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Extension of Two Parameter Gamma, Beta Functions and Its Properties

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Abstract

In this paper, we introduce the extension of the *p-k* Gamma function and the *p-k* Beta function. This extension of the *p-k* Gamma function is named as *p-k-b* Gamma function and an extension of the beta function is *p-k-b* Beta function. The new extension of the Gamma and Beta function has satisfied the usual properties. Also, we prove several identities of these functions.

Keywords: *p-k-b* Gamma and Beta functions; *p-k* Gamma and Beta functions; Euler's Gamma and Beta functions

MSC 2010 No.: 33B15, 33C15

1. Introduction

The Gamma function was introduced by the famous Swiss mathematician Euler (1707-1783) as a natural extension of the factorial operation from positive integers to real and even complex values of this argument. The Beta function in the form of integral is known as the first Eulerian integral and the integral form of Gamma function known as the second Eulerian integral. Later, because of its great importance, it was studied by other eminent mathematicians like Legendre (1752-1833), Gauss (1777-1855), Gudermann (1798-1852), Liouville (1809-1882), Weierstrass (1815-1897), Hermite (1822-1901), as well as many others. The first reported use of the Gamma symbol for this function was by Legendre in 1839. The first Eulerian integral was introduced by the Euler and is typically referred to by its more common name, the Beta function. The use of the Beta symbol for this function was first used in 1839 by Binet (1786-1856). At the same time as Legendre and Gauss, Kramp (1760-1826) worked on the generalized factorial function as it applied to non-integers.

During the twentieth century, the log gamma function was used in many works where the gamma function was applied or investigated. The appearance of computer systems at the end of the twentieth century demanded more careful attention to the structure of branch cuts for basic mathematical functions to support the validity of the mathematical relations everywhere in the complex plane. This leads to the appearance of a special log gamma function, which is equivalent to the logarithm of the gamma function as a multivalued analytic function, except that it is conventionally defined with a different branch cut structure and principal sheet. The log-gamma function was introduced by Keiper (1990). It allows a concise formulation of many identities related to the Riemann zeta function. Many generalizations and extensions of various functions are available in the literature (Atash et al. (2019); Atugba and Nantomah (2019); Duran and Acikgoz (2019); Qi et al. (2018); Tassaddiq (2019)). In this paper, we establish a new extension of the Beta and the Gamma function.

2. Preliminaries

The two parameter Gamma function for $x \in \mathbb{C}/k\mathbb{Z}^-; k, p \in \mathbb{R}^+ - \{0\}$, $Re(x) > 0, n \in \mathbb{N}$ denoted by ${}_p\Gamma_k(x)$ is introduced by Gehlot (2017b) and Gehlot (2018) and is defined by

$$ {}_p\Gamma_k(x) = \frac{1}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1}(np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}}. \tag{1} $$

The integral representation of (1) is given by

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \tag{2} $$

The *p-k* Pochhammer symbol is given for $x \in \mathbb{C}; k, p \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0, n \in \mathbb{N}$, the *p-k* Pochhammer Symbol (or the two parameter Pochhammer Symbol) denoted by ${}_p(x)_{n,k}$ and is defined by

$$ {}_p(x)_{n,k} = \left(\frac{xp}{k}\right)\left(\frac{xp}{k}+p\right)\left(\frac{xp}{k}+2p\right)...\left(\frac{xp}{k}+(n-1)p\right). \tag{3} $$

$$ {}_p(x)_{n,k} = \frac{{}_p\Gamma_k(x+nk)}{{}_p\Gamma_k(x)}. \tag{4} $$

The *p-k* Beta function, ${}_pB_k(x,y)$ is given for $x, y \in C/k\mathbb{Z}^-; k, p \in \mathbb{R} - \{0\}$ and $Re(x) > 0, Re(y) > 0$, as

$$ {}_pB_k(x,y) = \frac{{}_p\Gamma_k(x)\,{}_p\Gamma_k(y)}{{}_p\Gamma_k(x+y)}, Re(x) > 0, Re(y) > 0. \tag{5} $$

