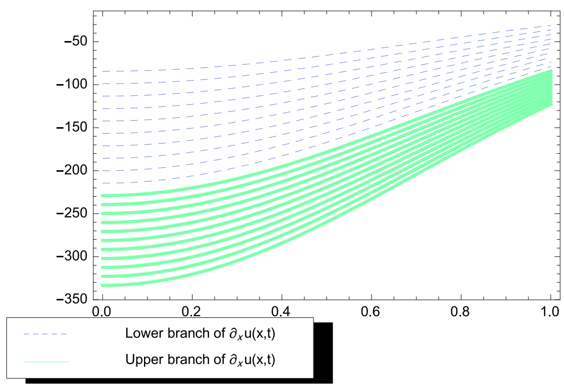
$$D \odot \frac{d^2_{ii.gH}U}{d\xi^2} \ominus r \odot U = 0. \tag{24}$$



(a) Graph of $u(x, t)$.



(b) Graph of $u_{t_{gH}}(x, t)$.



(c) Graph of $u_{x_{gH}}(x, t)$.

Figure 2. Graph of $u(x, t) = (2.1, 5.7, 8.3)e^{-x^2+3t}$ and its partial derivative for $t = 1$ and $0 \leq \alpha \leq 1$ and $0 \leq x \leq 1$.

By the method that described in Appendix 6 (B), Equation (24) has the following fuzzy solution

$$U(\xi) = C_1 \odot e^{-\sqrt{\frac{r}{D}}\xi} \oplus C_2 \odot e^{\sqrt{\frac{r}{D}}\xi}, \tag{25}$$

where C_1 and C_2 are constants to be determined.

The auxiliary condition $\lim_{\xi\to\infty} U(\xi) = 0$ implies that $C_2 = 0$. So, we have

$$U(\xi) = C_1 e^{-\sqrt{\frac{r}{D}}\xi}, \quad \Rightarrow \quad u(x, t) = C_1 e^{-\sqrt{\frac{r}{D}}(x-at)}. \tag{26}$$

Moreover, by initial condition $u(x, 0) = f(x)$ we can write

$$C_1 e^{-\sqrt{\frac{r}{D}}(x)} = f(x), \quad \Rightarrow \quad C_1 = f(x)e^{\sqrt{\frac{r}{D}}(x)}.$$

The traveling wave fuzzy solution for Equation (21) is

$$u(x, t) = f(x)e^{\sqrt{\frac{r}{D}}(x)} e^{-\sqrt{\frac{r}{D}}(x-at)}. \tag{27}$$

Now, consider the following fuzzy linear convection-diffusion-reaction partial differential equation

$$\begin{cases} u_{t_{gH}} = (-1)2 \odot u_{x_{gH}} \oplus 4 \odot u_{xx_{gH}} \ominus_{gH} 3 \odot u, \; (x, t) \in \mathbb{R} \times (0, \infty), \\ \\ u(x, 0) = f(x) = (1.3, 5.2, 9.6)\frac{1}{1+x^2}. \end{cases}$$

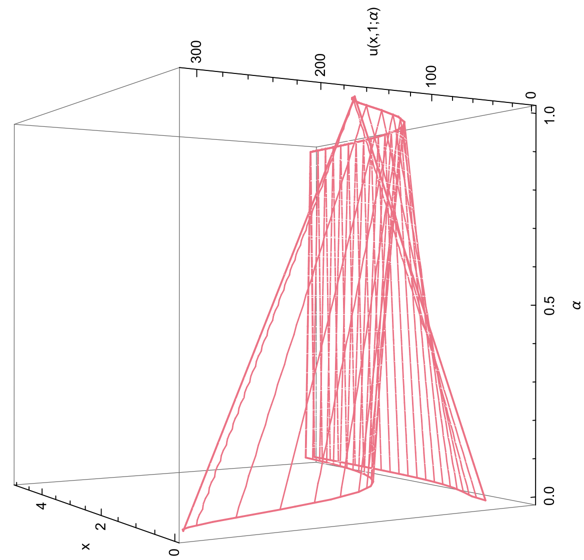


Figure 3. Graph of $u(x,t) = (1.3, 5.2, 9.6)\frac{1}{1+x^2}e^{2\sqrt{\frac{3}{4}}t}$ for $t = 1$ and $0 \leq \alpha \leq 1$ and $0 \leq x \leq 5$.

