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Finite and Infinite Integral Formulas Involving the Family of Incomplete H-Functions

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Abstract

Recent research focuses on the integral representations of the various type of special functions due to their potential applicability in different disciplines. In this line, we deal with several finite and infinite integrals involving the family of incomplete H-functions. Further, we point out some known and new special cases of these integrals. Finally, we establish the integral representation of incomplete H-functions.

Keywords: Incomplete H-functions; Gamma Function; Incomplete generalized hypergeometric function; Mellin-Barnes type integrals

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1. Introduction

A large number of integral formulas involving the various kind of special functions have been established by several authors (Saxena et al. (2018); Suthar et al. (2019); Kumar et al. (2018); Bansal et al. (2019b); Bansal et al. (2019a); Bansal, Kumar, Khan et al. (2019)). In recent years, these integral formulas have many applications in the potential field of physics, applied sciences, engineering and chemical science. In this article, we deal with those finite and infinite integral formulas which are most general in nature.

2. Preliminaries

Very often used *incomplete Gamma functions* $\Gamma(\mathsf{p},\nu)$ and $\gamma(\mathsf{p},\nu)$ are represented in the following way:

$$\gamma(\mathsf{p},\nu) := \int_0^{\nu} e^{-t} t^{\mathsf{p}-1} dt, \qquad (\Re(\mathsf{p}) > 0; \nu \geq 0), \tag{1}$$

and

$$\Gamma(\mathsf{p},\nu) := \int_{\nu}^{\infty} e^{-t} t^{\mathsf{p}-1} dt, \qquad (\nu \geq 0; \Re(\mathsf{p}) > 0 \quad \text{when} \quad \nu = 0). \tag{2}$$

The *incomplete Gamma functions* $\Gamma(\mathsf{p},\nu)$ and $\gamma(\mathsf{p},\nu)$ are holding the subsequent decomposition relation

$$\Gamma(\mathsf{p},\nu) + \gamma(\mathsf{p},\nu) = \Gamma(\mathsf{p}), \qquad (\Re(\mathsf{p}) > 0). \tag{3}$$

The condition that we have employed on the parameter ν and any place of the present study is unrestrained $\Re(\mathsf{w})$ $(\mathsf{w} \in \mathbb{C})$.

We recall here incomplete $H-$functions (IHF's) $\gamma_{p,q}^{m,n}(\mathsf{w})$ and $\Gamma_{p,q}^{m,n}(\mathsf{w})$ which was introduced by Srivastava et al. (2018), Equations (2.1)-(2.4) in the following manner:

$$\begin{aligned}
\Gamma_{p,q}^{m,n}(\mathsf{w}) &= \Gamma_{p,q}^{m,n}\left[\mathsf{w}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]\\
&= \Gamma_{p,q}^{m,n}\left[\mathsf{w}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_2,\mathcal{G}_2),\cdots,(\mathsf{g}_p,\mathcal{G}_p)\\ \\ (\mathsf{h}_1,\mathsf{H}_1),(\mathsf{h}_2,\mathsf{H}_2),\cdots,(\mathsf{h}_q,\mathsf{H}_q)\end{matrix}\right.\right]\\
&= \frac{1}{2\pi \mathrm{i}}\int_{\mathfrak{L}} f(\xi,\mathsf{u})\ \mathsf{w}^{-\xi} d\xi,
\end{aligned} \tag{4}$$

where

$$f(\xi, \mathsf{u}) = \frac{\Gamma(1-\mathsf{g}_1-\mathcal{G}_1\xi, \mathsf{u}) \prod_{j=1}^{m} \Gamma(\mathsf{h}_j+\mathsf{H}_j\xi) \prod_{j=2}^{n} \Gamma(1-\mathsf{g}_j-\mathcal{G}_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1-\mathsf{h}_j-\mathsf{H}_j\xi) \prod_{j=n+1}^{p} \Gamma(\mathsf{g}_j+\mathcal{G}_j\xi)},$$