The integral representations are given as

$$ {}_pB_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{6} $$

$$ {}_pB_k(x,y) = \int_0^1 t^{x-1}(1-t^k)^{y-1}dt. \tag{7} $$

$$ {}_pB_k(x,y) = \frac{1}{k}\int_0^1 \frac{t^{\frac{x}{k}-1}+t^{\frac{y}{k}-1}}{(t+1)^{\frac{x+y}{k}}}dt. \tag{8} $$

$$ {}_pB_k(x,y) = \int_0^\infty t^{x-1}(1+t^k)^{-\frac{x+y}{k}}dt. \tag{9} $$

Throughout this paper $\mathbb{C}, \mathbb{R}^+, Re(.), \mathbb{Z}^-$ and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, real part of complex number, negative integer and natural numbers respectively.

3. Generalized *p-k* Gamma function

Definition 3.1.

For $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0, n \in \mathbb{N}$, the ***p-k-b*** Gamma function, ${}_p\Gamma_k(x;b)$ is given by

$$ {}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty}\frac{(-1)^m b^m p^{\frac{x}{k}-m-1}}{m!}\frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(n)^{\frac{x}{k}-m-1}}{{}_p(x-km)_{n,k}}. \tag{10} $$

Theorem 3.2.

Let $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0$, then the integral representation of p-k-b Gamma function is given by

$$ {}_p\Gamma_k(x;b) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{p} - \frac{b}{t^k}} dt. \tag{11} $$

$$ {}_p\Gamma_k(x;b) = \frac{1}{k} \int_0^1 (t)^{\frac{1}{p}-1} \left(\log \frac{1}{t}\right)^{\frac{x}{k}-1} \exp\left(\frac{b}{\log t}\right) dt. \tag{12} $$

$$ {}_p\Gamma_k(x;b) = \frac{1}{x} \int_0^\infty \exp\left(-\frac{t^{\frac{k}{x}}}{p} - \frac{b}{t^{\frac{k}{x}}}\right) dt. \tag{13} $$

Proof (of (11)):

Consider the right hand side of Equation (11) denoting I_1,

$$ I_1 = \int_0^\infty t^{x-1} e^{-\frac{t^k}{p}} e^{-\frac{b}{t^k}} dt = \int_0^\infty t^{x-1} e^{-\frac{t^k}{p}} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \frac{b^m}{t^{km}} dt. $$

Interchanging the order of integration and summation, we get

$$ I_1 = \sum_{m=0}^\infty \frac{(-1)^m}{m!} b^m \int_0^\infty e^{-\frac{t^k}{p}} t^{x-km-1} dt. $$

Applying the Tannery's Theorem (Erdélyi et al. (1953), Page 2), we have

$$ \int_0^\infty t^{x-1} e^{-\frac{t^k}{p} - \frac{b}{t^k}} dt = \sum_{m=0}^\infty \frac{(-1)^m b^m}{m!} \lim_{n\to\infty} \int_0^{(np)^{\frac{1}{k}}} \left(1 - \frac{t^k}{n^p}\right)^n t^{x-mk-1} dt. $$

Let $A_{n,i}(x); i = 0, 1, 2, ..., n$ be given by

$$ A_{n,i}(x) = \lim_{n\to\infty} \int_0^{(np)^{\frac{1}{k}}} \left(1 - \frac{t^k}{n^p}\right)^i t^{x-mk-1} dt. $$

Applying the integration by part, we have

$$ A_{n,i}(x) = \frac{ki}{pn(x-km)} A_{n,i-1}(x+k). $$

Also,

$$A_{n,0}(x)=\int_0^{(np)^{\frac{1}{k}}} t^{x-km-1}dt,$$

$$A_{n,0}(x)=\frac{(np)^{\frac{x-km}{k}}}{(x-km)},$$

Therefore,

$$A_{n,n}(x)=\frac{1}{k}\frac{n!\,p^{n+1}(n)^{\frac{x}{k}-m-1}}{{}_p(x-km)_{n,k}}.$$

and

$$I_1=\sum_{m=0}^{\infty}\frac{(-1)^m b^m}{m!}\frac{1}{k}\lim_{n\to\infty}A_{n,n}(x),$$

gives

$$I_1=\sum_{m=0}^{\infty}\frac{(-1)^m b^m}{m!}\frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(n)^{\frac{x}{k}-m-1}}{{}_p(x-km)_{n,k}}.$$