Then, we have the following traveling wave solution

$$u(x,t) = (1.3, 5.2, 9.6)\frac{1}{1+x^2}e^{2\sqrt{\frac{3}{4}}t}. \tag{28}$$

4.4. The Fuzzy Linear Klein-Gordon Equation

A simple modification of the fuzzy Klein-Gordon equation is the following equation

$$\begin{cases} u_{tt_{gH}} \ominus_{gH} u_{xx_{gH}} \oplus u = 0, \; (x,t) \in \mathbb{R} \times (0,\infty), \\ \\ u(x,0) = f(x) \in \mathbb{E}. \end{cases} \tag{29}$$

To obtain the traveling wave solution for Equation (29), consider

$$u(x,t) = U(\xi), \quad \text{with} \quad \xi = x - ct,$$

where $U(\xi)$ is $[(ii) - gH]-$differentiable. According to the process described Section 3 and using Proposition 2.8,

$$c^2\frac{d^2_{ii.gH}U}{d\xi^2} \ominus_{gH} \frac{d^2_{ii.gH}U}{d\xi^2} \oplus U = 0.$$

Therefore,

$$(c^2 - 1)\frac{d^2_{ii.gH}U}{d\xi^2} \oplus U = 0. \tag{30}$$

Now, we determine a fuzzy solution for Equation (30) using method described in Appendix 6 (B) as follows:

- Consider $c^2 < 1$. Therefore, the fuzzy solution of Equation (30) is

$$U(\xi) = C_1 e^{-k\xi} \oplus C_2 e^{k\xi},$$

where C_1 and C_2 are integration constants and $k = \frac{1}{\sqrt{c^2-1}}$. The auxiliary condition $\lim_{\xi\to\infty} U(\xi) = 0$ implies that $C_2 = 0$, hence,

$$u(x,t) = C_1 e^{-k(x-ct)}.$$

Using the initial condition $u(x,0) = f(x)$, we obtain the traveling wave solution of Equation (29)

$$u(x,t) = f(x) e^{\frac{x}{\sqrt{c^2-1}}} e^{-(\frac{x-ct}{\sqrt{c^2-1}})}.$$

- Now consider $c^2 > 1$. By applying the method which is discussed in detail in the previous part,

$$U(\xi) = C_1 \cos(k\xi) \oplus \frac{C_2}{k} \sin(k\xi),$$

which by auxiliary condition $\lim_{\xi\to\infty} U(\xi) = 0$ we have $C_1 = C_2 = 0$. So when $c^2 > 1$, the fuzzy Klein-Gordon equation does not have any traveling wave solution.

5. Conclusion

In this paper, we obtain the fuzzy traveling wave solution of the partial differential equation by considering the type of gH-differentiability. To demonstrate the efficiency of the method, the fuzzy traveling wave solutions of the fuzzy Advection equation, fuzzy linear Diffusion equation, fuzzy Convection-Diffusion-Reaction equation, and fuzzy Klein-Gordon equation are obtained. All results show that this method is a very powerful and efficient method for obtaining an analytical solution for fuzzy partial differential equation.

6. Appendix

A. Linear Systems of Fuzzy Differential Equations

A system of equations of the form

$$\begin{aligned} y'_{1_{gH}}(t) &= a_{11}y_1(t) \oplus ... \oplus a_{1n}y_n(t) \oplus f_1(t), \\ y'_{2_{gH}}(t) &= a_{21}y_1(t) \oplus ... \oplus a_{2n}y_n(t) \oplus f_2(t), \\ &\vdots \\ y'_{n_{gH}}(t) &= a_{n1}y_1(t) \oplus ... \oplus a_{nn}y_n(t) \oplus f_n(t), \end{aligned} \tag{31}$$

where a_{ij} are real numbers and $f_i(t)$ are fuzzy functions, is called a first order linear system of fuzzy ordinary differential equations with constant coefficient. Moreover, if the following fuzzy initial conditions are satisfied in system (31),

$$y_1(t_0) = A_1,\ y_2(t_0) = A_2,\ ...,\ y_n(t_0) = A_n,$$

then it is called the fuzzy initial-valued linear system, where A_i are prescribed fuzzy constant values.

