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# A New Method to Solve Fractional Differential Equations: Inverse Fractional Shehu Transform Method

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## Abstract

In this paper, we propose a new method called the inverse fractional Shehu transform method to solve homogenous and non-homogenous linear fractional differential equations. Fractional derivatives are described in the sense of Riemann-Liouville and Caputo. Illustrative examples are given to demonstrate the validity, efficiency and applicability of the presented method. The solutions obtained by the proposed method are in complete agreement with the solutions available in the literature.

**Keywords:** Fractional differential equations; Riemann-Liouville fractional derivative; Caputo fractional derivative; Shehu transform

**MSC 2010 No.:** 34A08, 35A22, 33E12, 35C10

## 1. Introduction

The theory of fractional calculus plays an important role in many fields of pure and applied mathematics. Fractional integrals and derivatives, in association with different integral transforms, are used to solve different types of differential and integral equations. Fractional differential equations are widely used in interpretation and modeling in applied mathematics and physics including fluid mechanics, electrical circuits, diffusion, damping laws, relaxation processes, mathematical biology, and so on (see, for example, Herrmann (2014), Khader et al. (2018), Saad et al. (2019), and Khalouta et al. (2019a)). Therefore, the search for solutions to fractional differential equations is an important aspect of scientific research.

There are several mathematical methods to obtain the solutions of fractional differential equations, such as: Adomian decomposition method (Cheng et al. (2011)), variational iteration method (Ziane (2018)), new iterative method (Khalouta et al. (2019c)), differential transform method (Grover et al. (2017)), homotopy analysis method (Anber et al. (2014)), homotopy perturbation method (Hemed (2014)), fractional reduced differential transform method (Khalouta et al. (2019b)), fractional residual power series method (Khalouta et al. (2020)). Also, there are some other classical solution techniques such as Laplace transform method, fractional Green's function method, Mellin transform method and method of orthogonal polynomials (Podlubny (1999)).

The purpose of this paper is to present a new method called the inverse fractional Shehu transform method for solving fractional differential equations. Our aim is to extend the application of the proposed method to obtain the exact solutions to linear fractional differential equations.

This paper is organized as follows. In Section 2, we give some definitions and preliminaries of fractional calculus theory. In Section 3, we present six theorems with detailed proofs related to the inverse fractional Shehu transform method. In Section 4, we implement the inverse fractional Shehu transform method to some examples of homogenous and non-homogenous linear fractional differential equations. Section 5 is for conclusions of this paper.

## 2. Definitions and Preliminaries

There are several definitions of a fractional derivative of order $\alpha \geq 0$ (see Kilbas et al. (2006), Podlubny (1999)). The most commonly used definitions are the Riemann-Liouville and Caputo. In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

**Definition 2.1.**

A real function $f(t), t > 0$, is considered to be in the space $C_\mu$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h(t) \in C\left([0, \infty[\right)$, and it is said to be in the space $C_\mu^n$ if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$.

**Definition 2.2.**

The Riemann-Liouville fractional integral operator $I^{\alpha}$ of order $\alpha$ for a function $f \in C_{\mu}, \mu \geq -1$ is defined as follows,

$$I^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_{0}^{t} (t-\xi)^{\alpha-1} f(\xi) d\xi, \alpha > 0, t > 0, \\ f(t), \qquad \alpha = 0, \end{cases} \tag{1}$$

where $\Gamma(.)$ is the well-known Gamma function.

**Definition 2.3.**

The Riemann-Liouville fractional derivative operator ${}^{R}D^{\alpha}$ of order $\alpha$ for a function $f \in C_{\mu}, \mu \geq -1$ is defined as follows,

$${}^{R}D^{\alpha} f(t) = D^{n} I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int\limits_{0}^{t} (t-\xi)^{n-\alpha-1} f(\xi) d\xi, t > 0, \tag{2}$$

where $n-1 < \alpha \leq n, n \in \mathbb{N}$.

**Definition 2.4.**

The fractional derivative of $f(t)$ in the Caputo sense is defined as follows,

$${}^{c}D^{\alpha} f(t) = I^{n-\alpha} D^{n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{0}^{t} (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, t > 0, \tag{3}$$

where $n-1 < \alpha \leq n, n \in \mathbb{N}$, $f \in C_{-1}^{n}$.