Hence, we arrive at the result. ∎

Proof (of (12)):

Substituting $t^k=\log(\frac{1}{u})$ in Equation (11), we get

$$\begin{aligned}{}_p\Gamma_k(x;b)&=-\frac{1}{k}\int_1^0\big(\log\frac{1}{u}\big)^{\frac{x-1}{k}}e^{-\frac{\log 1/u}{p}-\frac{b}{\log 1/u}}\big(\log 1/u\big)^{\frac{1}{k}-1}\frac{1}{u}du\\&=\frac{1}{k}\int_0^1 u^{\frac{1}{p}-1}[\log(\frac{1}{u})]^{\frac{x}{k}-1}e^{\frac{b}{\log u}}du.\end{aligned}$$

Which is the required proof. ∎

Proof (of (13)):

Substituting $t^x=u$ in Equation (11), we get

$$\begin{aligned}{}_p\Gamma_k(x;b)&=\int_0^{\infty}(u^{\frac{1}{x}})^{x-1}e^{(-\frac{u^{k/x}}{p}-\frac{b}{u^{k/x}})}\frac{1}{x}u^{\frac{1}{x}-1}du\\&=\frac{1}{x}\int_0^{\infty}\exp\left(-\frac{u^{k/x}}{p}-\frac{b}{u^{k/x}}\right)du.\end{aligned}$$

Hence, the required result. ∎

Remark 3.3.

The integral in Equation (11) is convergent for $x < k; x \neq k\mathbb{Z}^-; p, k, b \in \mathbb{R}^+ - \{0\}; |\frac{b}{p}| < 1$ and divergent for $\left|\frac{b}{p}\right| > 1$.

Particular cases:

For some particular values of the parameters p, k and b we can obtain certain Gamma functions, defined earlier:

(a) For $b = 0$, Equation (11) reduces in p-k Gamma function defined by Gehlot (2017b).

$$ {}_p\Gamma_k(x; 0) = {}_p\Gamma_k(x). $$

(b) For $p = k, b = 0$, Equation (11) reduces in k Gamma function defined by Diaz and Pariguan (2007).

$$ {}_k\Gamma_k(x; 0) = \Gamma_k(x) $$

(c) For $p = k = 1, b = 0$, Equation (11) reduces in Gamma function.

$$ {}_1\Gamma_1(x; 0) = \Gamma(x). $$

(d) For $p = k = 1$, Equation (11) reduces in generalized Gamma function defined by Chaudhry et al. (1997)

$$ {}_1\Gamma_1(x; b) = \Gamma_b(x). $$

(e) Equation (11), reduces in ultra Gamma function defined in Gehlot (2017a); Anita et al. (2018).

$$ {}_p\Gamma_k(x; b) = \Gamma(k, p; k, \frac{1}{b})(x). $$

Theorem 3.4.

Given $x \in C/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0$, then the relation between *p-k-b* Gamma Function, *p-k* Gamma Function and classical Gamma function are given by,

$$ {}_p\Gamma_k(x; b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \, {}_p\Gamma_k(x - km), \tag{14} $$

$$ {}_p\Gamma_k(x; b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{p^{\frac{x-km}{k}}}{k} \Gamma\left(\frac{x - km}{k}\right). \tag{15} $$

Proof (of (14)):

From Equation (11),

$$
\begin{aligned}
{}_p\Gamma_k(x;b) &= \int_0^\infty t^{x-1} e^{-\frac{t^k}{p}-\frac{b}{t^k}}\,dt \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \int_0^\infty t^{x-km-1} e^{-\frac{t^k}{p}}\,dt.
\end{aligned}
$$

Using Theorem 2.4 of Gehlot (2017b), we get

$$
= \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!}\, {}_p\Gamma_k(x-km),
$$

which is the required result. ■

Proof (of (15)):

Using Theorem 2.9 of Gehlot (2017b), in Equation (14), we have

$$
{}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{p^{(x-km)/k}}{k} \Gamma\left(\frac{x-km}{k}\right),
$$

which is the required result. ■

Theorem 3.5.

