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Chebyshev Type Inequalities Involving the Fractional Integral Operator Containing Multi-Index Mittag-Leffler Function in the Kernel

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Abstract

Recently, several authors have investigated Chebyshev type inequalities for numerous fractional integral operators. Being motivated by the work done by earlier researchers and their numerous applications in probability, transform theory, numerical quadrature, statistical problems and its significance in fractional boundary value problems. We aim to evaluate Chebyshev type inequalities involving fractional integral operator containing multi-index Mittag-Leffler function in the kernel. Admissible connections of the results mentioned in this article to those associated with previously established familiar fractional integral operators have been pointed out.

Keywords: Chebyshev type inequalities; Mittag-Leffler function; Extended Mittag-Leffler functions; Fractional integral operators; Synchronous functions

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1. Introduction

Several fractional integral inequalities have proven to be a tool of great significance in the development of pure and applied mathematics. Numerous applications of these inequalities can be seen in probability, transform theory, numerical quadrature and statistical problems (for details, see Bansal, Kumar, Khan, et al. (2019); Lakshmikantham and Vatsala (2007); Ramírez and Vatsala (2009); Denton and Vatsala (2011); Debbouche et al. (2012); Sun et al. (2012); Zhao et al. (2013); Liu et al. (2013)) along with their significance in fractional boundary value problems where they are primarily used to establish uniqueness of solutions.

2. Preliminaries

First of all, we recall *Chebyshev inequality*, which is defined as follows (see Čebyšev (1982)):

$$\frac{1}{b-a}\int_a^b p(x)q(x)\ dx \geq \left(\frac{1}{b-a}\int_a^b p(x)\ dx\right)\left(\frac{1}{b-a}\int_a^b q(x)\ dx\right), \tag{1}$$

where p and q denotes synchronous and integrable functions on the interval $[a,b]$. Notably, two functions p and q are said to be *synchronous* on the interval $[a,b]$ if

$$(p(x)-p(y))\ (q(x)-q(y)) \geq 0, \qquad (x,y \in [a,b]). \tag{2}$$

Certain generalizations of the *Chebyshev inequality* (1) can be seen in literature, some examples of which are stated herewith. Niculescu and Roventa (2013) established that on considering two function $p,q \in L^{\infty}([a,b])$, the Chebyshev inequality will hold true under the following assumptions:

$$\left(p(x)-\frac{1}{x-a}\int_a^b p(x)dx\right)\left(q(x)-\frac{1}{x-a}\int_a^b q(x)dx\right) \geq 0. \tag{3}$$

Chebyshev inequality without using synchronous function was proven by Dahmani et al. (2016). Also, lately Chebyshev type inequalities including different integral operators have been presented by a number of authors (see, e.g. Belarbi and Dahmani (2009); Dahmani (2010); Dahmani (2011); Dahmani et al. (2016); Daiya et al. (2015); Purohit and Kalla (2014); Set et al. (2019a); Set et al. (2019b)).

For the sake of convenience, we recall here some definitions that we will be using in our study.

If $[a,b]$ $(-\infty < a < b < \infty)$ is a finite interval on the real axis $\mathbb{R}$, then the right-sided and left-sided Riemann-Liouville fractional integrals operators denoted by $I^{\gamma}_{a+}p$ and $I^{\gamma}_{b-}p$, respectively, are defined as follows (e.g., see Kilbas et al. (2006); Podlubny (1998)):

$$(I^{\gamma}_{a+}p)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x (x-t)^{\gamma-1} p(t)dt, \qquad (x > a; \Re(\gamma) > 0) \tag{4}$$

and

$$(I^{\gamma}_{b-}p)(x) = \frac{1}{\Gamma(\gamma)} \int_x^b (t-x)^{\gamma-1} p(t)dt, \qquad (x < b; \Re(\gamma) > 0). \tag{5}$$

Goš̈ta Mittag-Leffler (1903) introduced well known Mittag-Leffler function defined as:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad (\alpha \in \mathbb{C}, \, \Re(\alpha) > 0). \tag{6}$$

Since then numerous notable generalizations of popularly known Mittag-Leffler function (6) have been presented in literature.