$$\begin{aligned}\gamma_{p,q}^{m,n}(\mathsf{w}) &= \gamma_{p,q}^{m,n}\left[\mathsf{w}\,\middle|\, \begin{matrix}(\mathsf{g}_1, \mathcal{G}_1, \mathsf{u}), (\mathsf{g}_j, \mathcal{G}_j)_{2,p} \\ \\ (\mathsf{h}_j, \mathsf{H}_j)_{1,q}\end{matrix}\right] \\ &= \gamma_{p,q}^{m,n}\left[\mathsf{w}\,\middle|\, \begin{matrix}(\mathsf{g}_1, \mathcal{G}_1, \mathsf{u}), (\mathsf{g}_2, \mathcal{G}_2), \cdots, (\mathsf{g}_p, \mathcal{G}_p) \\ \\ (\mathsf{h}_1, \mathsf{H}_1), (\mathsf{h}_2, \mathsf{H}_2), \cdots, (\mathsf{h}_q, \mathsf{H}_q)\end{matrix}\right] \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} F(\xi, \mathsf{u})\ \mathsf{w}^{-\xi} d\xi, \end{aligned} \tag{5}$$

and

$$F(\xi, \mathsf{u}) = \frac{\Gamma(1-\mathsf{g}_1-\mathcal{G}_1\xi, \mathsf{u}) \prod_{j=1}^{m} \Gamma(\mathsf{h}_j+\mathsf{H}_j\xi) \prod_{j=2}^{n} \Gamma(1-\mathsf{g}_j-\mathcal{G}_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1-\mathsf{h}_j-\mathsf{H}_j\xi) \prod_{j=n+1}^{p} \Gamma(\mathsf{g}_j+\mathcal{G}_j\xi)}.$$

If $\mathsf{u} \geq 0$, then the IHF's $\Gamma_{p,q}^{m,n}(\mathsf{w})$ and $\gamma_{p,q}^{m,n}(\mathsf{w})$ are exist and conditions of IHF's are stated in various articles given by Kilbas et al. (2006), Mathai and Saxena (1978), and Mathai et al. (2009). A large number of special cases of IHF's are presented in the articles Srivastava et al. (2018), Bansal, Kumar, Khan et al. (2019), Bansal and Choi (2019), and Bansal et al. (2020).

Next, we call here Euler's Beta and Mellin Transform of IHF which was given by Srivastava et al. (2018) due to evaluate some integrals in Section 2.

Euler's Beta Transform

The *Euler's Beta* transform of incomplete H-function is given by Srivastava et al. (2018), p. 124, Equation (3.12) in the following way:

$$\begin{aligned}&\int_0^t \mathsf{w}^{-\alpha}(t-\mathsf{w})^{\alpha-\beta-1} \Gamma_{p,q}^{m,n}\left[\mathsf{w}\,\middle|\, \begin{matrix}(\mathsf{g}_1, \mathcal{G}_1, \mathsf{u}), (\mathsf{g}_j, \mathcal{G}_j)_{2,p} \\ \\ (\mathsf{h}_j, \mathsf{H}_j)_{1,q}\end{matrix}\right] d\mathsf{w} \\ &= \frac{\Gamma(\alpha-\beta)}{t^{\beta}} \Gamma_{p+1,q+1}^{m,n+1}\left[t\,\middle|\, \begin{matrix}(\mathsf{g}_1, \mathcal{G}_1, \mathsf{u}), (\alpha, 1), (\mathsf{g}_j, \mathcal{G}_j)_{2,p} \\ \\ (\mathsf{h}_j, \mathsf{H}_j)_{1,q}, (\beta, 1)\end{matrix}\right], \end{aligned} \tag{6}$$

provided that conditions are given in Srivastava et al. (2018).