Given $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b, a, q, r, s \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0$, then the following identities hold,

$$
{}_p\Gamma_k(x;b) = \frac{p^{\frac{x}{k}}}{k}\, {}_1\Gamma_1\left(\frac{x}{k}; \frac{b}{p}\right). \tag{16}
$$

$$
{}_p\Gamma_k(x;b) = \frac{s}{k}\, {}_p\Gamma_s\left(\frac{sx}{k}; b\right). \tag{17}
$$

$$
{}_p\Gamma_k(x;b) = \frac{s}{k}\left(\frac{p}{r}\right)^{\frac{x}{k}} {}_r\Gamma_s\left(\frac{sx}{k}; \frac{br}{p}\right). \tag{18}
$$

$$
{}_p\Gamma_k(x;b) = \left(\frac{p}{r}\right)^{\frac{x}{k}} {}_r\Gamma_k\left(x; \frac{br}{p}\right). \tag{19}
$$

$$
{}_p\Gamma_k(x;b) = \frac{1}{k}\, {}_p\Gamma_1\left(\frac{x}{k}; b\right). \tag{20}
$$

$$
{}_p\Gamma_k(-x;b) = {}_{\frac{1}{b}}\Gamma_k\left(x; \frac{1}{p}\right). \tag{21}
$$

$$ {}_p\Gamma_k(ax;b) = \frac{1}{a}\, {}_p\Gamma_{\frac{k}{a}}(x;b). \tag{22}$$

$$ {}_p\Gamma_k(x+a;b) = \frac{q}{x+a}\ {}_p\Gamma_{\frac{qk}{x+a}}(q;b). \tag{23}$$

$$ {}_p\Gamma_k(x;b) = a^x\ {}_{\frac{p}{a^k}}\Gamma_k\left(x;\frac{b}{a^k}\right). \tag{24}$$

Proof (of 16):

From Equation (11),

$$ {}_p\Gamma_k(x;b) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{p}-\frac{b}{t^k}}\,dt.$$

Substituting $\frac{t^k}{p} = u$, we get

$$\begin{aligned} {}_p\Gamma_k(x;b) &= \int_0^\infty (up)^{\frac{x-1}{k}} e^{-u-\frac{b}{up}} \frac{1}{k}(up)^{\frac{1}{k}-1} p\,du \\ &= \frac{p^{x/k}}{k}\int_0^\infty u^{\frac{x}{k}-1} e^{-u-\frac{b/p}{u}}\,du \\ &= \frac{p^{x/k}}{k}\,{}_1\Gamma_1\Big(\frac{x}{k};\frac{b}{p}\Big). \end{aligned}$$

Hence, proved. ■

Proof (of 17):

Substituting $t = u^{\frac{s}{k}}$ in Equation (11), we get the desired result (17). ■

Proof (of 18):

Substituting $t^k = \dfrac{p}{r}u^s$ in Equation (11), we get the desired result (18). ■

Proof (of 19):

Substituting $t^k = \dfrac{p}{r}u^k$ in Equation (11), we get the desired result (19). ■

Proof (of 20):

Substituting $t = u^{\frac{1}{k}}$ in Equation (11), we get the desired result (20). ■

Proof (of 21):

Substituting $t = \dfrac{1}{u}$ in Equation (11), we get the desired result (21). ■

Proof (of 22):

Substituting $t = u^{\frac{1}{a}}$ in Equation (11), we get the desired result (22). ■

Proof (of 23):

Substituting $t = u^{\frac{q}{(x+a)}}$ in Equation (11), we get the desired result (23). ■

Proof (of 24):

Substituting $t = au$ in Equation (11), we get the desired result (24). ■

Theorem 3.6.

