Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Special Issue No. 6 (April 2020), pp. 1 – 14

Certain Mathieu-type Series Pertaining to Incomplete H-Functions

[1]Nidhi Jolly, [2,*]Manish Kumar Bansal, [3]Devendra Kumar and [4]Jagdev Singh

[1]Department of Mathematics
Malaviya National Institute of Technology
Jaipur-302017, Rajasthan, India
nidhinj6@gmail.com

[2]Department of Applied Sciences
Government Engineering College
Banswara-327001, Rajasthan, India
bansalmanish443@gmail.com

[3]Department of Mathematics
University of Rajasthan
Jaipur 302004, Rajasthan, India
devendra.maths@gmail.com

[4]Department of Mathematics
JECRC University
Jaipur 303905, Rajasthan, India
jagdevsinghrathore@gmail.com

*Corresponding Author

Received: February 29, 2020; Accepted: March 15, 2020

Abstract

In the present article, we derive closed integral form expressions for a family of convergent Mathieu type a-series along with its alternating variants, whose terms contain incomplete H-functions, which are a notable generalization of familiar H-function. The results established herewith are very general in nature and provide an exquisite generalization of closed integral form expressions of aforementioned series whose terms contain H-function and Fox-Wright function, respectively. Next, we present some new and interesting special cases of our main results.

Keywords: Incomplete H-functions; Gamma function; Incomplete Gamma functions; Mellin-Barnes type contour

MSC 2010 No.: Primary 33B20, 44A10; Secondary 33E20, 44A40

1. Introduction

In order to consolidate and broaden the results for the Mathieu-type a-series whose terms contain the known functions namely Fox's H-function (HF), the Fox-Wright function (FWF) ${}_p\Psi_q$, generalized hypergeometric function(GHF) ${}_pF_q$, etc., studied in numerous papers by Srivastava and Tomovski (2004), Tomovski (2009), Tomovski and Tuan (2009), Pogány (2004, 2005, 2007), the authors introduce the Mathieu-type a-series and its alternative variant whose terms contain incomplete H-Functions (IHF's) $\Gamma^{m,n}_{p,q}(z)$ and $\gamma^{m,n}_{p,q}(z)$. IHF's $\Gamma^{m,n}_{p,q}(z)$ and $\gamma^{m,n}_{p,q}(z)$ are a notable generalization of well known H-function (HF). Most recently a number of extensions and generalizations of Mathieu-type a-series have been studied by several authors, namely Srivastava et al. (2018), Mehrez and Sitnik (2019), Mehrez and Tomovski (2019), Gerhold and Tomovski (2019), Choi et al. (2017), and Tomovski and Mehrez (2017). The results obtained in this paper serve the purpose of formulating key formulas for special functions with utility in Engineering, Science, and Technology scattered across the literature (see, for details, Bansal et al. (2019, 2019, 2019, 2020, 2019)). Numerous applications of IHF's $\Gamma^{m,n}_{p,q}(z)$ and $\gamma^{m,n}_{p,q}(z)$ can be seen in the analytic study of cumulative and survival probability density functions (in probability theory) along the lines previously investigated by Chaudhry and Qadir (2002), making use of incomplete exponential functions.

2. Preliminaries

We recall here frequently used *incomplete Gamma functions* (IGF) $\gamma(s,y)$ and $\Gamma(s,y)$ given by

$$\gamma(s,y)=\int_0^y t^{s-1}e^{-t}dt, \qquad (\Re(s)>0; y\geq 0), \tag{1}$$

and

$$\Gamma(s,y)=\int_y^\infty t^{s-1}e^{-t}dt, \qquad (y\geq 0; \Re(s)>0 \quad \text{when} \quad y=0), \tag{2}$$

respectively. Furthermore, Equations (1) and (2) satisfy the following decomposition formula:

$$\Gamma(s,y)+\gamma(s,y)=\Gamma(s), \qquad (\Re(s)>0). \tag{3}$$

The condition used on the parameter y throughout the current paper is unrestrained of $\Re(z)$ $(z\in\mathbb{C})$.

Incomplete generalized hypergeometric functions (IGHF) ${}_p\Gamma_q$ and ${}_p\gamma_q$ were defined in terms of IGF $\Gamma(s,y)$ and $\gamma(s,y)$ by Srivastava et al. (2012) with the help of Mellin-Barnes type integrals as follows:

$$
{}_p\Gamma_q\left[\begin{array}{r}(e_1,y)\,,e_2,\cdots,e_p;\\ \quad z\\ f_1,\cdots,f_q;\end{array}\right]=\frac{\prod\limits_{j=1}^{q}\Gamma(f_j)}{\prod\limits_{j=1}^{p}\Gamma(e_j)}\sum_{l=0}^{\infty}\frac{\Gamma(e_1+l,y)\prod\limits_{j=2}^{p}\Gamma(e_j+l)}{\prod\limits_{j=1}^{q}\Gamma(f_j+l)}\frac{z^l}{l!}
$$

$$
=\frac{1}{2\Pi i}\frac{\prod\limits_{j=1}^{q}\Gamma(f_j)}{\prod\limits_{j=1}^{p}\Gamma(e_j)}\int\limits_{\mathfrak{L}}\frac{\Gamma(e_1+s,y)\prod\limits_{j=2}^{p}\Gamma(e_j+s)}{\prod\limits_{j=1}^{q}\Gamma(f_j+s)}\Gamma(-s)(-z)^s ds, \tag{4}
$$

