F–Geometric Mean Graphs

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Received: November 1, 2014; Accepted: June 26, 2015

Abstract:

In a study of traffic, the labelling problems in graph theory can be used by considering the crowd at every junction as the weights of a vertex and expected average traffic in each street as the weight of the corresponding edge. If we assume the expected traffic at each street as the arithmetic mean of the weight of the end vertices, that causes mean labelling of the graph. When we consider a geometric mean instead of arithmetic mean in a large population of a city, the rate of growth of traffic in each street will be more accurate. The geometric mean labelling of graphs have been defined in which the edge labels may be assigned by either flooring function or ceiling function. In this, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function. To avoid this confusion, we establish the F-geometric mean labelling on graphs by considering the edge labels obtained only from the flooring function. An F-Geometric mean labelling of a graph \( G \) with \( q \) edges, is an injective function from the vertex set of \( G \) to \( \{1, 2, 3, \ldots, q + 1\} \) such that the edge labels obtained from the flooring function of geometric mean of the vertex labels of the end vertices of each edge, are all distinct and the set of edge labels is \( \{1, 2, 3, \ldots, q\} \). A graph is said to be an F–Geometric mean graph if it admits an F–Geometric mean labelling. In this paper, we study the F-geometric meanness of the graphs such as cycle, star graph, complete graph, comb, ladder, triangular ladder, middle graph of path, the graphs obtained from duplicating arbitrary vertex by a vertex as well as arbitrary edge by an edge in the cycle and subdivision of comb and star graph.

Keywords: Labelling, F-Geometric mean labelling, F-Geometric mean graph

MSC 2010 No.: 05C78
I. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. For notations and terminology, the readers are referred to the book of Harary (1972). For a detailed survey on graph labelling we refer the reader to the book of Gallian (2014).

A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. $K_{1,n}$ is called a star graph and it is denoted by $nS$. $\mathcal{G}K_2$ is the graph obtained from $G$ by attaching a new pendant vertex to each vertex of $G$. Let $G_1$ and $G_2$ be any two graphs with $p_1$ and $p_2$ vertices respectively. Then the Cartesian product $G_1 \times G_2$ has $p_1p_2$ vertices which are $\{(u, v) : u \in G_1, v \in G_2\}$. The edges are obtained as follows: $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and $v_1$ and $v_2$ are adjacent in $G_2$ or $u_1$ and $u_2$ are adjacent in $G_1$ and $v_1 = v_2$. The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is

$$\{e, e' : e_1, e_2 \in E(G) \text{ and } e_1 \text{ and } e_2 \text{ are adjacent edges of } G\}$$

$$\cup \{v e : v \in V(G), e \in E(G) \text{ and } e \text{ is incident with } v\}.$$

Let $G$ be a graph and let $v$ be a vertex of $G$. The duplicate graph $D(G, v')$ of $G$ is the graph whose vertex set is $V(G) \cup \{v'\}$ and edge set is

$$E(G) \cup \{v' : x : x \text{ is the vertex adjacent to } v \text{ in } G\}.$$

Let $G$ be a graph and let $e = uv$ be an edge of $G$. The duplicate graph $D(G, e' = u'v')$ of $G$ is the graph whose vertex set is $V(G) \cup \{u', v'\}$ and edge set is $E(G) \cup \{u'x, v'y : x \text{ and } y \text{ are the vertices adjacent with } u \text{ and } v \text{ in } G \text{ respectively}\}$. The triangular ladder $TL_n$, $n \geq 2$ is a graph obtained by completing the ladder $P_2 \times P_n$ by adding the edges $u_iv_{i+1}$ for $1 \leq i \leq n-1$. For a graph $G$ the graph $S(G)$ is obtained by subdividing each edge of $G$ by a vertex. An arbitrary subdivision of a graph $G$ is a graph obtained from $G$ by a sequence of elementary subdivisions forming edges into paths through new vertices of degree 2.