Given $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0$, then the recursion relation of *p-k-b* Gamma function is given by,

$$k\, {}_p\Gamma_k(x+k;b) = xp\, {}_p\Gamma_k(x;b) + bkp\, {}_p\Gamma_k(x-k;b).$$

Proof:

From Equation (11),

$${}_p\Gamma_k(x;b) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{p} - \frac{b}{t^k}} dt. \tag{25}$$

Integrating right hand side of Equation (11) by parts,

$$= \left\{\frac{t^x}{x}\, e^{-\frac{t^k}{p} - \frac{b}{t^k}}\right\}_0^\infty - \int_0^\infty \frac{t^x}{x}\, e^{-\frac{t^k}{p} - \frac{b}{t^k}} \{-\frac{kt^{k-1}}{p} + kbt^{-k-1}\} dt,$$

$$= 0 + \int_0^\infty \frac{k}{px} t^{x+k-1}\, e^{-\frac{t^k}{p} - \frac{b}{t^k}} dt - \int_0^\infty \frac{kb}{x} t^{x-k-1}\, e^{-\frac{t^k}{p} - \frac{b}{t^k}} dt,$$

$${}_p\Gamma_k(x;b) = \frac{k}{px}\, {}_p\Gamma_k(x+k;b) - \frac{kb}{x}\, {}_p\Gamma_k(x-k;b),$$

which is desired result. ■

Theorem 3.7.

Given $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0$, then the *p-k-b* Gamma function in terms of infinite product,

$${}_p\Gamma_k(x;b) = \sum_{m=0}^\infty \frac{(-1)^m b^m}{m!} \frac{p^{\frac{x-km}{k}}}{x-km} \prod_{n=1}^\infty [(1+\frac{1}{n})^{\frac{x-km}{k}} \times (1+\frac{x-km}{nk})^{-1}]. \tag{26}$$

Proof:

From Equation (14), we have

$$ {}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \, {}_p\Gamma_k(x-km), $$

using Theorem 2.5, Gehlot (2017b), we have desired result. ■

Theorem 3.8.

Given $x \in \mathbb{C}/k\mathbb{Z}^-; k,p,b \in \mathbb{R}^+ - \{0\}$, $x < k$ and $Re(x) > 0$, then the *p-k-b* Gamma function in terms of hypergeometric function.

$$ {}_p\Gamma_k(x;b) = \frac{p^{\frac{x}{k}}\Gamma(\frac{x}{k})}{k} \, {}_0F_1(-;1-\frac{x}{k};\frac{b}{p}). \tag{27} $$

Proof:

From Equation (15) we have,

$$ {}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{p^{\frac{x-km}{k}}}{k} \Gamma(\frac{x-km}{k}), $$

$$ {}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{p^{\frac{x-km}{k}}}{k} \frac{\Gamma(\frac{x}{k}-m)\Gamma(\frac{x}{k})}{\Gamma(\frac{x}{k})}, $$

$$ {}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{p^{\frac{x-km}{k}}}{k} \Gamma(\frac{x}{k})(\frac{x}{k})_{-m}, $$

using the identity

$$ (a)_{-r} = \frac{(-1)^r}{(1-a)_r}, $$

$$ {}_p\Gamma_k(x;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{p^{\frac{x-km}{k}}}{k} \Gamma(\frac{x}{k}) \frac{(-1)^m}{(1-\frac{x}{k})_m}, $$

$$ {}_p\Gamma_k(x;b) = \frac{p^{\frac{x}{k}}}{k} \Gamma(\frac{x}{k}) \sum_{m=0}^{\infty} \frac{(\frac{b}{p})^m}{m!\,(1-\frac{x}{k})_m}. $$

Hence, we arrive at the result. ■

Theorem 3.9.