$$
(|arg(-z)|<\pi)
$$

and

$$
{}_p\gamma_q\left[\begin{array}{r}(e_1,y)\,,e_2,\cdots,e_p;\\ \quad z\\ f_1,\cdots,f_q;\end{array}\right]=\frac{\prod\limits_{j=1}^{q}\Gamma(f_j)}{\prod\limits_{j=1}^{p}\Gamma(e_j)}\sum_{l=0}^{\infty}\frac{\gamma(e_1+l,y)\prod\limits_{j=2}^{p}\Gamma(e_j+l)}{\prod\limits_{j=1}^{q}\Gamma(f_j+l)}\frac{z^l}{l!}
$$

$$
=\frac{1}{2\Pi i}\frac{\prod\limits_{j=1}^{q}\Gamma(f_j)}{\prod\limits_{j=1}^{p}\Gamma(e_j)}\int\limits_{\mathfrak{L}}\frac{\gamma(e_1+s,y)\prod\limits_{j=2}^{p}\Gamma(e_j+s)}{\prod\limits_{j=1}^{q}\Gamma(f_j+s)}\Gamma(-s)(-z)^s ds, \tag{5}
$$

$$
(|arg(-z)|<\pi)
$$

where $\mathfrak{L}$ represents the contour integral, which starts at $\tau - i\infty$ and ends at $\tau + i\infty$ $(\tau \in \Re)$.

Very recently Srivastava et al. (2018) established and studied the IHF's $\Gamma^{m,n}_{p,q}(z)$ and $\gamma^{m,n}_{p,q}(z)$ as follows:

$$
\Gamma^{m,n}_{p,q}(z)=\Gamma^{m,n}_{p,q}\left[z\,\middle|\,\begin{array}{c}(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{array}\right]
$$

$$
=\Gamma^{m,n}_{p,q}\left[z\,\middle|\,\begin{array}{c}(e_1,E_1,y),(e_2,E_2),\cdots,(e_p,E_p)\\ \\ (f_1,F_1),(f_2,F_2),\cdots,(f_q,F_q)\end{array}\right]
$$

$$
=\frac{1}{2\pi i}\int\limits_{\mathfrak{L}} f(s,y)\ z^{-s}ds, \tag{6}
$$

where

$$
f(s,y)=\frac{\Gamma(1-e_1-E_1s,y)\prod\limits_{j=1}^{m}\Gamma(f_j+F_js)\prod\limits_{j=2}^{n}\Gamma(1-e_j-E_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-f_j-F_js)\prod\limits_{j=n+1}^{p}\Gamma(e_j+E_js)}.
$$

and

$$\gamma_{p,q}^{m,n}(z) = \gamma_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_1, E_1, y), (e_j, E_j)_{2,p} \\ \\ (f_j, F_j)_{1,q} \end{matrix} \right. \right]$$
$$= \gamma_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_1, E_1, y), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), (f_2, F_2), \cdots, (f_q, F_q) \end{matrix} \right. \right]$$
$$= \frac{1}{2\pi i} \int_{\mathfrak{L}} F(s, y)\ z^{-s} ds, \tag{7}$$

where

$$F(s, y) = \frac{\gamma(1 - e_1 - E_1 s, y) \prod_{j=1}^{m} \Gamma(f_j + F_j s) \prod_{j=2}^{n} \Gamma(1 - e_j - E_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - f_j - F_j s) \prod_{j=n+1}^{p} \Gamma(e_j + E_j s)}.$$

The IHF's $\Gamma_{p,q}^{m,n}(z)$ and $\gamma_{p,q}^{m,n}(z)$ exists for all $y \geq 0$ given by (6) and (7), respectively, under the same set of conditions which were given in the articles (see, for details, Kilbas et al. (2006), Mathai and Saxena (1978), Mathai et al. (2009)).

The above-mentioned IHF's have a large number of special cases out of which some of them are presented as follows:

(1) Considering $y = 0$ in (6), IHF $\Gamma_{p,q}^{m,n}(z)$ will reduce to the frequently used HF (Srivastava et al. (1982)) as follows:

$$\Gamma_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_1, E_1, 0), (e_j, E_j)_{2,p} \\ \\ (f_j, F_j)_{1,q} \end{matrix} \right. \right] = H_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_j, E_j)_{1,p} \\ \\ (f_j, F_j)_{1,q} \end{matrix} \right. \right]. \tag{8}$$