Wiman (1905) present a familiar generalization of $E_{\alpha}(z)$ as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \qquad (\alpha, \beta \in \mathbb{C}, \; \Re(\alpha) > 0). \tag{7}$$

Another, generalization was introduced by Prabhakar (1971), which is given as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \qquad (\alpha, \beta, \gamma \in \mathbb{C}, \; \Re(\alpha) > 0). \tag{8}$$

Srivastava and Tomovski (2009) introduced further generalization of $E^{\gamma}_{\alpha,\beta}(z)$ as

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\delta n}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{9}$$

$$(z, \beta, \gamma \in \mathbb{C}, \; \Re(\alpha) > \max(0, \Re(\delta) - 1), \Re(\delta) > 0),$$

which was considered by Shukla and Prajapati (2007), in the special case when

$$\delta = q \, (q \in (0,1) \cup \mathbb{N}) \quad \text{and} \quad \min(\Re(\beta), \Re(\gamma)) > 0.$$

Additionally, Salim and Faraj (2012) introduced and studied the following two generalizations of above mentioned Mittag-Leffler functions:

$$E_{\alpha,\beta,p}^{\eta,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_{pn}}, \tag{10}$$

$$(p, q \in \mathbb{R}^+;\ \alpha, \beta, \eta, \delta \in \mathbb{C};\ \Re(\alpha) > 0)$$

and

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\eta,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}(\delta)_{pn}} \frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{11}$$

$$(p, q \in \mathbb{R}^+;\ q \leq \Re(\alpha) + p;\ \alpha, \beta, \eta, \delta, \mu, \nu, \rho, \sigma \in \mathbb{C};\ \min\{\Re(\alpha), \Re(\rho), \Re(\sigma)\} > 0).$$

Saxena and Nishimoto (2010) defined generalized multi-index Mittag-Leffler function (GMIMLF) in the following manner:

$$E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}(z) = E_{\gamma,\kappa}[(\alpha_j, \beta_j)_{j=1}^m; z] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)} \frac{z^n}{n!}, \tag{12}$$

$$(\alpha_j, \beta_j, \gamma, \kappa, z \in \mathbb{C},\ \Re(\beta_j) > 0\ \ (j = 1, ..., m),\ \Re(\sum_{j=1}^{m} \alpha_j) > \max\{0, \Re(\kappa) - 1\}),$$

where $(\gamma)_n$ represents the Pochhammer symbol.

Various integral operators involving distinct generalizations of Mittag-Leffler functions have been studied by several mathematicians (for details, see Srivastava et al. (2018); Prabhakar (1971); Srivastava and Tomovski (2009); Salim and Faraj (2012); Bansal et al. (2019b); Bansal and Choi (2019); Bansal et al. (2019a); Bansal, Kumar, Singh, et al. (2020)). Among them, we recall the following integral operator introduced by Srivastava et al. (2018):

$$\left(\varepsilon_{a+;(\alpha_j,\beta_j)_m;\beta}^{\omega;\gamma,\kappa;\alpha}\varphi\right)(x) = \int_a^x (x-t)^{\beta-1} E_{(\alpha_j,\beta_j)_m}^{\gamma,\kappa}(\omega(x-t)^{\alpha})\varphi(t)dt, \qquad (x > a), \tag{13}$$

$$\left(\alpha, \beta, \alpha_j, \beta_j, \gamma, \kappa, \omega \in \mathbb{C}; \min\{\Re(\beta), \Re(\kappa)\} > 0; \Re\left(\sum_{j=1}^{m} \alpha_j\right) > \max\{0, \Re(\kappa)\}\right).$$

Motivated by the above cited work, we propose to investigate Chebyshev type inequalities involving fractional integral operator (13) containing GMIMLF in the kernel.

3. Chebyshev Type Inequalities

In this part, we investigate Chebyshev type inequalities involving synchronous functions p and q along with the fractional integral operator (13). We also point out some special cases of our main findings involving simpler integral operators.

Theorem 3.1.

