Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 13, Issue 2 (December 2018), pp. 997–1017

Applications and Applied Mathematics:
An International Journal
(AAM)

A de Casteljau Algorithm for Bernstein type Polynomials based on (p, q)-integers

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Received: March 9, 2017; Accepted: October 12, 2017

Abstract

In this paper, a de Casteljau algorithm to compute (p, q)-Bernstein B$\acute{e}$zier curves based on (p, q)-integers are introduced. The nature of degree elevation and degree reduction for (p, q)-B$\acute{e}$zier Bernstein functions are studied. The new curves have some properties similar to q-B$\acute{e}$zier curves. Moreover, we construct the corresponding tensor product surfaces over the rectangular domain $(u, v) \in [0, 1] \times [0, 1]$ depending on four parameters. De Casteljau algorithm and degree evaluation properties of the surfaces for these generalization over the rectangular domain are investigated. Furthermore, some fundamental properties for (p, q)-Bernstein B$\acute{e}$zier curves are discussed. We get q-B$\acute{e}$zier curves and surfaces for $(u, v) \in [0, 1] \times [0, 1]$ when we set the parameter $p_1 = p_2 = 1$. In comparison to q-B$\acute{e}$zier curves based on q-Bernstein polynomials, this generalization gives us more flexibility in controlling the shapes of curves.

Keywords: (p, q)-integers; (p, q)-Bernstein polynomials; Degree elevation; Degree reduction; de Casteljau algorithm; Tensor product; (p, q)-B$\acute{e}$zier curve; (p, q)-B$\acute{e}$zier surface; Shape preserving.

2010 MSC No.: 65D17, 41A10, 41A25, 41A36

1. Introduction

Recently, Mursaleen et al. (2015, 2016) applied the concept of (p,q)-calculus (which is considered as an extension of quantum-calculus) in Approximation theory and introduced the (p,q)-analogue of Bernstein operators based on (p,q)-integers. Apart from various generalisation of Bernstein operators, another paper by Mursaleen et al. (2016) which catches attention of researchers working in Approximation theory is Bleimann-Butzer-Hahn operators defined by (p,q)-integers. After this initiation, many researchers started working in this area. For similar works based on (p,q)-integers, one can refer [Acar (2016); Acar et al. (2016); Cai and Zhou (2016); Kadak (2016); Kadak (2017); Khan and Lobiyal (2017); Mursaleen et al. (2015); Mursaleen et al. (2016); Mursaleen et al. (2015); Mursaleen et al. (2016); Mishra and Pandey (2016); Mishra and Pandey (2017); Wafi and Rao (2016); Wafi and Rao (2017)].

Motivated by the above mentioned work, the idea of (p,q)-calculus (post quantum calculus) and its importance, we construct (p,q)-B$\acute{e}$zier curves and surfaces based on (p,q)-integers which is further generalization of q-B$\acute{e}$zier curves and surfaces.

Some of the advantages of using the extra parameter p has been discussed in [Khan and Lobiyal (2017); Mursaleen et al. (2016)].

It was S.N. Bernstein [Bernstein (1912)], who first introduced his famous operators $B_n : C[0,1] \to C[0,1]$ defined for any $n \in \mathbb{N}$ and for any function $f \in C[0,1]$

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1], \tag{1}$$

and named it Bernstein polynomials to provide a constructive proof of the Weierstrass theorem [Korovkin (1960)]. Later, it was found that Bernstein polynomials possess many remarkable properties and has various applications in areas such as approximation theory, numerical analysis, computer-aided geometric design, and solutions of differential equations due to its fine properties of approximation [Oruk and Phillips (2003)].

In Computer Aided Geometric Design (CAGD), Bernstein polynomials and its variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [Sederberg (2014); Farouki and Rajan (1988)] is the classical B$\acute{e}$zier curve [B$\acute{e}$zier (1972)] constructed with the help of Bernstein basis functions.

In recent years, generalization of the B$\acute{e}$zier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus, they generalized the B$\acute{e}$zier curves in [Farouki and Rajan (1988); Hana et al. (2014); Oruk and Phillips (2003); Rababah and Manna (2011)].

Lupaş (1987) introduced the first q-analogue of Bernstein operators (rational) as follows:

$$L_{n,q}(f;x)=\sum_{k=0}^{n}\frac{f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}\ x^k\ (1-x)^{n-k}}{\prod\limits_{j=1}^{n}\{(1-x)+q^{j-1}x\}},\tag{2}$$

and investigated its approximating and shape-preserving properties.

In 1996, Phillips (1997) proposed another q-variant of the classical Bernstein operators, the so-called Phillips q-Bernstein operators which attracted lots of investigations.

$$B_{n,q}(f;x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n-k-1}(1-q^sx)\ \ f\left(\frac{[k]_q}{[n]_q}\right),\ \ x\in[0,1],\tag{3}$$

where $B_{n,q}:C[0,1]\to C[0,1]$ defined for any $n\in\mathbb{N}$ and any function $f\in C[0,1]$.

The q-variants of Bernstein polynomials provide one shape parameter for constructing free-form curves and surfaces, Phillips q-Bernstein operator was applied well in this area.

In 2003, Oruk and Phillips [Oruk and Phillips (2003)] used the basis functions of Phillips q-Bernstein operator for construction of q-B$\acute{e}$zier curves, which they call Phillips q-B$\acute{e}$zier curves, and studied the properties of degree reduction and elevation.

Thus with the development of (p,q)-analogue of Bernstein operators and its variants, one natural question arises, how it can be used in order to preserve the shape of the curves or surfaces. In this way, it opens a new research direction which requires further investigations.