Mellin Transform

The *Mellin* transform of incomplete H-function is given by Srivastava et al. (2018), p. 122, Equation (3.4) in the following way:

$$\mathfrak{M}\left\{\Gamma_{p,q}^{m,n}\left[\mathsf{w}t\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right];\mathfrak{s}\right\}$$
$$=\mathsf{w}^{-\mathfrak{s}}\frac{\Gamma(1-\mathsf{g}_1-\mathcal{G}_1\mathfrak{s},\mathsf{u})\prod_{j=1}^{m}\Gamma(\mathsf{h}_j+\mathsf{H}_j\mathfrak{s})\prod_{j=2}^{n}\Gamma(1-\mathsf{g}_j-\mathcal{G}_j\mathfrak{s})}{\prod_{j=m+1}^{q}\Gamma(1-\mathsf{h}_j-\mathsf{H}_j\mathfrak{s})\prod_{j=n+1}^{p}\Gamma(\mathsf{g}_j+\mathcal{G}_j\mathfrak{s})}, \tag{7}$$

provided that conditions are given in Srivastava et al. (2018).

3. Main Results

In this section, we evaluate several finite and infinite integrals associated with family of incomplete $H-$functions (4) and (5).

Theorem 3.1.

If

$$\lambda>0,\qquad \mathsf{u}\geq 0,\qquad \mathsf{v}\geq 0,$$
$$\lambda\max_{1\leq j\leq n}\Re\left\{\frac{\mathsf{g}_j-1}{\mathcal{G}_j}\right\}-\min_{1\leq j\leq M}\Re\left\{\frac{b_j}{B_j}\right\}<\Re(\alpha)<\lambda\min_{1\leq j\leq m}\Re\left\{\frac{\mathsf{h}_j}{\mathsf{H}_j}\right\}+\max_{1\leq j\leq N}\Re\left\{\frac{1-a_j}{A_j}\right\},$$

then the following improper integral holds:

$$\int_0^\infty t^{\alpha-1}\Gamma_{p,q}^{m,n}\left[\mathsf{w}t^{-\lambda}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]\Gamma_{P,Q}^{M,N}\left[\mathfrak{s}t\left|\begin{matrix}(a_1,A_1,\mathsf{v}),(a_j,A_j)_{2,P}\\ \\ (b_j,B_j)_{1,Q}\end{matrix}\right.\right]dt$$
$$=\mathfrak{s}^{-\alpha}\Gamma_{p+P,q+Q}^{m+M,n+N}\left[\mathsf{w}\mathfrak{s}^{\lambda}\left|\begin{matrix}A^*\\ \\ B^*\end{matrix}\right.\right], \tag{8}$$

and

$$\int_0^\infty t^{\alpha-1}\gamma_{p,q}^{m,n}\left[\mathsf{w}t^{-\lambda}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]\gamma_{P,Q}^{M,N}\left[\mathfrak{s}t\left|\begin{matrix}(a_1,A_1,\mathsf{v}),(a_j,A_j)_{2,P}\\ \\ (b_j,B_j)_{1,Q}\end{matrix}\right.\right]dt$$

$$=\mathfrak{s}^{-\alpha}\gamma_{p+P,q+Q}^{m+M,n+N}\left[\mathsf{w}\mathfrak{s}^{\lambda}\left|\begin{matrix}A^*\\ \\ B^*\end{matrix}\right.\right], \tag{9}$$

where

$$A^*=(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(a_1+\alpha A_1,\lambda A_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,n},(a_j+\alpha A_j,\lambda A_j)_{2,P},(\mathsf{g}_j,\mathcal{G}_j)_{n+1,p},$$
$$B^*=(\mathsf{h}_j,\mathsf{H}_j)_{1,m},(b_j+\alpha B_j,\lambda B_j)_{1,Q},(\mathsf{h}_j,\mathsf{H}_j)_{m+1,q},$$

provided that conditions of IHF's $\Gamma_{p,q}^{m,n}(\mathsf{w})$ and $\gamma_{p,q}^{m,n}(\mathsf{w})$ in (4) and (5) are satisfied.