Given $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0$, then the relation between *p-k-b* Gamma function, *p-k* Gamma function and classical Gamma function is,

$$\int_0^\infty b^{s-1}\,{}_p\Gamma_k(x;b)db = \Gamma(s)_p\Gamma_k(x+ks) = \frac{\Gamma(s)p^{\frac{x}{k}+s}}{k}\,\Gamma(\frac{x}{k}+s;b). \tag{28}$$

Proof:

Multiplying Equation (11) by b^{s-1} and integrating with respect to b from $b = 0$ to $b = \infty$, we get

$$\int_0^\infty b^{s-1}\,{}_p\Gamma_k(x;b)db = \int_0^\infty b^{s-1}[\int_0^\infty t^{x-1}\exp(-\frac{t^k}{p} - \frac{b}{t^k})dt]db.$$

Changing the order of integration gives,

$$\int_0^\infty b^{s-1}\,{}_p\Gamma_k(x;b)db = \int_0^\infty t^{x-1}\exp(-\frac{t^k}{p})[\int_0^\infty b^{s-1}\exp(-\frac{b}{t^k})db]dt.$$

Using the basic definition of Gamma function, we have

$$\int_0^\infty b^{s-1}\,{}_p\Gamma_k(x;b)db = \int_0^\infty t^{x-1}\exp(-\frac{t^k}{p})[(\frac{1}{t^k})^{-s}\Gamma(s)]dt,$$

$$\int_0^\infty b^{s-1}\,{}_p\Gamma_k(x;b)db = \Gamma(s)\int_0^\infty t^{x+ks-1}\exp(-\frac{t^k}{p})dt,$$

$$\int_0^\infty b^{s-1}\,{}_p\Gamma_k(x;b)db = \Gamma(s)_p\Gamma_k(x+ks).$$

Finally using Equation (2.19) of Gehlot (2017b), we obtain the desired result. ■

4. Extension of *p-k* Beta function

In this section, we introduce an extension of the p-k beta function which is denoted as ${}_pB_k(x,y;b$ recently introduced by Gehlot (2017b).

Definition 4.1.

For $x, y \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(x) > 0, Re(y) > 0, n \in \mathbb{N}$, then the Extended p-k Beta Function is given by

$$ {}_pB_k(x,y;b) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\exp\left(\frac{-b}{t(1-t)}\right)dt. \tag{29} $$

or

$$ {}_pB_k(x,y;b) = \int_0^1 t^{x-1}(1-t^k)^{\frac{y}{k}-1}\exp\left(\frac{-b}{t^k(1-t^k)}\right)dt. \tag{30} $$

This extension will be very useful because of most of the properties of the *p-k* Beta and Beta function by considering particular values for the parameters and it provides connections with the hypergeometric function.

Particular cases:

For some particular values of the parameters p, k, b we can obtain certain Beta functions, defined earlier:

(a) For $b = 0$, equation (29), reduces to *p-k* Beta function defined in Gehlot (2017b)

$$ {}_pB_k(x,y;0) = {}_pB_k(x,y). $$

(b) For $p = k, b = 0$, equation (29), reduces to k-Beta function defined in Diaz and Pariguan (2007)

$$ {}_kB_k(x,y;0) = B_k(x,y). $$

(c) For $p = k = 1$, equation (29), reduces to extended Beta function defined in Chaudhry et al. (1997)

$$ {}_1B_1(x,y;b) = B(x,y;b). $$

(d) For $p = k = 1, b = 0$, equation (29), reduces to classical Beta function defined in Rainville (1963)

$$ {}_1B_1(x,y;0) = B(x,y). $$

Remark 4.2.

The n^{th} derivative of extended p-k Beta function with respect to parameter b can be expressed in terms of the function

$$ \frac{d^n}{db^n}\,{}_pB_k(x,y;b) = (-1)^n\,{}_pB_k(x-n,y-n;b),\ n = 0,1,2,... \tag{31} $$

4.1. Integral representation of extended *p-k* Beta function

In the following theorem, we give the relationship between the extended *p-k* Beta function, the *p-k* Beta function, and the classical Beta function.

