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Simplifying Coefficients in a Family of Ordinary Differential Equations Related to the Generating Function of the Laguerre Polynomials

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Abstract

In the paper, by virtue of the Faà di Bruno formula, properties of the Bell polynomials of the second kind, and the Lah inversion formula, the author simplifies coefficients in a family of ordinary differential equations related to the generating function of the Laguerre polynomials.

Keywords: Simplifying; coefficient; Laguerre polynomial; generating function; Faà di Bruno formula; ordinary differential equation; Bell polynomial of the second kind; Lah inversion formula

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1. Introduction

In (Kim et al., 2016, Theorem 1), it was established inductively and recursively that the family of differential equations

$$F^{(n)}(t) = F(t) \sum_{i=n}^{2n} \frac{a_{i-n}(n,x)}{(1-t)^i}, \quad n \in \mathbb{N}, \tag{1}$$

has a solution

$$F(t) = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right), \tag{2}$$

where $a_0(n,x) = n!$, $a_n(n,x) = (-x)^n$,

$$a_j(n,x) = (-x)^j \sum_{i_j=0}^{n-j} \sum_{i_{j-1}=0}^{n-i_j-j} \cdots \sum_{i_1=0}^{n-i_j-\cdots-i_2-j} \langle n+j \rangle_{i_j}$$
$$\times \prod_{k=2}^{j} \langle n - i_j - \cdots - i_k - [j-(2k-2)] \rangle_{i_{k-1}} (n - i_j - \cdots - i_1 - j)!, \quad 2 \le j \le n-1, \tag{3}$$

the falling factorial

$$\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1, \\ 1, & n = 0, \end{cases}$$

and the function $F(t)$ in (2) can be used to generate the Laguerre polynomials $L_n(x)$ by

$$F(t) = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) t^n.$$

Hereafter, the expression (3) was employed in (Kim et al., 2016, Theorem 2).

It is not difficult to see that

(1) the expression (3) is too complicated to be remembered, understood, and computed easily;
(2) the original proof of (Kim et al., 2016, Theorem 1) is long and tedious.

In this paper, we will provide a nice and standard proof for (Kim et al., 2016, Theorem 1) and, more importantly, discover a simple, meaningful, and significant form for $a_i(n,x)$.

2. Main results

Our main results can be stated as the following theorem.

Theorem 2.1.

For $n \ge 0$, the function $F(t)$ defined by (2) and its derivatives satisfy

$$F^{(n)}(t) = \frac{n!}{(1-t)^n} \left[\sum_{k=0}^{n} (-1)^k \binom{n-1}{k-1} \frac{x^{k-1}}{k!} \frac{x-k(1-t)}{(1-t)^k} \right] F(t), \tag{4}$$

and

$$F(t)=\frac{n!(1-t)^n}{x^{n-1}[x-n(1-t)]}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n-1}{k-1}(1-t)^kF^{(k)}(t), \tag{5}$$

where $\binom{-1}{-1}=1$ and $\binom{k}{-1}=0$ if $k\ge0$.

Proof:

The famous Faà di Bruno formula reads that

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr), \tag{6}$$

where $n\ge0$ and the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge0$ are defined (Comtet, 1974, p. 134, Theorem A) and (Comtet, 1974, p. 139, Theorem C) by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The function $F(t)$ in (2) can be rearranged as

$$F(t)=\frac{e^x}{x}\frac{x}{1-t}\exp\left(-\frac{x}{1-t}\right).$$

Applying $u=h(t)=\frac{x}{1-t}$ and $f(u)=\frac{u}{e^u}$ to (6) gives

$$\begin{aligned}\frac{x}{e^x}F^{(n)}(t)&=\sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left(\frac{u}{e^u}\right)\mathrm{B}_{n,k}\left(\frac{1!x}{(1-t)^2},\frac{2!x}{(1-t)^3},\dots,\frac{(n-k+1)!x}{(1-t)^{n-k+2}}\right)\\&=\sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left(\frac{u}{e^u}\right)x^k\left(\frac{1}{1-t}\right)^{n+k}\mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!)\\&=\sum_{k=0}^{n}(-1)^k\frac{u-k}{e^u}x^k\left(\frac{1}{1-t}\right)^{n+k}\frac{n!}{k!}\binom{n-1}{k-1}\\&=\left(\frac{1}{1-t}\right)^n\exp\left(-\frac{x}{1-t}\right)\sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}x^k\left(\frac{x}{1-t}-k\right)\left(\frac{1}{1-t}\right)^k,\end{aligned}$$

where we used the identities

$$\mathrm{B}_{n,k}\bigl(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\bigr)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}),$$

and

$$\mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!)=\frac{n!}{k!}\binom{n-1}{k-1},$$

in (Comtet, 1974, p. 135) and (Qi, 2016, Remark 3.5). The formula (4) is thus proved.

The Lah inversion theorem in (Aigner, 1979, p. 96, Corollary 3.38 (iii)) and (Aigner, 2007, pp. 60–61, Exercise 2.9) reads that

$$(-1)^na_n=\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{k-1}b_k\quad\Longleftrightarrow\quad(-1)^nb_n=\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{k-1}a_k. \tag{7}$$

Combining (7) with (4) arrives at

$$\frac{x^{n-1}[x-n(1-t)]}{(1-t)^n}=\sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}\frac{(1-t)^k}{F(t)}F^{(k)}(t),$$

which can be rewritten as (5). The proof of Theorem 2.1 is complete. ■

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark.

The equations (4) and (5) can be regarded as inversion formulas each other.

Remark.

The equation (5) is a new one which cannot be obtained inductively and recursively, as was in Kim et al. (2016).

Remark.

The equation (4) can be rewritten as

$$\begin{aligned}F^{(n)}(t)&=\frac{F(t)}{(1-t)^n}\sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}\left[\frac{x^k}{(1-t)^k}-\frac{kx^{k-1}}{(1-t)^{k-1}}\right]\\&=\frac{F(t)}{(1-t)^n}\left[\sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}\frac{x^k}{(1-t)^k}-\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}\frac{kx^{k-1}}{(1-t)^{k-1}}\right]\\&=\frac{F(t)}{(1-t)^n}\left[\sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1}\frac{n!}{k!}\frac{x^k}{(1-t)^k}+\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{(n-1)!}{k!}\frac{(k+1)x^k}{(1-t)^k}\right]\\&=\frac{F(t)}{(1-t)^n}\left[\frac{(-1)^nx^n}{(1-t)^n}+\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\frac{n!}{k!}\frac{x^k}{(1-t)^k}\right]\\&=\frac{F(t)}{(1-t)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{n!}{k!}\frac{x^k}{(1-t)^k}.\end{aligned}\tag{8}$$

The equation (1) can be reformulated as

$$F^{(n)}(t)=F(t)\sum_{i=0}^{n}\frac{a_i(n,x)}{(1-t)^{n+i}},\quad n\in\mathbb{N}.\tag{9}$$

Remark.

The motivations in the papers Qi (2018a,b,c), Qi and Guo (2018, 2017), Qi et al. (2019a, 2018a, 2019b, 2018b,c), Qi and Zhao (2018), Zhao et al. (2018) are same as the one in this paper.

Remark.

This paper is a slightly modified version of the preprint Qi (2017).

4. Conclusions

Comparing two equalities (8) and (9) reveals that

$$a_k(n,x) = (-1)^k \binom{n}{k} \frac{n!}{k!} x^k,$$

for $n \geq k \geq 0$. This form for $a_k(n,x)$ is apparently simpler, more meaningful, and more significant than the one (3) obtained in (Kim et al., 2016, Theorem 1).

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