Let us consider

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \qquad f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

Then system (31) can be written in matrix form as

$$y'_{gH}(t) = Ay(t) + f(t). \tag{32}$$

If $f(t) = 0$, the fuzzy linear differential system (32) is homogeneous, otherwise it is nonhomogeneous. In the following, we express the fundamental theorem for solving homogeneous system of fuzzy differential equations .

Definition 6.1. (Perko (2013))

Let A be an $n \times n$ matrix of real scalars. The matrix exponential, e^{At}, is defined by

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

The following statements hold

1. $e^{At}|_{t=0} = I$;
2. $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$;
3. e^{At} is an invertible matrix with inverse $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.

Theorem 6.2. (Adkins and Davidson (2012))

Let A be an $n \times n$ matrix of real constants. Then, e^{At} is a well-defined matrix-valued function and we have the following Laplace transform of e^{At}

$$\mathcal{L}[e^{At}] = (sI - A)^{-1}.$$

The Laplace inversion formula is given by

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}].$$

Theorem 6.3. (The Fundamental Theorem for Linear Systems)

Let A be an $n \times n$ matrix of real constants. Then for a given $y_0 \in \mathbb{E}^n$, the initial value problem

$$y' = Ay, \quad y(0) = y_0, \tag{33}$$

has a unique solution given by

$$y(t) = e^{At}y_0.$$

Proof:

We will show that $y(t) = e^{At}y_0$ is a solution for initial value problem (33). For this reason, first, by the properties of the matrix exponential that define in Definition 6.1, if $y(t) = e^{At}y_0$ we obtain

$$y'_{gH}(t) = \frac{d}{dt}\Big(e^{At}y_0\Big) = Ae^{At}y_0 = Ay(t),$$

for all $t \in \mathbb{R}$. Moreover,

$$y(0) = Iy_0 = y_0.$$

Thus, $y(t) = e^{At}y_0$ is a fuzzy solution of the initial value problem. Now we want to show that this is the only solution. Let $x(t)$ be any solution of the initial value problem (33) and

$$x(t) = e^{At}y_0. \tag{34}$$

Now, define

$$z(t) = e^{-At}x(t).$$

Using this fact that $x(t)$ is a fuzzy solution of (33), and Definition 6.1, we have the following situations:

1. If $x(t)$ is $[(i) - gH]-$differentiable, using Case 1-2 of Theorem 2.10 and Proposition 2.8,

$$\begin{aligned} z'_{i.gH}(t) &= e^{-At}x'_{i.gH}(t) \ominus_{gH} Ae^{-At}x(t) \\ &= e^{-At}Ax(t) \ominus_{gH} Ae^{-At}x(t) \\ &= \Big(Ae^{-At} - Ae^{-At}\Big)x(t) = 0. \end{aligned}$$

2. If $x(t)$ is $[(ii) - gH]-$differentiable, using Case 2-2 of Theorem 2.10 and Proposition 2.8 and Corollary 2.3,

$$\begin{aligned} z'_{ii.gH}(t) &= (-1)e^{-At}x'_{ii.gH}(t) \oplus Ae^{-At}x(t) \\ &= \ominus_{gH} Ae^{-At}x(t) \oplus Ae^{-At}x(t) \\ &= \Big(-Ae^{-At} + Ae^{-At}\Big)x(t) = 0. \end{aligned}$$

So $z(t)$ is a constant. On the other hand,

$$z(0) = x(0) = y_0.$$

Therefore, $z(t) = y_0$ and

$$z(t) = e^{-At}x(t) \qquad \Rightarrow \qquad x(t) = e^{At}y_0.$$

So any solution of the initial value problem (33) is given by $x(t) = y(t) = e^{At}y_0$. This completes the proof of the theorem. ∎

B. Eigenvalue Problem for the Second Order Fuzzy Differential Equation

Consider the following fuzzy differential equation

$$\begin{cases} u''_{gH}(t) \oplus \lambda u(t) = 0, \\ u(t_0) = u_0, \quad u'_{gH}(t_0) = u_1, \end{cases} \tag{35}$$

where λ is a real number and where u_1 and u_2 are two fuzzy numbers. We want to find a fuzzy solution for Equation (35) for different value of λ.