Proof:

To prove the assertion (8), firstly, we write the Mellin-Barnes contour integral form of IHF with the help of (4), we get (say Δ)

$$\Delta=\int_0^\infty t^{\alpha-1}\Gamma_{P,Q}^{M,N}\left[\mathfrak{s}t\left|\begin{matrix}(a_1,A_1,\mathsf{v}),(a_j,A_j)_{2,P}\\ \\ (b_j,B_j)_{1,Q}\end{matrix}\right.\right]\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}}f(\xi,\mathsf{u})(\mathsf{w}t^{-\lambda})^{-\xi}d\xi\right]dt.$$

Further, changing the order of integration and with the help of (7), we obtain

$$\Delta=\frac{\mathfrak{s}^{-\alpha}}{2\pi\mathrm{i}}\int_{\mathfrak{L}}f(\xi,\mathsf{u})(\mathsf{w}\mathfrak{s}^{\lambda})^{-\xi}$$
$$\times\frac{\Gamma(1-a_1-A_1(\alpha+\lambda\xi))\prod_{j=1}^{M}\Gamma(b_j+B_j(\alpha+\lambda\xi))\prod_{j=2}^{N}\Gamma(1-a_j-A_j(\alpha+\lambda\xi))}{\prod_{j=M+1}^{Q}\Gamma(1-b_j-B_j(\alpha+\lambda\xi))\prod_{j=N+1}^{P}\Gamma(a_j+A_j(\alpha+\lambda\xi))}d\xi.$$

With the help of Equation (4), we get the required result. ∎

Theorem 3.2.

If

$$\lambda>0,\qquad \beta>0,\qquad \mathsf{u}\geq 0,$$
$$-\lambda\min_{1\leq j\leq m}\Re\left\{\frac{\mathsf{h}_j}{\mathsf{H}_j}\right\}<\Re(\alpha)<\lambda\max_{1\leq j\leq n}\Re\left\{\frac{1-\mathsf{g}_j}{\mathcal{G}_j}\right\},$$

then the following improper integral holds:

$$\int_0^\infty t^{\alpha-1}(t+\mathfrak{a})^{-\beta}\Gamma_{p,q}^{m,n}\left[\mathsf{w}t^\lambda \,\middle|\, \begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right]dt$$
$$=\frac{\mathfrak{a}^{\alpha-\beta}}{\Gamma(\beta)}\Gamma_{p+1,q+1}^{m+1,n+1}\left[\mathsf{w}\mathfrak{a}^\lambda \,\middle|\, \begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(1-\alpha,\lambda),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\beta-\alpha,\lambda),(\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right], \tag{10}$$

and

$$\int_0^\infty t^{\alpha-1}(t+\mathfrak{a})^{-\beta}\gamma_{p,q}^{m,n}\left[\mathsf{w}t^\lambda \,\middle|\, \begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right]dt$$
$$=\frac{\mathfrak{a}^{\alpha-\beta}}{\Gamma(\beta)}\gamma_{p+1,q+1}^{m+1,n+1}\left[\mathsf{w}\mathfrak{a}^\lambda \,\middle|\, \begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(1-\alpha,\lambda),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\beta-\alpha,\lambda),(\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right], \tag{11}$$

provided that condition of incomplete $H-$functions $\Gamma_{p,q}^{m,n}(z)$ and $\gamma_{p,q}^{m,n}(z)$ in (4) and (5) are satisfied.

Proof:

To prove the result (10), firstly, we write the Mellin-Barnes contour integral form of IHF with the help of (4), we get (say Ω)

$$\Omega=\int_0^\infty t^{\alpha-1}(t+\mathfrak{a})^{-\beta}\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}} f(\xi,\mathsf{u})(\mathsf{w}t^\lambda)^{-\xi}d\xi\right]dt.$$

Further, changing the order of integration and with the help of Beta function definition, we obtained

$$\Omega=\frac{\mathfrak{a}^{\alpha-\beta}}{\Gamma(\beta)}\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}} f(\xi,\mathsf{u})\Gamma(\alpha-\lambda\xi)\Gamma(\beta-\alpha+\lambda\xi)(\mathsf{w}\mathfrak{a}^\lambda)^{-\xi}d\xi.$$

With the help of equation (4), we get the required result. ■

Theorem 3.3.