The study of graceful graphs and graceful labelling methods was first introduced by Rosa (1967). The concept of mean labelling was first introduced and developed by Somasundaram and Ponraj (2003). Further, it was studied by Vasuki et al. (2009, 2010, 2011). Vaidya and Lekha Bijukumar (2010) discussed the mean labelling of some graph operations. Mohanaselvi and Hemalatha (2014) discussed the super geometric mean labelling of various classes of some graphs.
A function $f$ is called an $F$-Geometric mean labelling of a graph $G(V,E)$ if $f: V(G) \to \{1, 2, 3, ..., q+1\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, 3, ..., q\}$ defined as

$$f^*(uv) = \left\lfloor\sqrt{f(u)f(v)}\right\rfloor,$$

is bijective. A graph that admits an $F$–Geometric mean labelling is called an $F$–Geometric mean graph.

Somasundaram et al. (2011) defined the geometric mean labelling as follows: A graph $G = (V,E)$ with $p$ vertices and $q$ edges is said to be a geometric mean graph if it possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, ..., q+1$ in such a way that when each edge $e = uv$ is labelled with $f(uv) = \left\lfloor\sqrt{f(u)f(v)}\right\rfloor$ or $\left\lceil\sqrt{f(u)f(v)}\right\rceil$, then the edge labels are distinct.

Somasundaram et al. (2012) have given the geometric mean labelling of the graph $C_5 \cup C_7$ as in the Figure 1.

From the above figure, for the edge $uv$, they have used flooring function $\left\lfloor\sqrt{f(u)f(v)}\right\rfloor$ and for the edge $vw$, they have used ceiling function $\left\lceil\sqrt{f(v)f(w)}\right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the flooring function $\left\lfloor\sqrt{f(u)f(v)}\right\rfloor$ for our discussion. Based on our definition, the $F$-Geometric mean labelling of the same graph $C_5 \cup C_7$ is given in Figure 2.
In this paper, we study the $F$-Geometric meanness of the graphs, namely, cycle $C_n$ for $n \geq 3$, the star graph $S_n$ for $n \leq 3$, the complete graph $K_n$ for $n \leq 3$, the comb $P_n \circ K_1$ for any positive integer $n$, the ladder $P_2 \times P_n$ for any positive integer $n$, the middle graph $M(P_n)$, the graphs obtained by duplicating an arbitrary vertex as well as arbitrary edge in the cycle $C_n$, the triangular ladder $TL_n$ for $n \geq 2$, the graph $S(P_n \circ K_1)$ and the arbitrary subdivision of $S_3$.

2. Main Results

To study the $F$-geometric meanness, some of the standard graphs, and graphs obtained from some graph operations are taken for discussion.

Lemma 2.1.

Let $G$ be a graph. If $|V(G)| > |E(G)| + 1$, then $G$ does not admit an $F$-Geometric mean labelling.

Proof:

If $|V(G)| > |E(G)| + 1$, then the vertex labelling will not be injective and hence the result follows.

Theorem 2.2.

The union of any two trees is not an $F$-Geometric mean graph.

Proof:

Let $G$ be the union of two trees $S$ and $T$. Then if $|V(G)| = |V(S)| + |V(T)|$ and $|E(G)| = |E(S)| + |E(T)| = |V(S)| + |V(T)| - 2$ then by Lemma 2.1, the result follows.

Corollary 2.3.

Any forest is not an $F$-Geometric mean graph.

Theorem 2.4.

Every cycle is an $F$-Geometric mean graph.

Proof:

Let $v_1, v_2, ..., v_n$ be the vertices of the cycle $C_n$.