**Definition 2.5.**

The Mittag-Leffler function is defined as follows,

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{4}$$

A further generalization of (4) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n\alpha+\beta)}, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0. \tag{5}$$

## 3. Theories of the inverse fractional Shehu transform method

In this section, we proves six theorems related to the inverse fractional Shehu transform method.

### 3.1. Shehu transform

Recently, Shehu Maitama Shehu et al. (2019) introduced a new integral transform, called Shehu transform, which is applied to solve an ordinary and partial differential equations.

**Definition 3.1.**

The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions

$$A = \left\{ f(t)/\exists N, \eta_1, \eta_2 > 0, |f(t)| < N \exp\left(\frac{|t|}{\eta_j}\right), \text{ if } t \in (-1)^i \times [0, \infty) \right\},$$

by the following integral

$$\mathbb{S}\left[f(t)\right] = F(s,u) = \int_0^\infty \exp\left(-\frac{st}{u}\right) f(t)dt, t > 0,$$

Some basic properties of the Shehu transform are given as follows

**Property 1.**

The Shehu transform is a linear operator. That is, if $\lambda$ and $\mu$ are non-zero constants, then

$$\mathbb{S}\left[\lambda f(t) \pm \mu g(t)\right] = \lambda \mathbb{S}\left[f(t)\right] \pm \mu \mathbb{S}\left[g(t)\right]. \tag{6}$$

**Property 2.**

If $f^{(n)}(t)$ is the $n$-th derivative of the function $f(t) \in A$ with respect to "$t$" then its Shehu transform is given by

$$\mathbb{S}\left[f^{(n)}(t)\right] = \frac{s^n}{u^n} F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} f^{(k)}(0). \tag{7}$$

**Property 3.**

Suppose $F(s,u)$ and $G(s,u)$ are the Shehu transforms of $f(t)$ and $g(t)$, respectively, both defined in the set $A$. Then the Shehu transform of their convolution is given by

$$\mathbb{S}\left[\left(f * g\right)(t)\right] = F(s,u)G(s,u),$$

where the convolution of two functions is defined by

$$\left(f * g\right)(t) = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t f(t-\xi)g(\xi)d\xi.$$

**Property 4.**

Some special Shehu transforms are

$$\begin{aligned} \mathbb{S}(1) &= \frac{u}{s}, \\ \mathbb{S}(t) &= \frac{u^2}{s^2}, \\ \mathbb{S}\left(\frac{t^n}{n!}\right) &= \left(\frac{u}{s}\right)^{n+1}, n = 0, 1, 2, ... \end{aligned} \tag{8}$$

**Property 5.**

The Shehu transform of $t^\alpha$ is given by

$$\mathbb{S}\left[t^\alpha\right] = \left(\frac{u}{s}\right)^{\alpha+1} \Gamma\left(\alpha+1\right).$$

### 3.2. Inverse Shehu transform

Now, we give the proof of Theorems 3.2–3.4, which are useful for finding the inverse Shehu transform function

$$f(t) = \mathbb{S}^{-1}\left[F(s,u)\right].$$

**Theorem 3.2.**

If $\alpha, \beta > 0, a \in \mathbb{R}$, and $|a| < \dfrac{s^\alpha}{u^\alpha}$, then we have the inverse Shehu transform formula

$$\mathbb{S}^{-1}\left[\frac{u^\beta s^{\alpha-\beta}}{s^\alpha + au^\alpha}\right] = t^{\beta-1}E_{\alpha,\beta}(-at^\alpha). \tag{9}$$

***Proof:***

First, we take the Shehu transform of the right-hand side of Equation (9) to get

$$\begin{aligned} \mathbb{S}\left[t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)\right] &= \int_0^\infty \exp\left(-\frac{st}{u}\right) t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)dt \\ &= \int_0^\infty \exp\left(-\frac{st}{u}\right) t^{\beta-1}\sum_{k=0}^\infty \frac{(-at^\alpha)^k}{\Gamma(k\alpha+\beta)}dt \\ &= \sum_{k=0}^\infty \frac{(-a)^k}{\Gamma(k\alpha+\beta)}\int_0^\infty \exp\left(-\frac{st}{u}\right) t^{\alpha k+\beta-1}dt. \end{aligned} \tag{10}$$