Theorem 4.3.

Given $x, s \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(s) > 0, Re(\frac{x}{s} + s) > 0, Re(\frac{y}{s} + s) > 0$ then

$$\int_0^\infty b^{s-1}\,{}_pB_k(x,y;b)db = \Gamma(s)\,{}_pB_k(x+ks, y+ks) = \frac{\Gamma(s)}{k}B(\frac{x}{k}+s, \frac{y}{k}+s). \tag{32}$$

Proof:

Multiplying (30) by b^{s-1} and integrating with respect to b from $b = 0$ to $b = \infty$, we get

$$\int_0^\infty b^{s-1}\,{}_pB_k(x,y;b)db = \int_0^\infty b^{s-1}[\int_0^1 t^{x-1}(1-t^k)^{\frac{y}{k}-1}\exp{(\frac{-b}{t^k(1-t^k)})}dt]db.$$

Changing the order of integration, we have

$$\int_0^\infty b^{s-1}\,{}_pB_k(x,y;b)db = \int_0^1 t^{x-1}(1-t^k)^{\frac{y}{k}-1}[\int_0^\infty b^{s-1}\exp{(\frac{-b}{t^k(1-t^k)})}db]dt.$$

Using the basic definition of Gamma function, we get

$$\int_0^\infty b^{s-1}\,{}_pB_k(x,y;b)db = \int_0^1 t^{x-1}(1-t^k)^{\frac{y}{k}-1}[\frac{1}{t^k(1-t^k)}]^{-s}\Gamma(s)dt,$$

$$\int_0^\infty b^{s-1}\,{}_pB_k(x,y;b)db = \Gamma(s)\int_0^1 t^{x+ks-1}(1-t^k)^{\frac{y}{k}+s-1}dt,$$

$$\int_0^\infty b^{s-1}\,{}_pB_k(x,y;b)db = \Gamma(s)\,{}_pB_k(x+ks, y+ks).$$

Using Equation (3.5) of Gehlot (2017b), we obtain the desired result. ■

Theorem 4.4.

Given $x \in \mathbb{C}/k\mathbb{Z}^-; k, p, b \in \mathbb{R}^+ - \{0\}$ and $Re(b) > 0, Re(x) > 0, Re(y) > 0$ then the integral representations are given by

$$ {}_pB_k(x,y;b)=\frac{2}{k}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{2x}{k}-1}(\sin\theta)^{\frac{2y}{k}-1}\exp\left(\frac{-4b}{(\sin 2\theta)^2}\right)d\theta. \tag{33}$$

$$ {}_pB_k(x,y;b)=\frac{1}{k}\int_0^{\infty}\frac{t^{\frac{x}{k}-1}}{(1+t)^{\frac{x+y}{k}}}\exp\left(\frac{-b(1+t)^2}{t}\right)dt. \tag{34}$$

$$ {}_pB_k(x,y;b)=\frac{2^{1-\frac{x+y}{k}}}{k}\int_{-1}^{1}(1+t)^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\exp\left(\frac{-4b}{(1-t^2)}\right)dt. \tag{35}$$

$$ {}_pB_k(x,y;b)=\frac{1}{k}\int_0^{1}\frac{t^{\frac{x}{k}-1}+t^{\frac{y}{k}-1}}{(1+t)^{\frac{x+y}{k}}}\exp\left(\frac{-b(1+t)^2}{t}\right)dt. \tag{36}$$

$$ {}_pB_k(x,y;b)=\frac{(c-a)^{1-\frac{x+y}{k}}}{k}\int_a^{c}(t-a)^{\frac{x}{k}-1}(c-t)^{\frac{y}{k}-1}\exp\left(\frac{-b(c-a)^2}{(t-a)(c-t)}\right)du. \tag{37}$$

$$ {}_pB_k(x,y;b)=\frac{2^{1-\frac{x+y}{k}}}{k}\int_{-\infty}^{\infty}\frac{\exp[\frac{\theta(y-x)}{k}-4b(\cosh\theta)^2]}{(\cosh\theta)^{\frac{(x+y)}{k}}}d\theta. \tag{38}$$