Recently, (p,q)-analogue of Lupaş Bernstein operators are defined as follows in [Khan and Lobiyal (2017)]:

For any $p>0$ and $q>0$, the linear operators $L^n_{p,q}:C[0,1]\to C[0,1]$

$$L^n_{p,q}(f;x)=\sum_{k=0}^{n}\frac{f\left(\frac{p^{n-k}\ [k]_{p,q}}{[n]_{p,q}}\right)\begin{bmatrix}n\\k\end{bmatrix}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\ x^k\ (1-x)^{n-k}}{\prod\limits_{j=1}^{n}\{p^{j-1}(1-x)+q^{j-1}x\}},\tag{4}$$

is (p,q)-analogue of Lupaş Bernstein operators.

Again when $p=1$, Lupaş (p,q)-Bernstein operators turns out to be Lupaş q-Bernstein operators as given in [Mahmudov and Sabancigil (2010)].

When $p = q = 1$, Lupaş (p,q)-Bernstein operators turns out to be classical Bernstein operators.

Let us recall certain notations of (p,q)-calculus .

For any $p > 0$ and $q > 0$, the (p,q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + ... + pq^{n-2} + q^{n-1} = \begin{cases} \frac{p^n - q^n}{p-q}, & \text{when} \quad p \neq q \neq 1, \\ n\ p^{n-1}, & \text{when } p = q \neq 1, \\ [n]_q, & \text{when } p = 1, \\ n, & \text{when } p = q = 1, \end{cases}$$

where $[n]_q$ denotes the q-integers and $n = 0, 1, 2, \cdots$.

It is obvious that $[n]_{p,q} = p^{n-1} \ \ [n]_{\frac{q}{p}}$.

The formula for (p,q)-binomial expansion is as follows:

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k} b^k x^{n-k} y^k.$$

For $a = b = 1$,

$$(x + y)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^{n-k} y^k.$$

On using,

$$(x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$

one can write,

$$(x)^n_{p,q} = (x)(px)(p^2x) \cdots (p^{n-1}x) = p^{\frac{n(n-1)}{2}} x^n,$$

which implies

$$(1)^n_{p,q} = (1)(p)(p^2) \cdots (p^{n-1}) = p^{\frac{n(n-1)}{2}}.$$

The (p,q)-analogue of Euler's identity is as follows:

$$(1-x)_{p,q}^n = \prod_{s=0}^{n-1}(p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x)$$
$$= \sum_{k=0}^{n}(-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k.$$

From (p,q)-binomial expansion, or by using induction on n, one can easily verify that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1}(p^s - q^s x) \ = p^{\frac{n(n-1)}{2}}, \ \ x \in [0,1]. \tag{5}$$

Again by some simple calculations and using the property of (p,q)-integers, one can find (p,q)-analogue of Pascal's relation as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \tag{6}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}. \tag{7}$$

Details on (p,q)-calculus can be found in [Hounkonnou et al. (2013); Jagannathan and Srinivasa (2005)].

The (p,q)-Bernstein Operators introduced by Mursaleen et al. (2015, 2016) for $0 < q < p \leq 1$ and $x \in [0,1]$ are as follows:

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1}(p^s - q^s x) \ \ f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right). \tag{8}$$

Note when $p = 1$, (p,q)-Bernstein Operators given by (8) turns out to be Phillips q-Bernstein Operators.

The operators given by (4) and (8) are respective generalization of q-Lupaş and q-Phillips operators.

For other works relevant to this field, one can see [Agarwal et al. (2013); Aral et al. (2013); Mishra and Deepmala (2013); Mursaleen and Khan (2013); Mursaleen and Nasiruzzaman (2017); Victor and Pokman (2002)].

We apply post-quantum calculus to introduce the (p, q)-B$\acute{e}$zier curves and surfaces based on (p, q)-integers constructed with the help of Bernstein basis of (8) which is further generalization of q-B$\acute{e}$zier curves and surfaces, for example, [Lupaş (1987); Hana et al. (2014); Oruk and Phillips (2003)].
The outline of this paper are as follows: Section 2 introduces a (p, q)-analogue of the Phillips q-Bernstein functions $B^{k,n}_{p,q}$ and its Properties. Section 3 introduces degree elevation and degree reduction properties for (p, q)-analogue of the Bernstein functions. Section 4 introduces a de Casteljau type algorithm for $B^{k,n}_{p,q}$. In Section 5, we define a tensor product patch based on algorithm 1 and its geometric properties as well as a degree elevation technique are investigated. Furthermore, tensor product of (p, q)-B$\acute{e}$zier surfaces on $[0, 1] \times [0, 1]$ for (p, q)-Bernstein polynomials are introduced and its properties that is inherited from the univariate case are being discussed. In section 6, we discuss Shape control of (p, q)-Bernstein curves.

2. (p, q)-Bernstein functions

For any $p > 0$ and $q > 0$ and $t \in [0, 1]$, the (p, q)-Bernstein functions are defined as follows:

$$B^{k,n}_{p,q}(t) = \frac{1}{p^{\frac{n(n-1)}{2}}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \; t^k (1-t)^{n-k}_{p,q}, \tag{9}$$

where

$$(1-t)^{n-k}_{p,q} = \prod_{s=0}^{n-k-1} (p^s - q^s t).$$

Note: The (p, q)-analogue of Bernstein operators introduced by Mursaleen et al. generate positive linear operators only if $0 < q < p \leq 1$.

2.1. Properties of the (p, q)-analogue of the Bernstein functions

Theorem 2.1.

The (p, q)-analogue of the Bernstein functions possess the following properties:

(1). Non-negativity: $B^{k,n}_{p,q}(t) \geq 0 \; k = 0, 1, ..., n, \quad t \in [0, 1]$.