(2) Taking suitable parameters in (6) and (7), then IHF's will reduce to IFWF's ${}_p\Psi_q^{(\Gamma)}$ and ${}_p\Psi_q^{(\gamma)}$ (see Srivastava et al. (2018) [p. 132, Equations (6.3) and (6.4)]:

$$\Gamma_{p,q+1}^{1,p}\left[-z \left| \begin{matrix} (1 - e_1, E_1, y), (1 - e_j, E_j)_{2,p} \\ \\ (0, 1), (1 - f_j, F_j)_{1,q} \end{matrix} \right. \right] = {}_p\Psi_q^{(\Gamma)}\left[\begin{matrix} (e_1, E_1, y), (e_j, E_j)_{2,p}; \\ \\ (f_j, F_j)_{1,q}; \end{matrix} \ z \right]. \tag{9}$$

and

$$\gamma_{p,q+1}^{1,p}\left[-z \left| \begin{matrix} (1 - e_1, E_1, y), (1 - e_j, E_j)_{2,p} \\ \\ (0, 1), (1 - f_j, F_j)_{1,q} \end{matrix} \right. \right] = {}_p\Psi_q^{(\gamma)}\left[\begin{matrix} (e_1, E_1, y), (e_j, E_j)_{2,p}; \\ \\ (f_j, F_j)_{1,q}; \end{matrix} \ z \right]. \tag{10}$$

(3) Further, considering $y = 0$ in (9) IFWF ${}_p\Psi_q^{(\Gamma)}$ will reduce to prominent FWF ${}_p\Psi_q$ (see Srivastava et al. (1982) [p. 39, Equation (2.6.11)]):

$$ {}_p\Psi_q^{(\Gamma)}\left[\begin{matrix}(e_1,E_1,0),(e_j,E_j)_{2,p}; \\ \\ (f_j,F_j)_{1,q};\end{matrix}\; z\right] = {}_p\Psi_q\left[\begin{matrix}(e_j,E_j)_{1,p}; \\ \\ (f_j,F_j)_{1,q};\end{matrix}\; z\right]. \tag{11}$$

(4) Again, small adjustment in the parameters of IFWF's (9) and (10), then it will be reduce to IGHF ${}_p\gamma_q$ and ${}_p\Gamma_q$(see Srivastava et al. (2012)):

$$ {}_p\Psi_q^{(\Gamma)}\left[\begin{matrix}(e_1,1,y),(e_j,1)_{2,p}; \\ \\ (f_j,1)_{1,q};\end{matrix}\; z\right] = {}_p\Gamma_q\left[\begin{matrix}(e_1,y),e_2,\cdots,e_p; \\ \\ f_1,\cdots,f_q;\end{matrix}\; z\right]. \tag{12}$$

and

$$ {}_p\Psi_q^{(\gamma)}\left[\begin{matrix}(e_1,1,y),(e_j,1)_{2,p}; \\ \\ (f_j,1)_{1,q};\end{matrix}\; z\right] = {}_p\gamma_q\left[\begin{matrix}(e_1,y),e_2,\cdots,e_p; \\ \\ f_1,\cdots,f_q;\end{matrix}\; z\right]. \tag{13}$$

(5) Small adjustment in the parameters of IHF $\Gamma_{p,q}^{m,n}(z)$, then it will reduce to multi index Mittag-Leffler function(MIMLF) $E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[z]$ (Saxena and Nishimoto (2010, 2010), see also Srivastava et al. (2018), Bansal and Choi (2019)):

$$\Gamma_{1,m+1}^{1,1}\left[-z\,\middle|\,\begin{matrix}(1-\gamma,\kappa,0) \\ \\ (0,1),(1-\beta_j,\alpha_j)_{1,m}\end{matrix}\right] = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_m\left[\begin{matrix}(\gamma,\kappa); \\ \\ (\beta_j,\alpha_j)_{1,m};\end{matrix}\; z\right] = E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[z], \tag{14}$$

where MIMLF is defined as follows:

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}[z] = E_{\gamma,\kappa}[(\alpha_j,\beta_j)_m;z] = \sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}\frac{z^n}{n!}, \tag{15}$$

$$\left(\alpha_j,\beta_j,\gamma,\kappa,z\in\mathbb{C},;\ \Re\Big(\sum_{j=1}^{m}\alpha_j\Big) > \max\{0,\Re(\kappa)-1\};\ \Re(\beta_j)>0\ (j=1,...,m)\right).$$

Now, by considering the IHF's $\Gamma_{p,q}^{m,n}(z)$ and $\gamma_{p,q}^{m,n}(z)$ input-kernel in place of HF and FWF ${}_p\Psi_q$ previously taken into account by Pogány (2007) and Pogány and Saxena (2011) respectively. We define Mathieu-type a-series $\Theta_{\lambda,\mu}$ and $\Omega_{\lambda,\mu}$ along with it's alternating variants $\tilde{\Theta}_{\lambda,\mu}$ and $\tilde{\Omega}_{\lambda,\mu}$ by the following series:

$$\Theta_{\lambda,\mu}\{\Gamma_{p+1,q}^{m,n+1};c,r\} = \sum_{j=1}^{\infty} \frac{\Gamma_{p+1,q}^{m,n+1}\left[\frac{r}{c_j}\left|\begin{matrix}(\alpha,\beta),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right.\right]}{c_j^{\lambda}(c_j+r)^{\mu}}, \tag{16}$$

$$(y \geq 0, \lambda, \mu, r \in \mathbb{R}^+)$$

$$\tilde{\Theta}_{\lambda,\mu}\{\Gamma_{p+1,q}^{m,n+1};c,r\} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\Gamma_{p+1,q}^{m,n+1}\left[\frac{r}{c_j}\left|\begin{matrix}(\alpha,\beta),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right.\right]}{c_j^{\lambda}(c_j+r)^{\mu}}, \tag{17}$$