Proof:

The result (33) follows when we use the transformation $t=\cos^2\theta$ in (30). The result (34) follows when we use the transformation $t^k=\frac{u}{u+1}$ in (30). If we substitute $t=\frac{1-u}{1+u}$ in (34), we get the result (35). To prove the result (36), we divide the integral given by equation (34) between 0 to 1 and 1 to ∞ and substitute $t=\frac{1}{u}$ in second integral. If we substitute $u=a+(c-a)t^k$ in equation (30), we get the result (37) and finally if we substitute $t=\tanh\theta$ in equation (35) we get the result (38). ∎

Theorem 4.5.

The following functional relation holds:

$$ {}_pB_k(x,y+k;b)+{}_pB_k(x+k,y;b)={}_pB_k(x,y;b). \tag{39}$$

Proof:

Denoting the left hand side of (39) by A, we have

$$A=\int_0^1[t^{x-1}(1-t^k)^{\frac{y+k}{k}-1}+t^{x+k-1}(1-t^k)^{\frac{y}{k}-1}]\exp\left(\frac{-b}{t(1-t^k)^{\frac{1}{k}}}\right)dt,$$

$$A = \int_0^1 [t^{x-1}(1-t^k)^{\frac{y}{k}-1}] \exp\left(\frac{-b}{t(1-t^k)^{\frac{1}{k}}}\right) dt,$$

which is the desire result. ■

Theorem 4.6.

The extended *p-k* Beta function in terms of the hypergeometric function can be expressed as follows:

$$ {}_pB_k(x,y;b) = {}_pB_k(x,y)\ {}_2F_2[\frac{2k-(x+y)}{2k}, \frac{k-(x+y)}{2k}; 1-\frac{x}{k}, 1-\frac{y}{k}; -4b]. \tag{40}$$

Proof:

Consider Equation (29),

$$ {}_pB_k(x,y;b) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} \exp\left(\frac{-b}{t(1-t)}\right) dt,$$

$$ {}_pB_k(x,y;b) = \frac{1}{k}\sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \int_0^1 t^{\frac{x-mk}{k}-1}(1-t)^{\frac{y-mk}{k}-1} dt.$$

Using Equation (6), we have

$$ {}_pB_k(x,y;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!}\ {}_pB_k(x-mk, y-mk).$$

Using Equation (3.5) of Gehlot (2017b), we get

$$ {}_pB_k(x,y;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{1}{k} B\left(\frac{x-mk}{k}, \frac{y-mk}{k}\right).$$

In view of the definition of the Beta function, we have

$$ {}_pB_k(x,y;b) = \sum_{m=0}^{\infty} \frac{(-1)^m b^m}{m!} \frac{1}{k} B\left(\frac{x}{k}, \frac{y}{k}\right) \frac{(\frac{x}{k})_{-m}(\frac{y}{k})_{-m}}{(\frac{x+y}{k})_{-2m}}.$$

Using the well known identity $(a)_{-n} = \frac{(-1)^n}{(1-a)_n}$ and $(a)_{2n} = 2^{2n}(\frac{a}{2})_n(\frac{a+1}{2})_n$, we get

$$ {}_pB_k(x,y;b) = {}_pB_k(x,y)\, {}_2F_2\left[\frac{2k-(x+y)}{2k}, \frac{k-(x+y)}{2k}; 1-\frac{x}{k}, 1-\frac{y}{k}; -4b\right]. $$

Hence, proved. ■

5. Concluding remark

In this paper, we use the relation between the two-parameter Gamma and Beta function and gave an extension of two-parameter Gamma and Beta function. The obtained results are different than the results of Özergin, Özarslan, and Altın, and Shadab, Jabee, and Choi.

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