(2). Partition of unity:

$$\sum_{k=0}^{n} B^{k,n}_{p,q}(t) = 1, \quad \text{for every } \; t \in [0, 1].$$

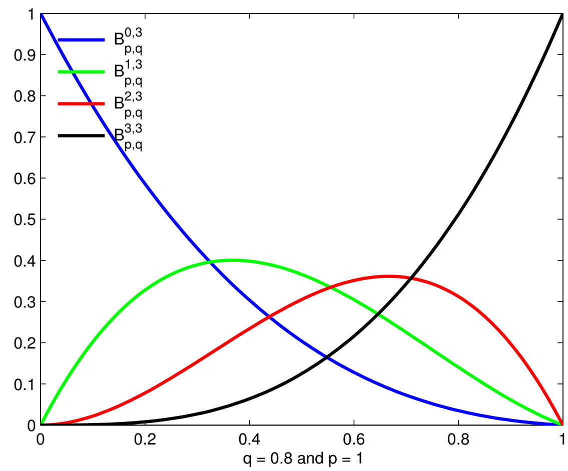

Figure 1. Cubic Bezier blending functions

(3). Both sided end-point property:

$$B_{p,q}^{k,n}(0) = \begin{cases} 1, & \text{if } k = 0, \\ \\ 0, & k \neq 0, \end{cases}$$

$$B_{p,q}^{k,n}(1) = \begin{cases} 1, & \text{if } k = n, \\ \\ 0, & k \neq n, \end{cases}$$

when $p = 1$, then, both side end point interpolation property holds.

(4). Reducibility: when $p = 1$, formula (2.1) reduces to the q-Bernstein bases.

Proof:

All these property can be deduced easily from equation (9).

Figure 2 shows the (p, q)-analogues of the Bernstein basis functions of degree 3 with $q = 0.5,\ p = 0.8$. Here we can observe that sum of blending fuctions is always unity and also end point interpolation property holds, when we put $p = 1$, it turns out to be q-Bernstein basis which is shown in Figure 1.

Apart from the basic properties above, the (p, q)-analogue of the Bernstein functions also satisfy some recurrence relations, as for the classical Bernstein basis.

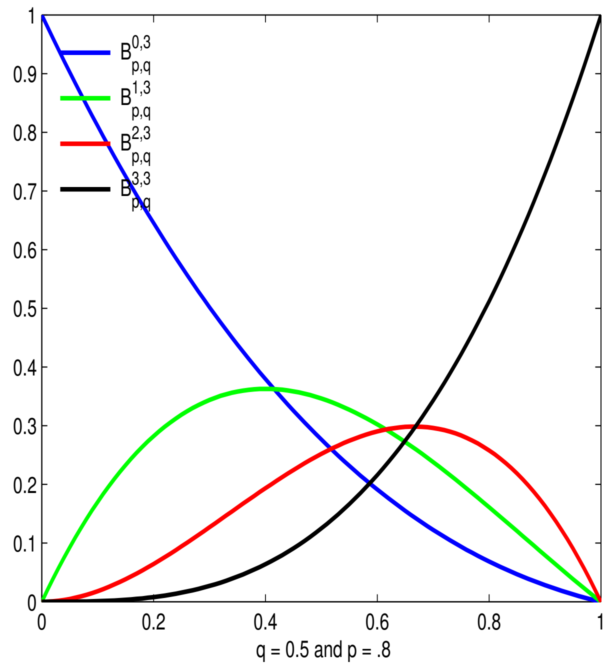

Figure 2. Cubic Bezier blending functions

3. Degree elevation and reduction for (p, q)-Bernstein functions

Technique of degree elevation has been used to increase the flexibility of a given curve. A degree elevation algorithm calculates a new set of control points by choosing a convex combination of the old set of control points which retains the old end points. For this purpose, the identities (12),(13) and Theorem (3.2) are useful.

Theorem 3.1.

Each (p, q)-Bernstein functions of degree n is a linear combination of two (p, q)-Bernstein functions of degree $n - 1$:

$$B_{p,q}^{k,n}(t) = q^{n-k}p^{k-1}\ t\ B_{p,q}^{k-1,n-1}(t) + (p^{n-1} - p^k q^{n-k-1}t)\ B_{p,q}^{k,n-1}(t), \tag{10}$$

$$B_{p,q}^{k,n}(t) = p^{n-1}\ t\ B_{p,q}^{k-1,n-1}(t) + (q^k p^{n-k-1} - q^{n-1}t)\ B_{p,q}^{k,n-1}(t). \tag{11}$$

Proof: On using Pascal's type relation based on (p,q)-integers i.e (6), we get

$$
\begin{aligned}
B^{k,n}_{p,q}(t) &= \frac{1}{p^{\frac{n(n-1)}{2}}}\left(q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} + p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}\right) p^{\frac{k(k-1)}{2}} \;\; t^k(1-t)^{n-k}_{p,q} \\
&= \frac{1}{p^{\frac{n(n-1)}{2}}} q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} \;\; t^k(1-t)^{n-k}_{p,q} \\
&+ \frac{1}{p^{\frac{n(n-1)}{2}}} p^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} t^k(1-t)^{n-k}_{p,q} \\
&= \frac{1}{p^{\frac{n(n-1)}{2}}} q^{n-k}p^{k-1} \;\; t \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{(k-1)(k-2)}{2}} t^{k-1}(1-t)^{n-k}_{p,q} \\
&+ \frac{1}{p^{\frac{n(n-1)}{2}}} \;\; p^k(p^{n-k-1} - q^{n-k-1}t) \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} t^k(1-t)^{n-k-1}_{p,q} \\
&= q^{n-k}p^{k-1} \; t \; B^{k-1,n-1}_{p,q}(t) + (p^{n-1} - p^k q^{n-k-1} t) \; B^{k,n-1}_{p,q}(t).
\end{aligned}
$$

Thus,

$$
B^{k,n}_{p,q}(t) = q^{n-k}p^{k-1} \; t \; B^{k-1,n-1}_{p,q}(t) + (p^{n-1} - p^k q^{n-k-1} t) \; B^{k,n-1}_{p,q}(t).
$$