Now, by integration by parts we have

$$\int_0^\infty \exp\left(-\frac{st}{u}\right) t^{\alpha k+\beta-1}dt = \left(\frac{u}{s}\right)^{\alpha k+\beta}\Gamma(k\alpha+\beta). \tag{11}$$

By substituting Equation (11) into Equation (10), we get

$$\begin{aligned}\mathbb{S}\left[t^{\beta-1}E_{\alpha,\beta}(-at^{\alpha})\right] &= \sum_{k=0}^{\infty}\frac{(-a)^k}{\Gamma(k\alpha+\beta)}\left(\frac{u}{s}\right)^{\alpha k+\beta}\Gamma(k\alpha+\beta)\\ &= \left(\frac{u}{s}\right)^{\beta}\sum_{k=0}^{\infty}\left(\frac{-au^{\alpha}}{s^{\alpha}}\right)^{k}\\ &= \left(\frac{u}{s}\right)^{\beta}\frac{1}{1-\left(\frac{-au^{\alpha}}{s^{\alpha}}\right)}\\ &= \left(\frac{u}{s}\right)^{\beta}\frac{s^{\alpha}}{s^{\alpha}+au^{\alpha}},\ \left|\frac{au^{\alpha}}{s^{\alpha}}\right|<1.\end{aligned} \tag{12}$$

Then, the inverse Shehu transform of Equation (12) is given by

$$\mathbb{S}^{-1}\left[\frac{u^{\beta}s^{\alpha-\beta}}{s^{\alpha}+au^{\alpha}}\right] = t^{\beta-1}E_{\alpha,\beta}(-at^{\alpha}).$$

The proof is complete. ■

**Theorem 3.3.**

If $\alpha \geq \beta > 0, a \in \mathbb{R}$, and $|a| < \left(\frac{s}{u}\right)^{\alpha-\beta}$, then we have the inverse Shehu transform formula

$$\mathbb{S}^{-1}\left[\frac{u^{(n+1)(\alpha+\beta)}}{(s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta})^{n+1}}\right] = t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}. \tag{13}$$

***Proof:***

Similarly to the proof of Theorem 3.2, we take the Shehu transform of the right-hand side of Equation (13) and by integration by parts, we get

$$\begin{aligned}&\mathbb{S}\left[t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}\right]\\ &= \left(\frac{u}{s}\right)^{\alpha(n+1)}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-a\frac{u^{\alpha-\beta}}{s^{\alpha-\beta}}\right)^{k}.\end{aligned} \tag{14}$$

Using the series expansion of $(1+t)^{-(n+1)}$ of the form

$$\frac{1}{(1+t)^{n+1}} = \sum_{k=0}^{\infty}\binom{n+k}{k}(-t)^k, \tag{15}$$

we have

$$\mathbb{S}\left[t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}\right] = \frac{u^{(n+1)(\alpha+\beta)}}{(s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta})^{n+1}},\ \left|a\frac{u^{\alpha-\beta}}{s^{\alpha-\beta}}\right|<1. \tag{16}$$

Then, the inverse Shehu transform of Equation (16) is given by

$$\mathbb{S}^{-1}\left[\frac{u^{(n+1)(\alpha+\beta)}}{(s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta})^{n+1}}\right]=t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^{k}\binom{n+k}{k}}{\Gamma(k\,(\alpha-\beta)+(n+1)\alpha)}t^{k(\alpha-\beta)}.$$

The proof is complete. ∎

**Theorem 3.4.**

If $\alpha\geq\beta,\alpha>\gamma,a\in\mathbb{R},|a|<\left(\frac{s}{u}\right)^{\alpha-\beta}$, and $|b|<\dfrac{s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta}}{u^{\alpha+\beta}}$, then we have the inverse Shehu transform formula

$$\mathbb{S}^{-1}\left[\frac{u^{\alpha+\beta-\gamma}s^{\gamma}}{s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta}+bu^{\alpha+\beta}}\right]=t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma(k\,(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha}. \quad (17)$$