Let us consider two synchronous functions p and q on $[0, \infty)$ and also let $\alpha, \beta, \alpha_j, \beta_j, \gamma, \kappa, \omega, x \in \mathbb{R}^+$ and $\sum_{j=1}^{m} \alpha_j > \kappa$. Then,

$$(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p\, q)(x) \geq \frac{1}{(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} 1)(x)} (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p)(x) (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, q)(x). \tag{14}$$

Proof:

By definition of synchronous functions (2), we have

$$(p(\sigma) - p(\mu))\ (q(\sigma) - q(\mu)) \geq 0, \qquad (\sigma, \mu \in \mathbb{R}_0^+). \tag{15}$$

Or, equivalently,

$$p(\sigma)q(\sigma) + p(\mu)q(\mu) \geq p(\sigma)q(\mu) + p(\mu)q(\sigma), \qquad (\sigma, \mu \in \mathbb{R}_0^+). \tag{16}$$

Notably,

$$(x - \sigma)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x - \sigma)^{\alpha}), \qquad (0 < \sigma < x), \tag{17}$$

is positive for all variables and parameters involved therein. Now, multiplying both the sides of (16) by (17), we obtain

$$\begin{aligned}
&(x - \sigma)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x - \sigma)^{\alpha}) p(\sigma)q(\sigma) \\
&\qquad + (x - \sigma)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x - \sigma)^{\alpha}) p(\mu)q(\mu) \\
&\qquad \geq (x - \sigma)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x - \sigma)^{\alpha}) p(\sigma)q(\mu) \\
&\qquad + (x - \sigma)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x - \sigma)^{\alpha}) p(\mu)q(\sigma).
\end{aligned} \tag{18}$$

Further, integrating both sides of (18) with respect to σ over 0 to x and applying (13), we have

$$
\begin{aligned}
&(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p\, q)(x) + p(\mu)q(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, 1)(x) \\
&\qquad \geq q(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p)(x) + p(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, q)(x). \qquad (19)
\end{aligned}
$$

Furthermore, multiplying each side of (19) by $(x-\mu)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\alpha}) \qquad (0<\mu<x)$, we get

$$
\begin{aligned}
&(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p\, q)(x)(x-\mu)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\alpha}) \\
&\quad + p(\mu)q(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, 1)(x)(x-\mu)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\alpha}) \\
&\qquad \geq q(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p)(x)(x-\mu)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\alpha}) \\
&\qquad + p(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, q)(x)(x-\mu)^{(\beta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\alpha}). \qquad (20)
\end{aligned}
$$

Now, integrating both sides of (20) with respect to μ over 0 to x and further simplifying with the help of (13), we finally obtain the required result (14). ∎

Remark 3.2.

Taking $m = 1$ in (14), we obtain the Chebyshev inequality mentioned in the above theorem for the fractional integral operator introduced by Srivastava and Tomovski (2009). Again, setting $m = \kappa = 1$ in (14), we establish Chebyshev inequality for the fractional integral operator suggested by Prabhakar (1971).

Theorem 3.3.

Let us consider two synchronous functions p and q on $[0,\infty)$ and also let $\sum\limits_{j=1}^{m} \alpha_j > \kappa$ and $\alpha, \beta, \alpha_j, \beta_j, \gamma, \kappa, \delta, \vartheta, \omega, x \in \mathbb{R}^{+}$. Then,

$$
\begin{aligned}
&(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p\, q)(x)(\varepsilon^{\omega;\gamma,\kappa;\delta}_{0+;(\alpha_j,\beta_j)_m;\vartheta}\, 1)(x) + (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, 1)(x)(\varepsilon^{\omega;\gamma,\kappa;\delta}_{0+;(\alpha_j,\beta_j)_m;\vartheta}\, p\, q)(x) \\
&\quad \geq (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} p)(x)(\varepsilon^{\omega;\gamma,\kappa;\delta}_{0+;(\alpha_j,\beta_j)_m;\vartheta}\, q)(x) + (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, q)(x)(\varepsilon^{\omega;\gamma,\kappa;\delta}_{0+;(\alpha_j,\beta_j)_m;\vartheta}\, p)(x). \quad (21)
\end{aligned}
$$