Similarly, if we use (7), we have

$$
B^{k,n}_{p,q}(t) = p^{n-1} \; t \; B^{k-1,n-1}_{p,q}(t) + (q^k p^{n-k-1} - q^{n-1} t) \; B^{k,n-1}_{p,q}(t).
$$

3.1. Degree elevation

$$
q^{n-k}p^k \; t \; B^{k,n}_{p,q}(t) = \left(1 - \frac{p^{k+1} \; [n-k]_{p,q}}{[n+1]_{p,q}}\right) B^{k+1,n+1}_{p,q}(t), \tag{12}
$$

$$
(p^n - p^k q^{n-k} \; t) \; B^{k,n}_{p,q}(t) = \left(\frac{p^k \; [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) B^{k,n+1}_{p,q}(t). \tag{13}
$$

Proof:

$$
\begin{aligned}
q^{n-k}p^k \; t \; B^{k,n}_{p,q}(t) &= \frac{1}{p^{\frac{n(n-1)}{2}}} q^{n-k}p^k \; t \left(\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} t^k \; (1-t)^{n-k}_{p,q} \right) \\
&= q^{n-k} \frac{[k+1]_{p,q}}{[n+1]_{p,q}} \frac{1}{p^{\frac{n(n-1)}{2}}} \left(\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{p,q} p^{\frac{(k+1)(k)}{2}} t^{k+1}(1-t)^{n-k}_{p,q} \right) \\
&= q^{n-k} \frac{[k+1]_{p,q}}{[n+1]_{p,q}} \; B^{k+1,n+1}_{p,q}(t).
\end{aligned}
$$

By some simple calculation, we have

$$q^{n-k}\frac{[k+1]_{p,q}}{[n+1]_{p,q}} = 1-\frac{p^{k+1}\,[n-k]_{p,q}}{[n+1]_{p,q}},$$

using this result, we get

$$q^{n-k}p^k\; t\; B_{p,q}^{k,n}(t) = \left(1-\frac{p^{k+1}\,[n-k]_{p,q}}{[n+1]_{p,q}}\right) B_{p,q}^{k+1,n+1}(t).$$

Similarly on considering,

$$\begin{aligned}(p^n-p^kq^{n-k}\;t)\;B_{p,q}^{k,n}(t) &= (p^n-p^kq^{n-k}\;t)\left(\frac{1}{p^{\frac{n(n-1)}{2}}}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}}t^k\;(1-t)_{p,q}^{n-k}\right)\\ &= \frac{(p^n-p^kq^{n-k}\;t)}{(p^{n-k}-q^{n-k}\;t)}\;\frac{1}{p^{\frac{n(n-1)}{2}}}\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\\ &\times\left(\begin{bmatrix} n+1\\ k\end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}}t^k\;(1-t)_{p,q}^{n+1-k}\right),\end{aligned}$$

finally we get

$$(p^n-p^kq^{n-k}\;t)\;B_{p,q}^{k,n}(t) = \left(\frac{p^k\,[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) B_{p,q}^{k,n+1}(t).$$

Theorem 3.2.

Each (p,q)-Bernstein function of degree n is a linear combination of two (p,q)-Bernstein functions of degree $n+1$.

$$B_{p,q}^{k,n}(t) = \left(\frac{p^{k-n}\,[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) B_{p,q}^{k,n+1}(t) + p^{-n}\left(1-\frac{p^{k+1}\,[n-k]_{p,q}}{[n+1]_{p,q}}\right) B_{p,q}^{k+1,n+1}(t). \tag{14}$$

Proof:

From equation (12) and (13), we have

$$B_{p,q}^{k,n}(t) = \left(\frac{p^{k-n}\,[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) B_{p,q}^{k,n+1}(t) + p^{-n}\left(1-\frac{p^{k+1}\,[n-k]_{p,q}}{[n+1]_{p,q}}\right) B_{p,q}^{k+1,n+1}(t).$$

For $n=3$, the blending functions are given by:

$$B_{p,q}^{0,3} = \frac{1}{p^3}\begin{bmatrix} 3\\ 0\end{bmatrix}_{p,q}\;(1-t)(p-qt)(p^2-q^2t),$$

$$B_{p,q}^{1,3} = \frac{1}{p^3} \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{p,q} t(1-t)(p-qt),$$

$$B_{p,q}^{2,3} = \frac{1}{p^3} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} pt^2(1-t),$$

$$B_{p,q}^{3,3} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{p,q} t^3.$$

We observed that the both sided end point interpolation property and partition of unity property always holds in case of (p, q)-Bernstein functions.
The de Casteljau algorithm describes how to subdivide a B$\acute{e}$zier curve, when a B$\acute{e}$zier curve is repeatedly subdivided, the collection of control polygons converge to the curve. Thus, the way of computing a B$\acute{e}$zier curve is to simply subdivide it an appropriate number of times and compute the control polygons.

4. (p, q)-Bernstein B$\acute{e}$zier curves:

Let us define the (p, q)-B$\acute{e}$zier curves of degree n using the (p, q)-analogues of the Bernstein functions as follows:

$$\mathbf{P}(t; p, q) = \sum_{i=0}^{n} \mathbf{P_i} \, B_{p,q}^{i,n}(t), \tag{15}$$

where $P_i \in R^3$ $(i = 0, 1, ..., n)$ and $p > q > 0$. P_i are control points. Joining up adjacent points P_i, $i = 0, 1, 2, ..., n$ to obtain a polygon which is called the control polygon of (p, q)-B$\acute{e}$zier curves.

4.1. Some basic properties of (p, q)-B$\acute{e}$zier curves.

Theorem 4.1.