$$\Omega_{\lambda,\mu}\{\gamma_{p+1,q}^{m,n+1};c,r\} = \sum_{j=1}^{\infty} \frac{\gamma_{p+1,q}^{m,n+1}\left[\frac{r}{c_j}\left|\begin{matrix}(\alpha,\beta),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right.\right]}{c_j^{\lambda}(c_j+r)^{\mu}}, \tag{18}$$

and

$$\tilde{\Omega}_{\lambda,\mu}\{\gamma_{p+1,q}^{m,n+1};c,r\} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\gamma_{p+1,q}^{m,n+1}\left[\frac{r}{c_j}\left|\begin{matrix}(\alpha,\beta),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right.\right]}{c_j^{\lambda}(c_j+r)^{\mu}}, \tag{19}$$

where the following sequence of real numbers $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$ is an increasing sequence tending to infinity, i.e.,

$$\mathbf{c} = 0 < c_1 < c_2 < \cdots < c_n \uparrow \infty. \tag{20}$$

The well-known Mathieu series defined by

$$S(r) = \sum_{m=1}^{\infty} \frac{2m}{(m^2+r^2)^2}, \tag{21}$$

was introduced and further studied by famous mathematician Émile Leonard Mathieu in his book Mathieu (1890) based on elasticity of solid bodies. The bounds of the series (21) are required for finding the solution of boundary value problems for the biharmonic equations present in two-dimensional rectangular domain Schröder (1949).

3. Integral representations of Mathieu type series involving incomplete H-functions

We establish a family of convergent Mathieu type a-series along with it's alternating variants containing incomplete $H-$functions.

Theorem 3.1.

If $\mu > 0, \lambda > 0, r > 0, \beta = \sigma = 1, \alpha = 1 - \lambda$ and $\mathbf{c}$ satisfies the condition (20). Then, the following result holds true:

$$\Theta_{\lambda,\mu}\{\Gamma^{m,n+1}_{p+1,q}; c, r\} = \Gamma^I_c(\lambda+1,\mu) + \mu\Gamma^I_c(\lambda,\mu+1), \tag{22}$$

$$\tilde{\Theta}_{\lambda,\mu}\{\Gamma^{m,n+1}_{p+1,q}; c, r\} = \tilde{\Gamma}_c^{\,I}(\lambda+1,\mu) + \mu\tilde{\Gamma}_c^{\,I}(\lambda,\mu+1), \tag{23}$$

where

$$\Gamma^I_c(u,v) = \int_{c_1}^{\infty} \Gamma^{m,n+1}_{p+1,q}\left[\frac{r}{x}\,\middle|\, \begin{matrix}(1-u,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right] \times \frac{[c^{-1}(x)]}{x^u(r+x)^v}dx,$$

$$\tilde{\Gamma}^I_c(u,v) = \int_{c_1}^{\infty} \Gamma^{m,n+1}_{p+1,q}\left[\frac{r}{x}\,\middle|\, \begin{matrix}(1-u,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right] \times \frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v}dx.$$

Here, $c : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ denotes an increasing function such that $c(x)|_{x\in\mathbb{N}} = \mathbf{c}$ and $[c^{-1}(x)]$ represents integer part of the quantity $c^{-1}(x)$.

Proof:

To prove the left hand side of (22), first we call well known GF formula

$$\Gamma(\mu)\zeta^{-\mu} = \int_0^{\infty} e^{-\zeta t}\, t^{\mu-1}dt, \qquad (\Re(\zeta) > 0,\, \Re(\mu) > 0). \tag{24}$$

Setting $\alpha = 1 - \lambda, \beta = 1$ in equation (16), we get

$$\Theta_{\lambda,\mu}\{\Gamma^{m,n+1}_{p+1,q}; c, r\} = \sum_{j=1}^{\infty} \frac{\Gamma^{m,n+1}_{p+1,q}\left[\frac{r}{c_j}\,\middle|\, \begin{matrix}(1-\lambda,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ \\ (f_j,F_j)_{1,q}\end{matrix}\right]}{c_j^{\lambda}(c_j+r)^{\mu}}. \tag{25}$$

Further, with the help of Srivastava et al. (2018)[p. 122, Theorem 3.2] and equation (24) alongwith applying $\zeta = c_j + r$, we obtain

$$\Theta_{\lambda,\mu}\{\Gamma^{m,n+1}_{p+1,q}; c, r\} = \sum_{j=1}^{\infty} \int_0^{\infty} s^{\lambda-1}e^{-c_j s}\ \Gamma^{m,n}_{p,q}[rs]ds \int_0^{\infty} \frac{t^{\mu-1}}{\Gamma(\mu)}\, e^{-(c_j+r)t}dt. \tag{26}$$