- If $\lambda = K^2 > 0$, then the differential equation $u''_{gH}(t) \oplus K^2 u(t) = 0$ can be rewritten as a system of 2 first order linear equations by defining

$$y_1(t) := u(t), \quad y_2(t) := u'_{gH}(t).$$

  Therefore, we can write

$$y'_{1_{gH}}(t) = u'_{gH}(t),$$

$$y'_{2_{gH}}(t) = u''_{gH}(t) = \ominus_{gH} K^2 u(t),$$

  and we obtain the following system of fuzzy first-order equations

$$y'_{1_{gH}}(t) = y_2(t), \qquad y_1(t_0) = u_0,$$

$$y'_{2_{gH}}(t) = \ominus_{gH} K^2 y_1(t), \qquad y_2(t_0) = u_1.$$

  In this case

$$A = \begin{bmatrix} 0 & 1 \\ -K^2 & 0 \end{bmatrix} \qquad \Rightarrow e^{At} = \begin{bmatrix} \cos(Kt) & \frac{1}{K}\sin(Kt) \\ -K\sin(Kt) & \cos(Kt) \end{bmatrix}.$$

  Using Theorem 6.3, we have

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \cos(Kt) & \frac{1}{K}\sin(Kt) \\ -K\sin(Kt) & \cos(Kt) \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

  Then we obtain the following fuzzy solution for Equation (35) when $\lambda = K^2 > 0$

$$u(t) = u_0 \cos(Kt) \oplus \frac{u_1}{K} \sin(Kt).$$

- $\lambda = -K^2 < 0$, we have the fuzzy differential equation $u''_{gH}(t) \ominus_{gH} K^2 u = 0$. Using the same method, we have the following system of fuzzy differential equation

$$y'_{1_{gH}}(t) = y_2(t), \qquad y_1(t_0) = u_0,$$

$$y'_{2_{gH}}(t) = K^2 y_1(t), \qquad y_2(t_0) = u_1.$$

In this case

$$A = \begin{bmatrix} 0 & 1 \\ K^2 & 0 \end{bmatrix} \qquad \Rightarrow e^{At} = \begin{bmatrix} \cosh(Kt) & \frac{1}{K}\sinh(Kt) \\ K\sinh(Kt) & \cosh(Kt) \end{bmatrix}.$$

Therefore, the fuzzy solution of Equation (35) when $\lambda = -K^2 < 0$ is obtained as follows

$$u(t) = u_0 \cosh(Kt) \oplus \frac{u_1}{K} \sinh(Kt).$$

Moreover, $\cosh(Kt) = \frac{e^{Kt}+e^{-Kt}}{2}$ and $\sinh(Kt) = \frac{e^{Kt}-e^{-Kt}}{2}$, hence we have

$$\begin{aligned} u(t) &= u_0\Big(\frac{e^{Kt}+e^{-Kt}}{2}\Big) \oplus u_1(\frac{e^{Kt}-e^{-Kt}}{2K}) \\ &= \Big(\frac{u_0}{2} \ominus_{gH} \frac{u_1}{2K}\Big)e^{-Kt} \oplus \Big(\frac{u_0}{2} \oplus \frac{u_1}{2K}\Big)e^{Kt}. \end{aligned}$$

Consider $c_1 = \Big(\frac{u_0}{2} \ominus_{gH} \frac{u_1}{2K}\Big)$ and $c_2 = \Big(\frac{u_0}{2} \oplus \frac{u_1}{2K}\Big)$. Therefore,

$$u(t) = c_1 e^{-Kt} \oplus c_2 e^{Kt}.$$

- If $\lambda = 0$, then we have the second order fuzzy differential equation $u''_{gH} = 0$. For solving this equation we use Theorem 2.7,

$$u'_{gH}(t) \ominus_{gH} 1u_1 = 0 \quad \Rightarrow \quad u'_{gH}(t) = u_1,$$

where c_0 is integration constant. Using this Theorem again gets

$$u(t) \ominus_{gH} u_0 = u_1 t,$$

so if $u(t)$ is a $[(i) - gH]-$differentiable function, we have

$$u(t) = u_1 t \oplus u_0.$$

And if $u(t)$ is a $[(ii) - gH]-$differentiable function, we obtain

$$u(t) = u_1 t \ominus (-1)u_0.$$

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