***Proof:***

We take the Shehu transform of the right-hand side of Equation (17), by integration by parts and using the series expansion (15), we get

$$\begin{aligned}&\mathbb{S}\left[t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(a)^{k}\binom{n+k}{k}}{\Gamma(k\,(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha}\right]\\&=\left(\frac{s}{u}\right)^{\gamma}\sum_{n=0}^{\infty}(-b)^{n}\left(\frac{u}{s}\right)^{\alpha(n+1)}\frac{1}{\left(1+a\dfrac{u^{\alpha-\beta}}{s^{\alpha-\beta}}\right)^{n+1}},\ \left|a\frac{u^{\alpha-\beta}}{s^{\alpha-\beta}}\right|<1\\&=\frac{u^{\alpha+\beta-\gamma}s^{\gamma}}{s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta}}\sum_{n=0}^{\infty}\left(\frac{-bu^{\alpha+\beta}}{s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta}}\right)^{n}\\&=\frac{v^{\alpha-\gamma+1}}{s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta}+bu^{\alpha+\beta}},\ \left|\frac{-bu^{\alpha+\beta}}{s^{\alpha}u^{\beta}+au^{\alpha}s^{\beta}}\right|<1.\end{aligned} \quad (18)$$

Then, the inverse Shehu transform of Equation (18) is given by

$$\mathbb{S}^{-1}\left[\frac{v^{\alpha-\gamma+1}}{1+av^{\alpha-\beta}+bv^{\alpha}}\right]=t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma(k\,(\alpha-\beta)+(n+1)\alpha-\gamma)}t^{k(\alpha-\beta)+n\alpha}.$$

The proof is complete. ∎

### 3.3. Shehu transform for fractional derivatives

**Theorem 3.5.**

If $F(s,u)$ is the Shehu transform of $f(t)$, then the Shehu transform of the Riemann-Liouville fractional integral for the function $f(t)$ of order $\alpha$, is given by

$$\mathbb{S}\left[I^{\alpha}f(t)\right]=\left(\frac{u}{s}\right)^{\alpha}F(s,u). \quad (19)$$

***Proof:***

The Riemann-Liouville fractional integral for the function $f(t)$ as in (1) can be expressed as the convolution

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t). \tag{20}$$

Applying the Shehu transform in Equation (20) and using Properties 3 and 5, we have

$$\begin{aligned} \mathbb{S}\left[I^{\alpha} f(t)\right] = \mathbb{S}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)\right] &= \mathbb{S}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right] \mathbb{S}\left[f(t)\right] \\ &= \left(\frac{u}{s}\right)^{\alpha} F(s,u). \end{aligned}$$

The proof is complete. ■

**Theorem 3.6.**

Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n - 1 < \alpha \leq n$, and $F(s,u)$ be the Shehu transform of the function $f(t)$, then the Shehu transform denoted by $F_{\alpha}^{R}(s,u)$ of the Riemann-Liouville fractional derivative of $f(t)$ of order $\alpha$, is given by

$$\mathbb{S}\left[{}^{R}D^{\alpha} f(t)\right] = F_{\alpha}^{R}(s,u) = \frac{s^{\alpha}}{u^{\alpha}} F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{k} \left[{}^{R}D^{\alpha-k-1} f(t)\right]_{t=0}. \tag{21}$$

***Proof:***

Since

$${}^{R}D^{\alpha} f(t) = D^{n} I^{n-\alpha} f(t) = \frac{d^{n}}{dt^{n}} I^{n-\alpha} f(t).$$

Let

$$g(t) = I^{n-\alpha} f(t), \tag{22}$$

then,

$${}^{R}D^{\alpha} f(t) = \frac{d^{n}}{dt^{n}} g(t) = g^{(n)}(t).$$

Applying the Shehu transform on both sides of Equation (22) and using Theorem 3.5, we get

$$G(s,u) = \mathbb{S}\left[g(t)\right] = \mathbb{S}\left[I^{n-\alpha} f(t)\right] = \left(\frac{u}{s}\right)^{n-\alpha} F(s,u). \tag{23}$$

Also, we have from Property 2

$$\begin{aligned}
\mathbb{S}\left[{}^{R}D^{\alpha}f(t)\right] &= \mathbb{S}\left[\frac{d^{n}}{dt^{n}}g(t)\right] \\
&= \frac{s^{n}}{u^{n}}G(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)}\left[g^{(k)}(t)\right]_{t=0} \\
&= \frac{s^{n}}{u^{n}}G(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{k}\left[g^{(n-k-1)}(t)\right]_{t=0}.
\end{aligned} \tag{24}$$