Next, we interchanging the order of integration with summation under the permissible conditions, we get

$$\Theta_{\lambda,\mu}\{\Gamma^{m,n+1}_{p+1,q};c,r\} = \frac{1}{\Gamma(\mu)}\int_0^\infty\int_0^\infty \left(\sum_{j=1}^{\infty} e^{-c_j(s+t)}\right) s^{\lambda-1}e^{-rt}\,t^{\mu-1}\,\Gamma^{m,n}_{p,q}[rs]\,dsdt, \tag{27}$$

where $\Re(\mu) > 0$. The inside *Dirichlet series*

$$\mathbf{D_c}(s+t) = \sum_{j=1}^{\infty} e^{-c_j(s+t)},$$

has Laplace form Integral representation Pogány (2004, 2007) such that it can be expressed as follows

$$\begin{aligned}\mathbf{D_c}(s+t) &= (s+t)\int_0^\infty e^{-(s+t)x}\left(\sum_{j\,:\,c_j\le x} 1\right)dx\\ &= (s+t)\int_0^\infty e^{-(s+t)x}\,[c^{-1}(x)]\,dx,\end{aligned}$$

where $[c^{-1}(x)] = 0$ for all $x \in [0, c_1)$. By using the above expression in (27), we obtain

$$\begin{aligned}\Theta_{\lambda,\mu}\{\Gamma^{m,n+1}_{p+1,q};c,r\} &= \frac{1}{\Gamma(\mu)}\int_0^\infty\int_0^\infty\int_{c_1}^\infty s^{\lambda}t^{\mu-1}e^{-(r+x)t-xs}\Gamma^{m,n}_{p,q}[rs][c^{-1}(x)]dsdtdx\\ &+\frac{1}{\Gamma(\mu)}\int_0^\infty\int_0^\infty\int_{c_1}^\infty s^{\lambda-1}t^{\mu}e^{-(r+x)t-xs}\Gamma^{m,n}_{p,q}[rs][c^{-1}(x)]dsdtdx.\end{aligned}$$

Since,

$$\mathbf{I}_s = \int_{c_1}^\infty\left(\int_0^\infty s^{\lambda}e^{-sx}\Gamma^{m,n}_{p,q}[rs]ds\right)\left(\int_0^\infty t^{\mu-1}e^{-(r+x)t}dt\right)\frac{[c^{-1}(x)]}{\Gamma(\mu)}dx \tag{28}$$

$$= \int_{c_1}^\infty \Gamma^{m,n+1}_{p+1,q}\left[\frac{r}{x}\left|\begin{matrix}(-\lambda,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q}\end{matrix}\right.\right]\frac{[c^{-1}(x)]}{x^{\lambda+1}(r+x)^{\mu}}dx, \tag{29}$$

the auxiliary integral is as follows

$$\begin{aligned}\Gamma^{I}_{c}(u,v) = \int_{c_1}^\infty &\Gamma^{m,n+1}_{p+1,q}\left[\frac{r}{x}\left|\begin{matrix}(1-u,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q}\end{matrix}\right.\right]\\ &\times\frac{[c^{-1}(x)]}{x^{u}(r+x)^{v}}dx.\end{aligned}$$

It is easily established that

$$\mathbf{I}_s = \Gamma^{I}_{c}(\lambda+1,\mu) \qquad \text{and} \qquad \mathbf{I}_t = \mu\,\Gamma^{I}_{c}(\lambda,\mu+1). \tag{30}$$

This proves assertion (22).

It can be readily noted that the interesting proof of (23) is identical to that of (22), now implementing the definition of $\tilde{D}_c(.)$ which is known as the new alternating *inner* Dirichlet series Pogány et al. (2006) [p. 77, Section 4] is defined by

$$\begin{aligned}\tilde{\mathbf{D}}_{\mathbf{c}}(s+t) &= \sum_{j=1}^{\infty}(-1)^{j-1}e^{-c_j(s+t)} \\ &= (s+t)\int_0^{\infty} e^{-(s+t)x}\sum_{j\,:\,c_j\le x}(-1)^{j-1}dx \\ &= \frac{s+t}{2}\int_0^{\infty} e^{-(s+t)x}\left(1-(-1)^{[c^{-1}(x)]}\right)dx \\ &= (s+t)\int_{c_1}^{\infty} e^{-(s+t)x}\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)dx. \qquad (31)\end{aligned}$$

The proof is completed by utilization of (31) to (23). ∎

Theorem 3.2.

If $\mu > 0, \lambda > 0, r > 0, \beta = \sigma = 1, \alpha = 1 - \lambda$ and $\mathbf{c}$ satisfies the condition (20). Then, the following result holds true:

$$\Omega_{\lambda,\mu}\{\gamma_{p+1,q}^{m,n+1}; c, r\} = \gamma_c^I(\lambda+1,\mu) + \mu\gamma_c^I(\lambda,\mu+1), \qquad (32)$$

$$\tilde{\Omega}_{\lambda,\mu}\{\gamma_{p+1,q}^{m,n+1}; c, r\} = \tilde{\gamma}_c{}^I(\lambda+1,\mu) + \mu\tilde{\gamma}_c{}^I(\lambda,\mu+1), \qquad (33)$$

where

$$\begin{aligned}\gamma_c^I(u,v) = \int_{c_1}^{\infty}\gamma_{p+1,q}^{m,n+1}&\left[\frac{r}{x}\,\middle|\,\begin{matrix}(1-u,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q}\end{matrix}\right] \\ &\times\frac{[c^{-1}(x)]}{x^u(r+x)^v}dx,\end{aligned}$$

$$\begin{aligned}\tilde{\gamma}_c^I(u,v) = \int_{c_1}^{\infty}\gamma_{p+1,q}^{m,n+1}&\left[\frac{r}{x}\,\middle|\,\begin{matrix}(1-u,1),(e_1,E_1,y),(e_j,E_j)_{2,p}\\ (f_j,F_j)_{1,q}\end{matrix}\right] \\ &\times\frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v}dx.\end{aligned}$$

Here, $c:\mathbb{R}_+\longmapsto\mathbb{R}_+$ denotes an increasing function such that $c(x)|_{x\in\mathbb{N}} = \mathbf{c}$ and $[c^{-1}(x)]$ represents integer part of the quantity $c^{-1}(x)$.