If

$$\lambda>0, \qquad \mu>0, \qquad \mathsf{u}\geq 0,$$
$$\Re(\alpha)+\lambda\min_{1\leq j\leq m}\Re\left(\frac{\mathsf{h_j}}{\mathsf{H}_j}\right)>0, \qquad \Re(\beta)+\mu\min_{1\leq j\leq m}\Re\left(\frac{\mathsf{h_j}}{\mathsf{H}_j}\right)>0,$$

then the following improper integral is hold:

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{\lambda}(t-x)^{\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=t^{\alpha+\beta-1}\Gamma_{p+2,q+1}^{m,n+2}\left[\mathsf{w}t^{\lambda+\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(1-\alpha,\lambda),(1-\beta,\mu),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\beta-\alpha,\lambda),(\mathsf{h}_j,\mathsf{H}_j)_{1,q},(1-\alpha-\beta,\lambda+\mu)\end{matrix}\right.\right], \tag{12}$$

and

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\gamma_{p,q}^{m,n}\left[\mathsf{w}x^{\lambda}(t-x)^{\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=t^{\alpha+\beta-1}\gamma_{p+2,q+1}^{m,n+2}\left[\mathsf{w}t^{\lambda+\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(1-\alpha,\lambda),(1-\beta,\mu),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\beta-\alpha,\lambda),(\mathsf{h}_j,\mathsf{H}_j)_{1,q},(1-\alpha-\beta,\lambda+\mu)\end{matrix}\right.\right], \tag{13}$$

provided that the condition of incomplete $H-$functions $\Gamma_{p,q}^{m,n}(\mathsf{w})$ and $\gamma_{p,q}^{m,n}(\mathsf{w})$ in (4) and (5) are satisfied.

Proof:

To prove the assertion (12), firstly, we write the Mellin-Barnes contour integral form of IHF with the help of (4), we get (say Ξ)

$$\Xi=\int_0^t x^{\alpha-1}(t-x)^{-\beta}\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}} f(\xi,\mathsf{u})(\mathsf{w}x^{\lambda}(t-x)^{\mu})^{-\xi}d\xi\right]dt.$$

Further, changing the order of integration and with the help of Beta function definition, we obtained

$$\Xi=\frac{t^{\alpha+\beta-1}}{2\pi\mathrm{i}}\int_{\mathfrak{L}} f(\xi,\mathsf{u})\frac{\Gamma(\alpha-\lambda\xi)\Gamma(\beta-\mu\xi)}{\Gamma(\alpha+\beta-(\lambda+\mu)\xi)}(\mathsf{w}t^{\lambda+\mu})^{-\xi}d\xi.$$

With the help of (4), we get the required result. ■

Theorem 3.4.

If

$$\rho>0,\qquad \sigma>0,\qquad \mathsf{u}\geq 0,\qquad \mathsf{v}\geq 0,\qquad \lambda+\mu\geq 1$$
$$\Re(\alpha)+\lambda\min_{1\leq j\leq m}\Re\left(\frac{\mathsf{h_j}}{\mathsf{H}_j}\right)>0,\qquad \Re(\beta)+\mu\min_{1\leq j\leq m}\Re\left(\frac{\mathsf{h_j}}{\mathsf{H}_j}\right)>0,$$

then the following improper integral is hold:

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_P\Gamma_Q\left[\begin{matrix}(a_1,\mathsf{u}),a_2,\cdots,a_P;\\ b_1,\cdots,b_Q;\end{matrix}\ \mathfrak{a}x^{\lambda}(t-x)^{\mu}\right]$$

$$\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{\rho}(t-x)^{\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx=t^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}$$

$$\times\sum_{r=0}^{\infty}f(r)\Gamma_{p+2,q+1}^{m,n+2}\left[\mathsf{w}t^{\rho+\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(1-\alpha-\lambda r,\rho),(1-\beta-\mu r,\sigma),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q},(1-\alpha-\beta-(\mu+\lambda)r,\rho+\sigma)\end{matrix}\right.\right]t^{(\lambda+\mu)r},\tag{14}$$

where

$$f(r)=\frac{\Gamma(a_1+r,\mathsf{u})\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{\mathfrak{a}^r}{r!},$$