From the definition of the Riemann-Liouville fractional derivative as in (2), we obtain

$$\begin{aligned}
\left[g^{(n-k-1)}(t)\right]_{t=0} &= \left[\frac{d^{n-k-1}}{dt^{n-k-1}}g(t)\right]_{t=0} \\
&= \left[D^{n-k-1}I^{n-\alpha}f(t)\right]_{t=0} \\
&= \left[{}^{R}D^{\alpha-k-1}f(t)\right]_{t=0}.
\end{aligned} \tag{25}$$

Hence, by using Equations (25) and (23) in (24), we get

$$\begin{aligned}
\mathbb{S}\left[{}^{R}D^{\alpha}f(t)\right] &= \frac{s^{\alpha}}{u^{\alpha}}F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{k}\left[{}^{R}D^{\alpha-k-1}f(t)\right]_{t=0} \\
&= F^{c}_{\alpha}(s,u), \qquad -1 < n-1 < \alpha \leq n.
\end{aligned}$$

The proof is complete. ∎

**Theorem 3.7.**

Let $n \in \mathbb{N}^{*}$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $F(s,u)$ be the Shehu transform of the function $f(t)$. Then the Shehu transform denoted by $F^{c}_{\alpha}(s,u)$ of the Caputo fractional derivative of $f(t)$ of order $\alpha$, is given by

$$\mathbb{S}\left[{}^{c}D^{\alpha}f(t)\right] = F^{c}_{\alpha}(s,u) = \frac{s^{\alpha}}{u^{\alpha}}F(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{\alpha-(k+1)}\left[D^{k}f(t)\right]_{t=0}. \tag{26}$$

***Proof:***

Let

$$g(t) = f^{(n)}(t),$$

then, by the definition of the Caputo fractional derivative as in (3), we obtain

$$\begin{aligned} {}^cD^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} g(\xi) d\xi \\ &= I^{n-\alpha} g(t). \end{aligned} \tag{27}$$

Applying the Shehu transform on both sides of (27) and using Theorem 3.5, we get

$$\mathbb{S}\left[{}^cD^\alpha f(t)\right] = \mathbb{S}\left[I^{n-\alpha} g(t)\right] = \left(\frac{u}{s}\right)^{n-\alpha} G(s,u). \tag{28}$$

Also, we have from the Property 2

$$\begin{aligned} &\mathbb{S}\left[g(t)\right] = \mathbb{S}\left[f^{(n)}(t)\right], \\ &G(s,u) = \frac{s^n}{u^n} F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} \left[f^{(k)}(t)\right]_{t=0}. \end{aligned} \tag{29}$$

Hence, (28) becomes

$$\begin{aligned} \mathbb{S}\left[{}^cD^\alpha f(t)\right] &= \left(\frac{u}{s}\right)^{n-\alpha} \left(\frac{s^n}{u^n} F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{n-(k+1)} \left[f^{(k)}(t)\right]_{t=0}\right) \\ &= \frac{s^\alpha}{u^\alpha} F(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{\alpha-(k+1)} \left[D^k f(t)\right]_{t=0} = F_\alpha^c(s,u), \\ &-1 < n-1 < \alpha \leq n. \end{aligned}$$

The proof is complete. ∎

## 4. Illustrative examples

In this section, we shall illustrate the applicability of the inverse fractional Shehu transform method to some linear fractional differential equations.

**Example 4.1.**

Consider the following linear fractional initial value problem (Li (2010)),

$${}^RD^{1/2} y(t) + y(t) = 0, \tag{30}$$

subject to the initial condition

$$\left[{}^RD^{-1/2} y(t)\right]_{t=0} = 2. \tag{31}$$

Applying the Shehu Transform on both sides of Equation (30) and using Theorem 3.6, we get

$$\left(\frac{s}{u}\right)^{1/2} Y(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{k} \left[{}^{R}D^{1/2-k-1} f(t)\right]_{t=0} + Y(s,u) = 0. \tag{32}$$