From the definition, we can derive some basic properties of (p, q)-B$\acute{e}$zier curves:

1. (p, q)-B$\acute{e}$zier curves have geometric and affine invariance.
2. (p, q)-B$\acute{e}$zier curves lie inside the convex hull of its control polygon.
3. The end-point interpolation property: $\mathbf{P}(0; p, q) = \mathbf{P_0}$, $\mathbf{P}(1; p, q) = \mathbf{P_n}$.
4. Reducibility: when $p = 1$, formula 15 gives the q-B$\acute{e}$zier curves.

Proof:

These properties of (p, q)-B$\acute{e}$zier curves can be easily deduced from corresponding properties of the (p, q)-analogue of the Bernstein functions.

4.2. Degree elevation for (p, q)-Bézier curves

(p,q)-Bézier curves have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. Using the technique of degree elevation, we can increase the flexibility of a given curve.

$$\mathbf{P}(t;p,q) = \sum_{k=0}^{n} \mathbf{P_k}\ B_{p,q}^{k,n}(t),$$

$$\mathbf{P}(t;p,q) = \sum_{k=0}^{n+1} \mathbf{P'_k}\ B_{p,q}^{k+1,n}(t),$$

where

$$\mathbf{P_k}^* = p^n \mathbf{P'} = \left(1 - \frac{p^k\ [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_{k-1}} + p^k \left(\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_k}. \tag{16}$$

The statement above can be derived using the identities (12) and (13). Consider

$$p^n \mathbf{P}(t;p,q) = (p^n - p^k q^{n-k} t)\ \ \mathbf{P}(t;p,q) + p^k q^{n-k} t\ \ \mathbf{P}(t;p,q).$$

We obtain

$$\begin{aligned} p^n \mathbf{P}(t;p,q) = &\sum_{k=0}^{n} \left(p^k\ \ \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_k^0} B_{p,q}^{k,n+1}(t) \\ &+ \sum_{k=0}^{n} \left(1 - \frac{p^{k+1}\ [n-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_k^0} B_{p,q}^{k+1,n+1}(t). \end{aligned}$$

Now by shifting the limits, we have

$$\begin{aligned} p^n \mathbf{P}(t;p,q) = &\sum_{k=0}^{n+1} \left(p^k \frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_k^0} B_{p,q}^{k,n+1}(t) \\ &+ \sum_{k=0}^{n+1} \left(1 - \frac{p^k\ [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_{k-1}^0} B_{p,q}^{k,n+1}(t), \end{aligned}$$

where $\mathbf{P_{-1}^0}$is defined as the zero vector. Comparing coefficients on both side, we have

$$\mathbf{P_k}^* = p^n \mathbf{P'} = \left(1 - \frac{p^k\ [n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_{k-1}} + p^k \left(\frac{[n+1-k]_{p,q}}{[n+1]_{p,q}}\right) \mathbf{P_k},$$

where $k = 0, 1, 2, ..., n + 1$ and $\mathbf{P_{-1}} = \mathbf{P_{n+1}} = 0$.

When $p = 1$, formula (16) reduces to the degree evaluation formula of the q-B$\acute{e}$zier curves. Let $P = (P_0, P_1, ..., P_n)^T$ denote the vector of control points of the initial (p, q)-B$\acute{e}$zier curve of degree n, and $\mathbf{P^{(1)}} = (P_0^*, P_1^*, ..., P_{n+1}^*)$ the vector of control points of the degree elevated (p, q)-B$\acute{e}$zier curve of degree $n + 1$, then, we can represent the degree elevation procedure as:

$$\mathbf{P^{(1)}} = T_{n+1}\mathbf{P},$$

where

$$T_{n+1} = \frac{1}{[n+1]_{p,q}} \times \begin{bmatrix} [n+1]_{p,q} & 0 & \dots & 0 & 0 \\ [n+1]_{p,q} - p[n]_{p,q} & p[n]_{p,q} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & [n+1]_{p,q} - p^{n-1}[2]_{p,q} & p^{n-1}[2]_{p,q} & 0 \\ 0 & 0 & \dots & [n+1]_{p,q} - p^n[1]_{p,q} & p^n[1]_{p,q} \\ 0 & 0 & \dots & 0 & [n+1]_{p,q} \end{bmatrix}_{(n+2)\times(n+1).}$$

For any $l \in \mathbb{N}$, the vector of control points of the degree elevated (p, q)-B$\acute{e}$zier curves of degree $n+l$ is: $\mathbf{P^{(l)}} = T_{n+l}\, T_{n+2}, ..., T_{n+1}\mathbf{P}$. As $l \longrightarrow \infty$, the control polygon $\mathbf{P^{(l)}}$ converges to a (p, q)-B$\acute{e}$zier curve.

4.3. de Casteljau algorithm:

(p, q)-B$\acute{e}$zier curves of degree n can be written as two kinds of linear combination of two (p, q)-B$\acute{e}$zier curves of degree $n-1$, and we can get the two selectable algorithms to evaluate (p, q)-B$\acute{e}$zier curves. The algorithms can be expressed as:

Algorithm 1.

$$\begin{cases} \mathbf{P_i^0}(t; p, q) \equiv \mathbf{P_i^0} \equiv \mathbf{P_i} \quad i = 0, 1, 2, ..., n \\ \\ \mathbf{P_i^r}(t; p, q) = p^{r-1}t\, \mathbf{P_{i+1}^{r-1}}(t; p, q) + (q^k p^{r-i-1} - q^{r-1}t)\, \mathbf{P_i^{r-1}}(t; p, q) \\ r = 1, ..., n, \quad i = 0, 1, 2, ..., n - r., \end{cases} \tag{17}$$

or

$$\begin{cases} \mathbf{P_i^0}(t;p,q) \equiv \mathbf{P_i^0} \equiv \mathbf{P_i} \quad i=0,1,2,...,n \\ \mathbf{P_i^r}(t;p,q) = p^i q^{r-i-1} t \, \mathbf{P_{i+1}^{r-1}}(t;p,q) + (p^{r-1} - p^i q^{r-i-1} t) \, \mathbf{P_i^{r-1}}(t;p,q). \\ r=1,...,n, \quad i=0,1,2,...,n-r. \end{cases} \tag{18}$$