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}{}_P\gamma_Q\left[\begin{matrix}(a_1,\mathsf{u}),a_2,\cdots,a_P;\\ b_1,\cdots,b_Q;\end{matrix}\ \mathfrak{a}x^{\lambda}(t-x)^{\mu}\right]$$

$$\gamma_{p,q}^{m,n}\left[\mathsf{w}x^{\rho}(t-x)^{\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx=t^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}$$

$$\times\sum_{r=0}^{\infty}g(r)\gamma_{p+2,q+1}^{m,n+2}\left[\mathsf{w}t^{\rho+\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(1-\alpha-\lambda r,\rho),(1-\beta-\mu r,\sigma),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q},(1-\alpha-\beta-(\mu+\lambda)r,\rho+\sigma)\end{matrix}\right.\right]t^{(\lambda+\mu)r},\tag{15}$$

and

$$g(r)=\frac{\gamma(a_1+r,\mathsf{u})\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{\mathfrak{a}^r}{r!},$$

provided that the condition of incomplete $H-$functions $\Gamma_{p,q}^{m,n}(\mathsf{w})$ and $\gamma_{p,q}^{m,n}(\mathsf{w})$ in (4) and (5) are satisfied.

Proof:

To prove the assertion (14), firstly, we express the incomplete generalized hypergeometric function in series form, we get(say Υ)

$$\Upsilon = \int_0^t x^{\alpha-1}(t-x)^{\beta-1}\left[\frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}\sum_{r=0}^{\infty}\frac{\Gamma(a_1+r,\mathsf{u})\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{(\mathfrak{a}x^{\lambda}(t-x)^{\mu})^r}{r!}\right]$$
$$\times\,\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{\rho}(t-x)^{\sigma}\,\middle|\,\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right]dx.$$

Further, changing the order of integration and summation (under the permissible conditions) and with the help of (12), we obtain

$$\Upsilon = \frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}\sum_{r=0}^{\infty}\frac{\Gamma(a_1+r,\mathsf{u})\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{\mathfrak{a}^r}{r!}$$
$$\times\int_0^t x^{\alpha+\lambda r-1}(t-x)^{\beta+\mu r-1}\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{\rho}(t-x)^{\sigma}\,\middle|\,\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right]dx.$$

Next, with the help of (12), we get the required result after a little simplification. ■

Theorem 3.5.

$$\lambda>0, \qquad \mu>0, \qquad \mathsf{u}\geq 0,$$
$$\Re(\alpha)-\lambda\max_{1\leq j\leq n}\Re\left\{\frac{\mathsf{g}_j-1}{\mathcal{G}_j}\right\}>0, \qquad \Re(\beta)-\mu\max_{1\leq j\leq n}\Re\left\{\frac{\mathsf{g}_j-1}{\mathcal{G}_j}\right\}>0,$$

then the following improper integral is hold:

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{-\lambda}(t-x)^{-\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=t^{\alpha+\beta-1}\Gamma_{p+1,q+2}^{m+2,n}\left[\mathsf{w}t^{-\lambda-\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p},(\alpha+\beta,\mu+\lambda)\\ \\ (\alpha,\lambda),(\beta,\mu),(\mathsf{h}_j,\mathsf{H}_j)_{1,q},\end{matrix}\right.\right], \tag{16}$$

and

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\gamma_{p,q}^{m,n}\left[\mathsf{w}x^{-\lambda}(t-x)^{-\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx$$
$$=t^{\alpha+\beta-1}\gamma_{p+1,q+2}^{m+2,n}\left[\mathsf{w}t^{-\lambda-\mu}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p},(\alpha+\beta,\mu+\lambda)\\ \\ (\alpha,\lambda),(\beta,\mu),(\mathsf{h}_j,\mathsf{H}_j)_{1,q},\end{matrix}\right.\right], \tag{17}$$

provided that the condition of incomplete $H-$functions $\Gamma_{p,q}^{m,n}(\mathsf{w})$ and $\gamma_{p,q}^{m,n}(\mathsf{w})$ in (4) and (5) are satisfied.