Substituting Equation (31) into Equation (32), we get

$$\left[\left(\frac{s}{u}\right)^{1/2} + 1\right] Y(s,u) - 2 = 0.$$

So

$$Y(s,u) = \mathbb{S}\left[y(t)\right] = \frac{2u^{1/2}}{s^{1/2} + u^{1/2}}.$$

Using the Theorem 3.2, the exact solution of this problem can be obtained as

$$y(t) = 2t^{-1/2} E_{\frac{1}{2},\frac{1}{2}}(-t^{1/2}).$$

**Example 4.2.**

Consider the initial value problem for a non-homogeneous fractional differential equation (Li (2010)),

$${}^{R}D^{\alpha} y(t) - \lambda y(t) = h(t), t > 0, n-1 < \alpha \leq n, \tag{33}$$

subject to the initial condition

$$\left[{}^{R}D^{\alpha-k-1} y(t)\right]_{t=0} = b_k, k = 0, 1, 2, ..., \tag{34}$$

where $\lambda$ and $b_k$ are constants and fractional derivative ${}^{R}D^{\alpha}$ denotes the Riemann-Liouville fractional derivative.

Applying the Shehu Transform on both sides of Equation (33) and using Theorem 3.6, we get

$$\left(\frac{s}{u}\right)^{\alpha} Y(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{k} \left[{}^{R}D^{\alpha-k-1} f(t)\right]_{t=0} - \lambda Y(s,u) = H(s,u). \tag{35}$$

Substituting Equation (34) into Equation (35), we get

$$\left[\left(\frac{s}{u}\right)^{\alpha} - \lambda\right] Y(s,u) - \sum_{k=0}^{n-1} \left(\frac{s}{u}\right)^{k} b_k = H(s,u).$$

So

$$Y(s,u) = \mathbb{S}\left[y(t)\right] = \frac{u^{\alpha}}{s^{\alpha} - \lambda u^{\alpha}} H(s,u) + \sum_{k=0}^{n-1} \frac{s^{k} u^{\alpha-k}}{(s^{\alpha} - \lambda u^{\alpha})} b_k.$$

Using Theorem 3.2 and the convolution property (3), we get

$$y(t) = \left[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) * h(t)\right] + \sum_{k=0}^{n-1} b_k t^{\alpha-k-1} E_{\alpha,\alpha-k}(\lambda t^{\alpha})$$
$$= \int_0^{\infty} (t-\xi)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-\xi)^{\alpha}) h(\xi) d\xi + \sum_{k=0}^{n-1} b_k t^{\alpha k-1} E_{\alpha,\alpha-k}(\lambda t^{\alpha}).$$

This is the exact solution of this problem.

**Example 4.3.**

Consider the initial value problem of non-homogeneous Bagley-Torvik equation (Bansal et al. (2016)),

$$y''(t) + D^{3/2}y(t) + y(t) = 1 + t, \tag{36}$$

subject to the initial conditions

$$y(0) = y'(0) = 1. \tag{37}$$

Applying the Shehu Transform on both sides of Equation (36) and using Theorem 3.7, we get

$$\frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - 1 + \frac{s^{3/2}}{u^{3/2}}Y(s,u) - \frac{s^{1/2}}{u^{1/2}}y(0) - \frac{s^{-1/2}}{u^{-1/2}}y'(0) + Y(s,u) = \frac{u}{s} + \frac{u^2}{s^2}. \tag{38}$$

Substituting Equation (37) into Equation (38), we get

$$Y(s,u)\left[\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right] = \frac{u}{s} + \frac{u^2}{s^2} + \frac{s}{u} + 1 + \frac{s^{1/2}}{u^{1/2}} + \frac{s^{-1/2}}{u^{-1/2}}. \tag{39}$$

Then Equation (39) becomes

$$Y(s,u)\left[\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right] = \left(\frac{u}{s} + \frac{u^2}{s^2}\right)\left(\frac{s^2}{u^2} + \frac{s^{3/2}}{u^{3/2}} + 1\right). \tag{40}$$

So

$$Y(s,u) = \mathbb{S}\left[y(t)\right] = \frac{u}{s} + \frac{u^2}{s^2}. \tag{41}$$

Taking the inverse Shehu transform of Equation (41), we have

$$y(t) = 1 + t.$$

This is the exact solution of this problem.

**Example 4.4.**

Consider the following linear fractional initial value problem (Hashim et al. (2009), Kumar et al. (2006), Saadatmandi et al. (2010)),

$$^{c}D^{\alpha}y(t)+y(t)=0, 0<\alpha\leq 2, \tag{42}$$

subject to the initial conditions

$$y(0)=1, y'(0)=0. \tag{43}$$

The second initial condition is for $\alpha>1$ only.