Then,

$$\mathbf{P}(t;p,q) = \sum_{i=0}^{n-1} \mathbf{P_i^1}(t;p,q) = ... = \sum \mathbf{P_i^r}(t;p,q) \; B_{p,q}^{i,n-r}(t) = ... = \mathbf{P_0^n} \; (t;p,q). \tag{19}$$

It is clear that the results can be obtained from Theorem (3.2). When $p = 1$, formula (17) and (18) recover the de Casteljau algorithms of classical q-B$\acute{e}$zier curves. Let $P^0 = (P_0, P_1, ..., P_n)^T$, $P^r = (P_0^r, P_1^r, ..., P_{n-r}^r)^T$, then, de Casteljau algorithm can be expressed as:

Algorithm 2.

$$\mathbf{P^r}(t;p,q) = M_r(t;p,q)...M_2(t;p,q)M_1(t;p,q)\mathbf{P^0}, \tag{20}$$

where $M_r(t;p,q)$ is a $(n-r+1) \times (n-r+2)$ matrix and

$M_r(t;p,q) =$

$$\begin{bmatrix} (p^{r-1} - q^{r-1}t) & p^{r-1}t & \dots & 0 & 0 \\ 0 & (qp^{r-2} - q^{r-1}t) & p^{r-1}t & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & (q^{n-r-1}p^{2r-n-2} - q^{r-1}t) & p^{r-1}t & 0 \\ 0 & 0 & \dots & (q^{n-r}p^{2r-n-1} - q^{r-1}t) & p^{r-1}t \end{bmatrix}$$

or

$M_r(t;p,q) =$

$$\begin{bmatrix} (p^{r-1} - q^{r-1}t) & q^{r-1}t & \dots & 0 & 0 \\ 0 & (p^{r-1} - pq^{r-2}t) & pq^{r-2}t & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & & & 0 \\ 0 & 0 & \dots & (p^{r-1} - p^{n-r}q^{-n-1}t) & p^{n-r}q^{-n-1}t \end{bmatrix}.$$

5. Tensor product (p, q)-Bernstein B$\acute{e}$zier surfaces on $[0, 1] \times [0, 1]$

We define a two-parameter family $\mathbf{P}(u, v)$ of tensor product surfaces of degree $m \times n$ as follows:

$$\mathbf{P}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{i,j}\;B^{i,m}_{p_1,q_1}(u)\;\;B^{j,n}_{p_2,q_2}(v),\;\;(u,v)\in[0,1]\times[0,1], \tag{21}$$

where $\mathbf{P}_{i,j}\in\mathbb{R}^3\;\;(i=0,1,...,m, j=0,1,...,n)$ and two real numbers $p_1>q_1>0,\;p_2>q_2>0$, $B^{i,m}_{p_1,q_1}(u),\;\;B^{j,n}_{p_2,q_2}(v)$ are (p,q)-analogue of Bernstein functions respectively with the parameter p_1,q_1 and p_2,q_2. We call the parameter surface tensor product (p,q)-B$\acute{e}$zier surface with degree $m\times n$. We refer to the $\mathbf{P}_{i,j}$ as the control points. By joining up adjacent points in the same row or column to obtain a net which is called the control net of tensor product (p,q)-B$\acute{e}$zier surface.

5.1. Properties

1. **Geometric invariance and affine invariance property:**

Since

$$\sum_{i=0}^{m}\sum_{j=0}^{n}B^{i,m}_{p_1,q_1}(u)\;\;B^{j,n}_{p_2,q_2}(v)=1, \tag{22}$$

$\mathbf{P}(u,v)$ is an affine combination of its control points.

2. **Convex hull property:** $\mathbf{P}(u,v)$ is a convex combination of $\mathbf{P}_{i,j}$ and lies in the convex hull of its control net.

3. **Isoparametric curves property:** The isoparametric curves $v=v^*$ and $u=u^*$ of a tensor product (p,q)-B$\acute{e}$zier surface are respectively the (p,q)-B$\acute{e}$zier curves of degree m and degree n, namely,

$$\mathbf{P}(u,v^*)=\sum_{i=0}^{m}\left(\sum_{j=0}^{n}\mathbf{P}_{i,j}\;B^{j,n}_{p_2,q_2}(v^*)\right)B^{i,m}_{p_1,q_1}(u)\;\;,\;\;u\in[0,1],$$

$$\mathbf{P}(u^*,v)=\sum_{j=0}^{n}\left(\sum_{i=0}^{m}\mathbf{P}_{i,j}\;B^{j,n}_{p_1,q_1}(u^*)\right)B^{i,m}_{p_2,q_2}(v)\;\;,\;\;v\in[0,1].$$

The boundary curves of $\mathbf{P}(u,v)$ are evaluated by $\mathbf{P}(u,0)$, $\mathbf{P}(u,1)$, $\mathbf{P}(0,v)$ and $\mathbf{P}(1,v)$.

4. **Corner point interpolation property:** The corner control net coincide with the four corners of the surface. Namely, $\mathbf{P}(0,0)=\mathbf{P}_{0,0}$, $\mathbf{P}(0,1)=\mathbf{P}_{0,n}$, $\mathbf{P}(1,0)=\mathbf{P}_{m,0}$, $\mathbf{P}(1,1)=\mathbf{P}_{m,n}$.

5. **Reducibility:** When $p_1=p_2=1$, formula (21) reduces to a tensor product q-B$\acute{e}$zier patch.