Proof:

To prove the assertion (16), firstly, we write the Mellin-Barnes contour integral form of IHF with the help of (4), we get (say Ξ)

$$\Xi=\int_0^t x^{\alpha-1}(t-x)^{-\beta}\left[\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}}f(\xi,\mathsf{u})(\mathsf{w}x^{-\lambda}(t-x)^{-\mu})^{-\xi}d\xi\right]dt.$$

Further, changing the order of integration and with the help of Beta function definition, we obtained

$$\Xi=\frac{t^{\alpha+\beta-1}}{2\pi\mathrm{i}}\int_{\mathfrak{L}}f(\xi,\mathsf{u})\frac{\Gamma(\alpha+\lambda\xi)\Gamma(\beta+\mu\xi)}{\Gamma(\alpha+\beta+(\lambda+\mu)\xi)}(\mathsf{w}t^{-\lambda-\mu})^{-\xi}d\xi.$$

With the help of equation (4), we get the required result. ■

Theorem 3.6.

$$\rho>0,\qquad \sigma>0,\qquad \mathsf{u}\geq 0,\qquad \mathsf{v}\geq 0,\qquad \lambda+\mu\geq 1,$$
$$\Re(\alpha)-\rho\max_{1\leq j\leq n}\Re\left\{\frac{\mathsf{g}_j-1}{\mathcal{G}_j}\right\}>0,\qquad \Re(\beta)-\sigma\max_{1\leq j\leq n}\Re\left\{\frac{\mathsf{g}_j-1}{\mathcal{G}_j}\right\}>0,$$

then the following improper integral is hold:

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1} {}_P\Gamma_Q \left[\begin{matrix} (a_1,\mathsf{u}), a_2, \cdots, a_P; \\ b_1, \cdots, b_Q; \end{matrix} \mathfrak{a} x^{\lambda}(t-x)^{\mu} \right]$$

$$\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{-\rho}(t-x)^{-\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right] dx = t^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}$$

$$\times\sum_{r=0}^{\infty} f(r)\Gamma_{p+1,q+2}^{m+2,n}\left[\mathsf{w}t^{-\rho-\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p},(\alpha+\beta+(\mu+\lambda)r,\rho+\sigma)\\ (\alpha+\lambda r,\rho),(\beta+\mu r,\sigma),(\mathsf{h}_j,\mathsf{H}_j)_{1,q},\end{matrix}\right.\right] t^{(\lambda+\mu)r}, \quad (18)$$

where

$$f(r)=\frac{\Gamma(a_1+r,u)\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{\mathfrak{a}^r}{r!},$$

and

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1} {}_P\gamma_Q \left[\begin{matrix} (a_1,\mathsf{u}), a_2, \cdots, a_P; \\ b_1, \cdots, b_Q; \end{matrix} \mathfrak{a} x^{\lambda}(t-x)^{\mu} \right]$$

$$\gamma_{p,q}^{m,n}\left[\mathsf{w}x^{-\rho}(t-x)^{-\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right] dx = t^{\alpha+\beta-1}\frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}$$

$$\times\sum_{r=0}^{\infty} g(r)\cdot\gamma_{p+1,q+2}^{m+2,n}\left[\mathsf{w}t^{-\rho-\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p},(\alpha+\beta+(\mu+\lambda)r,\rho+\sigma)\\ (\alpha+\lambda r,\rho),(\beta+\mu r,\sigma),(\mathsf{h}_j,\mathsf{H}_j)_{1,q},\end{matrix}\right.\right] t^{(\lambda+\mu)r}, \quad (19)$$

where

$$g(r)=\frac{\Gamma(a_1+r,u)\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{\mathfrak{a}^r}{r!},$$

provided that the condition of incomplete $H-$functions $\Gamma_{p,q}^{m,n}(\mathsf{w})$ and $\gamma_{p,q}^{m,n}(\mathsf{w})$ in (4) and (5) are satisfied.