Proof:

Proceeding on similar lines as in proof of the Theorem 3.1, we establish

$$
\begin{aligned}
&(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p\, q)(x)(x-\mu)^{(\vartheta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\delta}) \\
&\quad + p(\mu)q(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}(1))(x)(x-\mu)^{(\vartheta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\delta}) \\
&\qquad \geq q(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p)(x)(x-\mu)^{(\vartheta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\delta}) \\
&\qquad + p(\mu)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, q)(x)(x-\mu)^{(\vartheta-1)} E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(\omega(x-\mu)^{\delta}), \qquad (22)
\end{aligned}
$$

the desired result (21) can be easily obtained by integrating the above expression with respect to μ over 0 to x and applying (13). ∎

Remark 3.4.

Considering, $m = 1$ in (21), we obtain the Chebyshev inequality mentioned Theorem 3.3 for the fractional integral operator introduced by Srivastava and Tomovski (2009). Again, taking $m = \kappa = 1$ in (21), we establish Chebyshev inequality for the fractional integral operator given by Prabhakar (1971).

Theorem 3.5.

Let us consider a sequence of positive increasing functions $\{p_i\}$ $(i = 1, \cdots, n;\ n \in \mathbb{N})$ on $[0, \infty)$ and also let $\alpha, \beta, \alpha_j, \beta_j, \gamma, \kappa, \omega, x \in \mathbb{R}^+$ and $\sum_{j=1}^{m} \alpha_j > \kappa$. Then,

$$\left(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} \prod_{i=1}^{n} p_i\right)(x) \geq \{(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} 1)(x)\}^{1-n} \prod_{i=1}^{n} (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p_i)(x). \tag{23}$$

Proof:

The proof of this theorem can be established by induction on n. Firstly, taking a note that two real valued increasing functions on a given interval are synchronous on that particular interval.

If $n = 1$, then the result (23) holds true.

If $n = 2$, then by using (14), we obtain

$$\begin{aligned}\left(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} p_1 p_2\right)(x) \geq &\{(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} 1)(x)\}^{-1} \times \\ &(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p_1)(x)(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p_2)(x).\end{aligned} \tag{24}$$

Assuming that,

$$\left(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} \prod_{i=1}^{n-1} p_i\right)(x) \geq \{(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta} 1)(x)\}^{2-n} \prod_{i=1}^{n-1} (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\, p_i)(x). \tag{25}$$

Since, $\{p_i\}$ $(i = 1, \cdots, n;\ n \in \mathbb{N})$ are given to be positive increasing functions, then $\prod_{i=1}^{n-1} p_i$ is also an increasing function. Henceforth, applying Theorem 3.1 to the functions $\prod_{i=1}^{n-1} p_i = q,\ p_n = p$, we obtain

$$\left(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\prod_{i=1}^{n}p_i\right)(x) = (\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\,p\,q)(x)$$
$$\geq \{(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}1)(x)\}^{-1}(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\,p_n)(x)\left(\varepsilon^{\omega;\gamma,\kappa;\alpha}_{0+;(\alpha_j,\beta_j)_m;\beta}\prod_{i=1}^{n-1}p_i\right)(x). \quad (26)$$

Finally, by replacing the last factor of the right hand side (RHS) of the above inequality (26) by the RHS of the inequality (25), we obtain the required result. ■

Remark 3.6.

Taking $m = 1$ in (23), we obtain the Chebyshev inequality mentioned in the above theorem for the fractional integral operator introduced by Srivastava and Tomovski (2009). Again, setting $m = \kappa = 1$ in (23), we establish Chebyshev inequality for the fractional integral operator suggested by Prabhakar (1971).

4. Conclusion

We conclude this paper by commenting on the development of new general extensions of Chebyshev type inequalities involving fractional integral operators containing multi-index Mittag-Leffler function in the kernel. Through properly specialization the parameters, additional integral inequalities concerning the variety of fractional integral operators can further be easily obtained from these key results.

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