Proof:

Theorem 3.2 can be easily proved on the similar lines, so we omit the details. ∎

4. Consequences and special cases

A number of frequently used special functions namely the Fox-Wright Ψ-functions, GHF ${}_pF_q$, Meijer G-functions and Bessel functions are contained in the well known class of function which can be expressed effortlessly in terms of HF. Now, we present some special cases related to the above-mentioned results.

(i) If we reduce IHF to HF by using equation (8) in Theorem 3.1, we obtain the result established by Pogány and Saxena (2011) [p. 119, Corollary 2].

(ii) If we reduce IHF to well known FWF (11) in Theorem 3.1, we obtain the result established by Pogány (2007) [p. 765, Theorem 1].

(iii) If we reduce IHF to MIMLF by using equation (14) in Theorem 3.1, then the following result holds true:

$$\Theta_{\lambda,\mu}\{E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_{m+1}};c,r\} = E^I_c(\lambda+1,\mu) + \mu E^I_c(\lambda,\mu+1), \tag{34}$$

$$\tilde{\Theta}_{\lambda,\mu}\{E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_{m+1}};c,r\} = \tilde{E_c}^I(\lambda+1,\mu) + \mu\tilde{E_c}^I(\lambda,\mu+1), \tag{35}$$

where

$$E^I_c(u,v) = \frac{1}{\Gamma(\gamma)}\int_{c_1}^{\infty} H^{1,2}_{2,m+1}\left[-\frac{r}{x}\,\middle|\,\begin{matrix}(1-u,1),(1-\gamma,\kappa)\\ \\ (0,1),(1-\beta_j,\alpha_j)_{1,m}\end{matrix}\right] \times \frac{[c^{-1}(x)]}{x^u(r+x)^v}dx,$$

$$\tilde{E}^I_c(u,v) = \frac{1}{\Gamma(\gamma)}\int_{c_1}^{\infty} H^{1,2}_{2,m+1}\left[-\frac{r}{x}\,\middle|\,\begin{matrix}(1-u,1),(1-\gamma,\kappa)\\ \\ (0,1),(1-\beta_j,\alpha_j)_{1,m}\end{matrix}\right] \times \frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v}dx.$$

Here, $c(x)$, $c^{-1}(x)$ and $[c^{-1}(x)]$ will retain the same meanings as discussed previously.

(iv) If we reduce IHF to IFWF by using Equation (9) in Theorem 3.1, then the following result holds true:

$$\Theta_{\lambda,\mu}\{{}_{p+1}\Psi^{(\Gamma)}_q;c,r\} = \Psi^I_c(\lambda+1,\mu) + \mu\Psi^I_c(\lambda,\mu+1), \tag{36}$$

$$\tilde{\Theta}_{\lambda,\mu}\{{}_{p+1}\Psi^{(\Gamma)}_q;c,r\} = \tilde{\Psi}^I_c(\lambda+1,\mu) + \mu\tilde{\Psi}^I_c(\lambda,\mu+1), \tag{37}$$

where

$$\Psi_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\Psi_q^{(\Gamma)} \left[-\tfrac{r}{x} \left| \begin{matrix} (u,1),(1-e_1,E_1,y),(1-e_j,E_j)_{2,p} \\ (1-f_j,F_j)_{1,q} \end{matrix} \right. \right] \times \frac{[c^{-1}(x)]}{x^u(r+x)^v} dx,$$

$$\tilde{\Psi}_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\Psi_q^{(\Gamma)} \left[-\tfrac{r}{x} \left| \begin{matrix} (u,1),(1-e_1,E_1,y),(1-e_j,E_j)_{2,p} \\ (1-f_j,F_j)_{1,q} \end{matrix} \right. \right] \times \frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v} dx.$$

Here, $c(x)$, $c^{-1}(x)$ and $[c^{-1}(x)]$ will retain the same meanings as discussed previously.

(v) If we reduce IHF to IGHF by using Equations (9) and (12) in Theorem 3.1, then the following result holds true:

$$\Theta_{\lambda,\mu}\{{}_{p+1}\Gamma_q; c, r\} = \delta_c^I(\lambda+1,\mu) + \mu\delta_c^I(\lambda,\mu+1), \tag{38}$$

$$\tilde{\Theta}_{\lambda,\mu}\{{}_{p+1}\Gamma_q; c, r\} = \tilde{\delta}_c^I(\lambda+1,\mu) + \mu\tilde{\delta}_c^I(\lambda,\mu+1), \tag{39}$$

where

$$\delta_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\Gamma_q \left[\begin{matrix} (u,1),(1-e_1,y),1-e_2,\cdots,1-e_p; \\ 1-f_1,\cdots,1-f_q; \end{matrix} \; -\frac{r}{x} \right] \times \frac{[c^{-1}(x)]}{x^u(r+x)^v} dx,$$

$$\tilde{\delta}_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\Gamma_q \left[\begin{matrix} (u,1),(1-e_1,y),1-e_2,\cdots,1-e_p; \\ 1-f_1,\cdots,1-f_q; \end{matrix} \; -\frac{r}{x} \right] \times \frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v} dx.$$

Here, $c(x)$, $c^{-1}(x)$ and $[c^{-1}(x)]$ will retain the same meanings as discussed previously.