5.2. Degree elevation and de Casteljau algorithm

Let $\mathbf{P}(u,v)$ be a tensor product (p,q)-B$\acute{e}$zier surface of degree $m \times n$. As an example, let us take obtaining the same surface as a surface of degree $(m+1)\times(n+1)$. Hence we need to find new control points $\mathbf{P}^*_{i,j}$ such that

$$\begin{aligned}\mathbf{P}(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{i,j}\ B^{i,m}_{p_1,q_1}(u)\ \ B^{j,n}_{p_2,q_2}(v) \\ &= \sum_{i=0}^{m+1}\sum_{j=0}^{n+1}\mathbf{P^*}_{\mathbf{i,j}}\ B^{i,m+1}_{p_1,q_1}(u)\ \ B^{j,n+1}_{p_2,q_2}(v).\end{aligned} \tag{23}$$

Let $\alpha_i = 1 - \frac{p_1^i\ [m+1-i]_{p_1,q_1}}{[m+1]_{p_1,q_1}}, \quad \beta_j = 1 - \frac{p_2^j\ [n+1-j]_{p_2,q_2}}{[n+1]_{p_2,q_2}}$.

Then,

$$\mathbf{P}^*_{i,j} = \alpha_i\ \beta_j\ \mathbf{P}_{i-1,j-1} + \alpha_i\ (1-\beta_j)\ \mathbf{P}_{i-1,j} + (1-\alpha_i)\ (1-\beta_j)\ \mathbf{P}_{i,j}\ \ , \tag{24}$$

which can be written in matrix form as

$$\begin{bmatrix}1 - \frac{p_1^i\ [m+1-i]_{p_1,q_1}}{[m+1]_{p_1,q_1}} & \frac{p_1^i\ [m+1-i]_{p_1,q_1}}{[m+1]_{p_1,q_1}}\end{bmatrix}\begin{bmatrix}\mathbf{P}_{i-1,j-1} & \mathbf{P}_{i-1,j} \\ \mathbf{P}_{i,j-1} & \mathbf{P}_{i,j}\end{bmatrix}\begin{bmatrix}1 - \frac{p_2^j\ [n+1-j]_{p_2,q_2}}{[n+1]_{p_2,q_2}} \\ \frac{p_2^j\ [n+1-j]_{p_2,q_2}}{[n+1]_{p_2,q_2}}\end{bmatrix}.$$

The de Casteljau algorithms are also easily extended to evaluate points on a (p,q)-B$\acute{e}$zier surface. Given the control net $\mathbf{P}_{i,j} \in \mathbb{R}^3, i = 0,1,...,m,\ \ j = 0,1,...,n$, we have

$$\begin{cases}\mathbf{P^{0,0}_{i,j}}(u,v) \equiv \mathbf{P^{0,0}_{i,j}} \equiv \mathbf{P_{i,j}} \quad i = 0,1,2,...,m; \ \ j = 0,1,2,...,n. \\ \mathbf{P^{r,r}_{i,j}}(u,v) = \begin{bmatrix}(q_1^i p_1^{r-i-1} - q_1^{r-1}u) & p_1^{r-1}u\end{bmatrix}\begin{bmatrix}\mathbf{P}^{r-1,r-1}_{i,j} & \mathbf{P}^{r-1,r-1}_{i,j+1} \\ \mathbf{P}^{r-1,r-1}_{i+1,j} & \mathbf{P}^{r-1,r-1}_{i+1,j+1}\end{bmatrix}\begin{bmatrix}(q_2^j p_2^{r-j-1} - q_2^{r-1}v) \\ p_2^{r-1}v\end{bmatrix} \\ r = 1,...,k,\ k = \min(m,n) \quad i = 0,1,2...,m-r; \ \ j = 0,1,...,n-r\end{cases} \tag{25}$$

or

$$\begin{cases}\mathbf{P^{0,0}_{i,j}}(u,v) \equiv \mathbf{P^{0,0}_{i,j}} \equiv \mathbf{P_{i,j}} \quad i = 0,1,2,...,m; \ \ j = 0,1,2,...,n. \\ \mathbf{P^{r,r}_{i,j}}(u,v) = \begin{bmatrix}(p_1^{r-1} - p_1^i q_1^{r-i-1}u) & p_1^i q_1^{r-i-1}u\end{bmatrix}\begin{bmatrix}\mathbf{P}^{r-1,r-1}_{i,j} & \mathbf{P}^{r-1,r-1}_{i,j+1} \\ \mathbf{P}^{r-1,r-1}_{i+1,j} & \mathbf{P}^{r-1,r-1}_{i+1,j+1}\end{bmatrix}\begin{bmatrix}(p_2^{r-1} - p_2^j q_2^{r-j-1}v) \\ p_2^j q_2^{r-j-1}v\end{bmatrix}. \\ r = 1,...,k,\ k = \min(m,n) \quad i = 0,1,2,...,m-r; \ \ j = 0,1,...,n-r.\end{cases} \tag{26}$$

When $m = n$, one can directly use the algorithms above to get a point on the surface. When $m \neq n$, to get a point on the surface after k applications of formula (25) or (26), we perform formula (20) for the intermediate point $\mathbf{P}^{k,k}_{i,j}$.

Note: We get q-B$\acute{e}$zier curves and surfaces for $(u,v) \in [0,1]\times[0,1]$ when we set the parameter $p_1 = p_2 = 1$ as proved in [Disibuyuk and Oruc (2008)] and [Disibuyuk (2009)].