In two cases of $\alpha$, $\mathbb{S}\left[D^{\alpha}y(t)\right]$ is obtained as

1- For $\alpha<1$

$$\mathbb{S}\left[{}^{c}D^{\alpha}y(t)\right]=\frac{s^{2}Y(s,u)}{u^{\alpha}s^{2-\alpha}}-\frac{s}{u^{\alpha-1}s^{2-\alpha}}=\frac{s^{\alpha}}{u^{\alpha}}Y(s,u)-\left(\frac{s}{u}\right)^{\alpha-1}.$$

2- For $\alpha>1$

$$\mathbb{S}\left[{}^{c}D^{\alpha}y(t)\right]=\frac{sY(s,u)}{u^{\alpha}s^{1-\alpha}}-\frac{1}{u^{\alpha-1}s^{1-\alpha}}=\frac{s^{\alpha}}{u^{\alpha}}Y(s,u)-\left(\frac{s}{u}\right)^{\alpha-1},$$

which are the same.

Applying the Shehu transform to both sides of Equation (42) and using Theorem 3.7, we get

$$\frac{s^{\alpha}}{u^{\alpha}}Y(s,u)-\left(\frac{s}{u}\right)^{\alpha-1}+Y(s,u)=0.$$

So

$$Y(s,u)=\mathbb{S}\left[y(t)\right]=\frac{us^{\alpha-1}}{s^{\alpha}+u^{\alpha}}.$$

Using the Theorem (3.2), the exact solution of this problem can be obtained as

$$y(t)=E_{\alpha}(-t^{\alpha}).$$

**Example 4.5.**

Consider the following linear fractional initial value problem (Odibat et al. (2008)),

$$^{c}D^{\alpha}y(t)=y(t)+1, 0<\alpha\leq 1, \tag{44}$$

subject to the initial condition

$$y(0)=0. \tag{45}$$

Applying the Shehu transform to both sides of Equation (44) and using Theorem 3.7, we get

$$\frac{s^{\alpha}}{u^{\alpha}}Y(s,u)=Y(s,u)+\frac{u}{s}.$$

So

$$Y(s,u) = \mathbb{S}\left[y(t)\right] = \frac{u^{\alpha+1}s^{-1}}{s^{\alpha}+u^{\alpha}}.$$

Using the Theorem (3.2, the exact solution of this problem can be obtained as

$$y(t) = t^{\alpha}E_{\alpha,\alpha+1}(t^{\alpha}).$$

**Example 4.6.**

Consider the composite fractional oscillation equation (Odibat et al. (2008)),

$$y''(t) - a^{c}D^{\alpha}y(t) - by(t) = 8, \qquad 1 < \alpha \leq 2, \tag{46}$$

subject to the initial conditions

$$y(0) = y'(0) = 0. \tag{47}$$

Applying the Shehu transform to both sides of Equation (46) and using Theorem 3.7, we get

$$\frac{s^2}{u^2}Y(s,u) - a\frac{s^{\alpha}}{u^{\alpha}}Y(s,u) - bY(s,u) = 8\frac{u}{s}.$$

So

$$Y(s,u) = \mathbb{S}\left[y(t)\right] = \frac{u^{\alpha+3}s^{-1}}{s^2u^{\alpha} - au^2s^{\beta}bu^{\alpha+2}}.$$

Using the Theorem 3.4, the exact solution of this problem can be obtained as

$$y(t) = 8t^2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^n a^k\binom{n+k}{k}}{\Gamma\left(k\left(2-\alpha\right)+2(n+1)+1\right)}t^{k(2-\alpha)+2n}.$$

## 5. Conclusion

In this paper, a new method called the inverse fractional Shehu transform method have been successfully applied to homogenous and non-homogenous linear fractional differential equations. We proved six theorems related to this method. The resolution of some examples show that the inverse fractional Shehu transform method is a powerful and efficient technique for finding exact solution of linear fractional differential equations.

In the next studies, we shall extend this approach by combining the proposed method with semi-analytical methods such as: Adomian decomposition method (ADM), homotopy perturbation method (HPM), homotopy analysis method (HAM), and variational iteration method (VIM) to study the solutions of another set of nonlinear fractional differential equations with high-order fractional derivatives where $n-1<\alpha<n$ and $n \geq 1$.

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