(vi) If we reduce IHF to IFWF by using Equation (10) in Theorem 3.2, then the following result holds true:

$$\Omega_{\lambda,\mu}\{{}_{p+1}\Psi_q^{(\gamma)}; c, r\} = \psi_c^I(\lambda+1,\mu) + \mu\psi_c^I(\lambda,\mu+1), \tag{40}$$

$$\tilde{\Omega}_{\lambda,\mu}\{{}_{p+1}\Psi_q^{(\gamma)}; c, r\} = \tilde{\psi}_c^I(\lambda+1,\mu) + \mu\tilde{\psi}_c^I(\lambda,\mu+1), \tag{41}$$

where

$$\psi_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\Psi_q^{(\gamma)} \left[-\tfrac{r}{x} \middle| \begin{matrix} (u,1),(1-e_1,E_1,y),(1-e_j,E_j)_{2,p} \\ (1-f_j,F_j)_{1,q} \end{matrix} \right] \times \frac{[c^{-1}(x)]}{x^u(r+x)^v} dx,$$

$$\tilde{\psi}_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\Psi_q^{(\gamma)} \left[-\tfrac{r}{x} \middle| \begin{matrix} (u,1),(1-e_1,E_1,y),(1-e_j,E_j)_{2,p} \\ (1-f_j,F_j)_{1,q} \end{matrix} \right] \times \frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v} dx.$$

Here, $c(x)$, $c^{-1}(x)$ and $[c^{-1}(x)]$ will retain the same meanings as discussed previously.

(vii) If we reduce IHF to IGHF by using Equations (10) and (13) in Theorem 3.2, then the following result holds true:

$$\Theta_{\lambda,\mu}\{{}_{p+1}\gamma_q; c, r\} = \sigma_c^I(\lambda+1,\mu) + \mu\sigma_c^I(\lambda,\mu+1), \tag{42}$$

$$\tilde{\Theta}_{\lambda,\mu}\{{}_{p+1}\gamma_q; c, r\} = \tilde{\sigma_c^I}(\lambda+1,\mu) + \mu\tilde{\sigma_c^I}(\lambda,\mu+1), \tag{43}$$

where

$$\sigma_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\gamma_q \left[\begin{matrix} (u,1),(1-e_1,y),1-e_2,\cdots,1-e_p; \\ 1-f_1,\cdots,1-f_q; \end{matrix} -\frac{r}{x} \right] \times \frac{[c^{-1}(x)]}{x^u(r+x)^v} dx,$$

$$\tilde{\sigma}_c^I(u,v) = \int_{c_1}^{\infty} {}_{p+1}\gamma_q \left[\begin{matrix} (u,1),(1-e_1,y),1-e_2,\cdots,1-e_p; \\ 1-f_1,\cdots,1-f_q; \end{matrix} -\frac{r}{x} \right] \times \frac{sin^2(\frac{\pi}{2}[c^{-1}(x)])}{x^u(r+x)^v} dx.$$

Here, $c(x)$, $c^{-1}(x)$ and $[c^{-1}(x)]$ will retain the same meanings as discussed previously.

5. Conclusion

In this work, we have derived closed integral form expressions for a family of convergent Mathieu type a-series along with its alternating variants, whose terms contain IHF's $\Gamma_{p,q}^{m,n}(z)$ and $\gamma_{p,q}^{m,n}(z)$, which are a notable generalization of well-known HF. The results established in the present study are very general in nature and give an exquisite generalization of closed integral form expressions of aforementioned series available in literature.

REFERENCES

Bansal, M.K. and Choi, J. (2019). A note on pathway fractional integral formulas associated with the incomplete $H-$functions, Int. J. Appl. Comput. Math., Vol. 5, No. 5, Article 133.

Bansal, M.K., Kumar, D. and Jain, R. (2019a). Interrelationships between Marichev-Saigo-Maeda fractional integral operators, the Laplace transform and the $\bar{H}$-function, Int. J. Appl. Comput. Math, Vol. 5, No. 4, Article 103.

Bansal, M.K., Kumar, D. and Jain, R. (2019b). A study of Marichev-Saigo-Maeda fractional integral operators associated with S-generalized Gauss hypergeometric function, Kyungpook Math. J., Vol. 59, No. 3, pp. 433–443.

Bansal, M.K., Kumar, D., Khan, I., Singh, J. and Nisar, K.S. (2019). Certain unified integrals associated with product of $M-$series and incomplete $H-$functions, Mathematics, Vol. 7, No. 12, Article 1191.

Bansal, M.K., Kumar, D. , Singh, J., Tassaddiq, A. and Nisar, K.S. (2020). Some new results for the Srivastava-Luo-Raina M-transform pertaining to the incomplete H-Functions, AIMS-Mathematics, Vol. 5, No. 1, pp. 717-722.