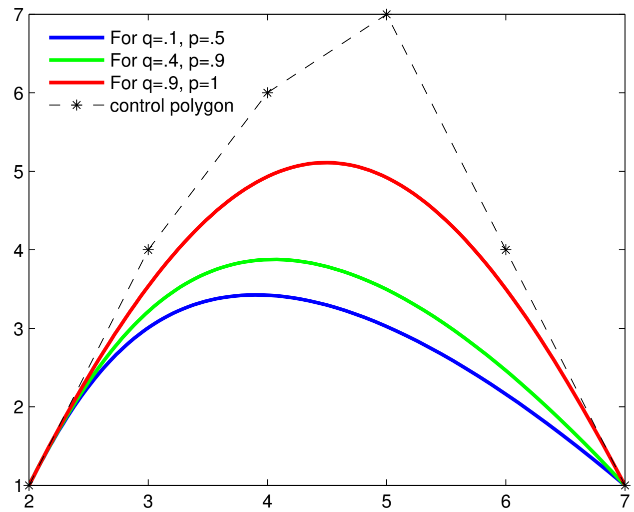

Figure 3. The effect of the shape of (p, q)-B$\acute{e}$zier curves.

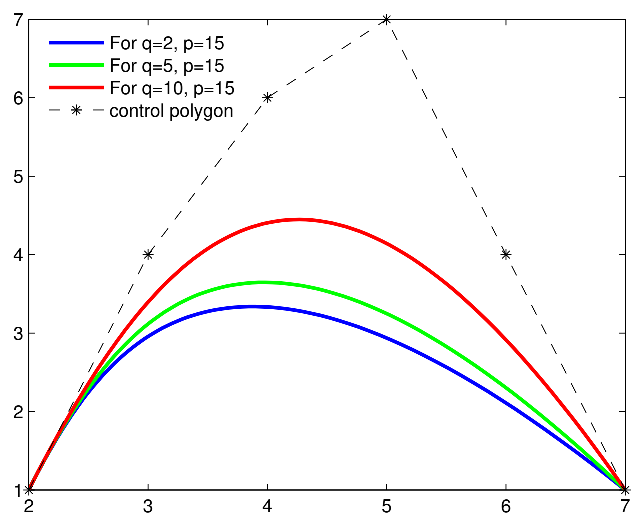

Figure 4. The effect of the shape of (p, q)-B$\acute{e}$zier curves.

6. Shape control of (p, q)-Bernstein curves

We have constructed (p, q)-Bernstein functions which holds both the end point interpolation property as shown in Figure 1 and 2. Parameter p and q have been used to control the shape of curves which can be observed through Figures 3-7.

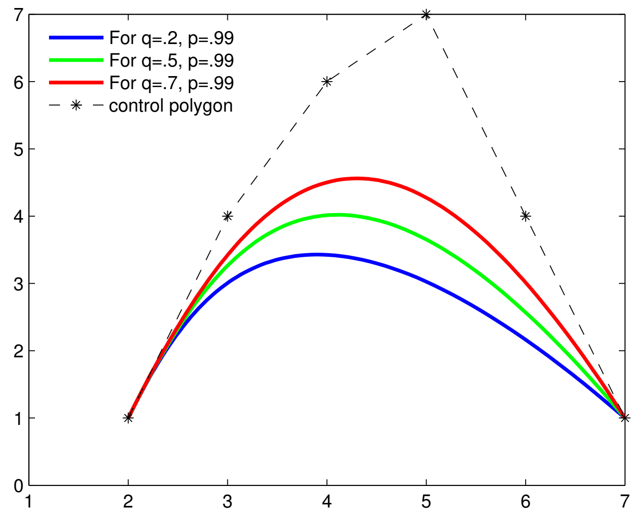

Figure 5. The effect of the shape of (p, q)-B$\acute{e}$zier curves.

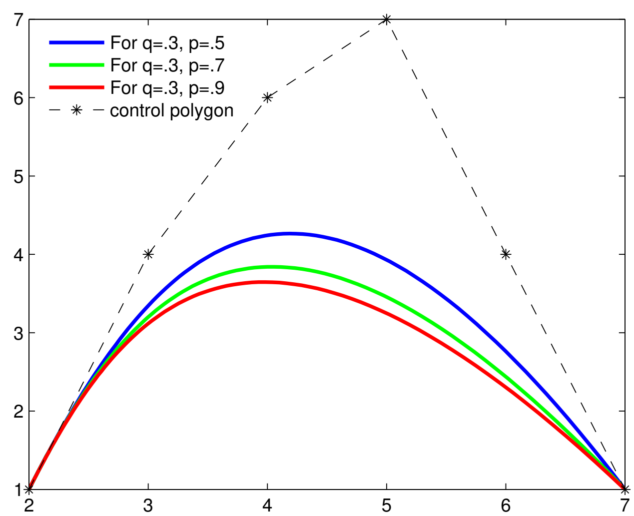

Figure 6. The effect of the shape of (p, q)-B$\acute{e}$zier curves.

7. Conclusion

In this paper, we have further studied various results for (p, q)-Bernstein basis function used in (p, q)-Bernstein operators (8). In comparison to q-B$\acute{e}$zier curves based on q-Bernstein polynomials, this generalization gives us more flexibility in controlling the shapes of curves.

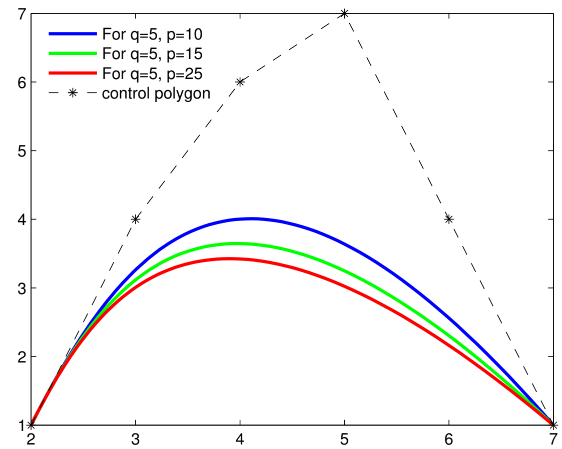

Figure 7. The effect of the shape of (p, q)-B$\acute{e}$zier curves.

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