Proof:

To prove the assertion (18), firstly, we express the incomplete generalized hypergeometric function in series form, we get(say Υ)

$$\Upsilon = \int_0^t x^{\alpha-1}(t-x)^{\beta-1}\left[\frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}\sum_{r=0}^{\infty}\frac{\Gamma(a_1+r,\mathsf{u})\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{(\mathfrak{a}x^{\lambda}(t-x)^{\mu})^r}{r!}\right]$$
$$\times\,\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{-\rho}(t-x)^{-\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx.$$

Further, changing the order of integration and summation (under the permissible conditions) and with the help of (12), we obtained

$$\Upsilon = \frac{\prod_{j=1}^{Q}\Gamma(b_j)}{\prod_{j=1}^{P}\Gamma(a_j)}\sum_{r=0}^{\infty}\frac{\Gamma(a_1+r,\mathsf{u})\prod_{j=2}^{P}\Gamma(a_j+r)}{\prod_{j=1}^{Q}\Gamma(b_j+r)}\frac{\mathfrak{a}^r}{r!}$$
$$\times\int_0^t x^{\alpha+\lambda r-1}(t-x)^{\beta+\mu r-1}\Gamma_{p,q}^{m,n}\left[\mathsf{w}x^{-\rho}(t-x)^{-\sigma}\left|\begin{matrix}(\mathsf{g}_1,\mathcal{G}_1,\mathsf{v}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dx.$$

Next, with the help of (16), we get the required result after a little simplification. ■

4. Integral representation of incomplete H-functions

Recently, Srivastava et al. (2012) established the integral representation of the incomplete Gauss hypergeometric functions. So, motivated by the work of Srivastava et al. (2012), we give the integral representation of incomplete $H-$functions (4) and (5).

Theorem 4.1.

If $\mathsf{u}\geq 0$ and $\Re(\mathsf{g}_1)>0$, then the following integral representation formula holds:

$$\Gamma_{p,q}^{m,n}\left[\mathsf{w}\left|\begin{matrix}(1-\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]=\int_{\mathsf{u}}^{\infty}t^{\mathsf{g}_1-1}e^{-t}\mathcal{H}_{p-1,q}^{m,n-1}\left[\mathsf{w}t^{\mathcal{G}_1}\left|\begin{matrix}(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right]dt. \qquad (20)$$

Proof:

To prove the assertion (20), we write the Mellin-Barnes contour integral form of well known $\mathcal{H}-$function and then with the help of (2), we get the desired result. ∎

Theorem 4.2.

If $\mathsf{u} \geq 0$ and $\Re(\mathsf{g}_1) > 0$, then the following integral representation formula holds:

$$\gamma_{p,q}^{m,n}\left[\mathsf{w}\left|\begin{matrix}(1-\mathsf{g}_1,\mathcal{G}_1,\mathsf{u}),(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right] = \int_0^{\mathsf{u}} t^{\mathsf{g}_1-1}e^{-t}\mathcal{H}_{p-1,q}^{m,n-1}\left[\mathsf{w}t^{\mathcal{G}_1}\left|\begin{matrix}(\mathsf{g}_j,\mathcal{G}_j)_{2,p}\\ \\ (\mathsf{h}_j,\mathsf{H}_j)_{1,q}\end{matrix}\right.\right] dt. \qquad (21)$$

Proof:

To prove the assertion (21), we write the Mellin-Barnes contour integral form of well known $\mathcal{H}-$function and then with the help of (1), we get the desired result. ∎

Remark 4.3.

If incomplete H-function reduces to the familiar of Fox's $\mathcal{H}$-function and incomplete generalized hypergeometric function reduce into generalised hypergeometric function, then results are recorded in the text book of Srivastava et al. (1982).

5. Conclusions

Many authors produced a large number of research articles on the integral formulas involving the various kind of special functions. In this regard, we deal with several finite and infinite integrals involving the family of incomplete $H-$functions(IHF's). Further, we point out some known and new special cases of these integrals. Finally, we established the integral representation of IHF's.

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