Chaudhry, M.A. and Qadir, A. (2002). Incomplete exponential and hypergeometric functions with applications to non-central χ^2-distribution, Comm. Statist. Theory Methods, Vol. 34, pp. 525–535.

Choi, J., Parmar, R.K. and Pogány, T.K. (2017). Mathieu-type series built by $(p,q)-$ extended Gaussian hypergeometric function, Bull. Korean Math. Soc., Vol. 54, No. 3, pp. 789–797.

Gerhold, S. and Tomovski, Ž. (2019). Asymptotic expansion of Mathieu power series and trigonometric Mathieu series, J. Math. Anal. Appl., Vol. 479, pp. 1882-1892.

Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations* (North-Holland Mathematical Studies), Elsevier (North Holland) Science Publishers, Amsterdam, London and New York.

Mathai, A.M. and Saxena, R.K. (1978). *The $H-$ function with Applications in Statistics Other Disciplines*, Wiley Eastern, New Delhi & Wiley Halsted, New York.

Mathai, A.M., Saxena, R.K. and Haubold, H.J. (2009). *The H-function: Theory and Applications*, Springer, New York.

Mathieu, É.L. (1890). *Traité de Physique Mathématique. VI-VII: Théory de l'Elasticité des Corps Solidés (Part 2)*, Gauthier-Villars, Paris.

Mehrez, K. and Sitnik, S.M. (2019). Generalized Volterra functions its integral representations and applications to the Mathieu-type series, Appl. Math. Comput., Vol. 347, pp. 578–589.

Mehrez, K. and Tomovski,Ž. (2019). On a new (p,q)-Mathieu-type power series and it's applications, Appl. Anal. Discrete Math., Vol. 13, pp. 309–324.

Pogány, T.K. (2004). Integral representation of a series which includes the Mathieu a-series, J. Math. Anal. Appl., Vol. 296, pp. 309–313.

Pogány, T.K. (2005). Integral representation of Mathieu a, λ-series, Integral Transfroms Spec. Funct., Vol. 16, No. 8, pp. 685–689.

Pogány, T.K. (2007). Integral expressions of of Mathieu-type series whose terms contain Fox's

H-function, Appl. Math. Letters, Vol. 20, pp. 764–769.

Pogány, T.K. and Saxena, R.K. (2011). Some Mathieu-type series for the I-function occurring in the Fokker-Plank equation, Proyecciones Journal of Mathematics, Vol. 30, No. 1, pp. 111–122.

Pogány, T.K., Srivastava, H.M. and Tomovski, Ž. (2006). Some families of Mathieu **a**-series and alternating Mathieu **a**-series, Appl. Math. Comput., Vol. 173, No. 1, pp. 69–108.

Saxena, R.K. and Nishimoto, K. (2010a). Further results on generalized Mittag-Leffler functions of fractional calculus, J. Fract. Calc., Vol. 39, pp. 29–41.

Saxena, R.K. and Nishimoto, K. (2010b). N-fractional calculus of generalized Mittag-Leffler functions, J. Fract. Calc., Vol. 37, pp. 43–52.

Schröder, K. (1949). Das Problem der eingespannten rechteckigen elastischen Platte, Math. Ann., Vol. 121, pp. 247–326.

Srivastava, H.M., Bansal, M.K. and Harjule, P. (2018). A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function, Math. Meth. Appl. Sci., Vol. 41, No. 16, pp. 6108–6121.

Srivastava, H.M., Chaudhary, M.A. and Agarwal, R.P. (2012). The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms Spec. Funct., Vol. 23, pp. 659–683.

Srivastava, H.M., Gupta, K.C. and Goyal, S.P. (1982). *The H- Functions of one and Two Variables with Applications*, South Asian Publishers, New Delhi.

Srivastava, H.M., Mehrez, K. and Tomovski, Ž. (2018). New inequalities for some generalized Mathieu type series and the Riemann zeta function, J. Math. Ineq., Vol. 12, No. 1, pp. 163–174.

Srivastava, H.M., Saxena, R.K. and Parmar, R.K. (2018). Some families of the incomplete $H-$functions and the incomplete $\overline{H}-$functions and associated integral transforms and operators of fractional calculus with applications, Russ. J. Math. Phys., Vol. 25, No. 1, pp. 116–138.

Srivastava, H.M. and Tomovski, Ž. (2004). Some problems and solutions involving Mathieu's series and it's generalizations, J. Inequal. Pure and Appl. Math., Vol. 5, No. 2, Article 45.

Tomovski, Ž. (2009). On Hankel transform of generalized Mathieu series, Fract. Calc. Appl. Anal., Vol. 12, No. 1, pp. 97–107.

Tomovski, Ž. and Mehrez, K. (2017). Some families of generalized Mathieu-type power series, associated probability distributions and related inequalities involving complete monotonicity and log-convexity, Math. Ineq. Appl., Vol. 20, No. 4, pp. 973–986.

Tomovski, Ž and Tuan, V.K. (2009). On Fourier transforms and summation formulas of generalized Mathieu series, Math. Sci. Res. J., Vol. 13, No. 1, pp. 1–10.