We define
as follows:

\[ f(v_i) = \begin{cases} 
  i, & 1 \leq i \leq \left\lfloor \sqrt{n+1} \right\rfloor - 1, \\
  i+1, & \left\lfloor \sqrt{n+1} \right\rfloor \leq i \leq n. 
\end{cases} \]

The induced edges labelling is as follows:

\[ f^*(v_iv_{i+1}) = \begin{cases} 
  i, & 1 \leq i \leq \left\lfloor \sqrt{n+1} \right\rfloor - 1, \\
  i+1, & \left\lfloor \sqrt{n+1} \right\rfloor \leq i \leq n-1, 
\end{cases} \]

and

\[ f^*(v_1v_n) = \left\lceil \sqrt{n+1} \right\rceil. \]

Hence, \( f \) is an \( F \)-Geometric mean labelling of the cycle \( C_n \). Thus the cycle \( C_n \) is an \( F \)-Geometric mean graph.

An \( F \)-Geometric mean labelling of \( C_6 \) is shown in Figure 3.

![Figure 3. An F-Geometric mean labelling of C_6 and its edge labelling](image-url)

**Theorem 2.5.**

The star graph \( S_n \) is an \( F \)-Geometric mean graph if and only if \( n \leq 3 \).

**Proof:**

The number of vertices and edges of \( S_n \) are \( n+1 \) and \( n \) respectively. If \( f \) is an \( F \)-geometric mean labelling of \( S_n \), then it is a bijective function from \( V(S_n) \) to \( \{1,2,3,...,n+1\} \). As we
have to label 1 for an edge, the vertex labels of its pair of adjacent vertices are either 1 and 2 or 1 and 3. So, the central vertex of \(S_n\) is labelled as either 1 or 2 or 3. 1 cannot be a label for the central vertex in the case of \(n \geq 2\), since two of the pendant vertices of \(S_n\) are to be labelled as 2 and 3. When \(n \geq 3\), 2 cannot be the label for the central vertex, since two of its pendant vertices having the labels 3 and 4. When \(n \geq 4\), the pendant vertices are labelled to be 4 and 5 if the label of central vertex is 3.

The \(F\)-Geometric mean labelling of \(S_n\), \(n \leq 3\) are given in Figure 4.

**Theorem 2.6.**

Every complete graph \(K_n, n \geq 4\) is not an \(F\)-Geometric mean graph.

**Proof:**

To get the edge label \(q\), \(q\) and \(q+1\) should be the vertex labels for two of the vertices of \(K_n\), say \(x\) and \(y\). Also to obtain the edge label 1, 1 is to be a vertex label of a vertex of \(K_n\), say \(v\). Since \(q = nC_2\) in \(K_n\) and \(q + 1 < (n - 1)^2\) for \(n \geq 4\), the edge labels of the edges \(vx\) and \(vy\) are one and the same. Hence \(K_n\) is not an \(F\)-Geometric mean graph. While \(n = 2\) and 3, the \(F\)-geometric mean labelling of \(K_n\) are given in Figure 5.

**Theorem 2.7.**

Every comb graph is an \(F\)-Geometric mean graph.
Proof:

Let $G = P_n \circ K_1$ be a comb graph for any positive integer $n$ having $2n$ vertices and $2n - 1$ edges. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ and $v_i$ be the pendant vertices attached at each $u_i$, for $1 \leq i \leq n$. Then, the edge set of $G$ is $\{u_i u_{i+1} ; 1 \leq i \leq n-1\} \cup \{u_i v_i ; 1 \leq i \leq n\}$.

We define $f : V(G) \to \{1, 2, 3, \ldots, 2n\}$ as follows:

$$f(u_i) = 2i, \text{ for } 1 \leq i \leq n \text{ and } f(v_i) = 2i - 1, \text{ for } 1 \leq i \leq n.$$ 

The induced edge labelling is as follows:

$$f^*(u_i u_{i+1}) = 2i, \text{ for } 1 \leq i \leq n-1 \text{ and } f^*(u_i v_i) = 2i - 1, \text{ for } 1 \leq i \leq n.$$ 

Hence, $f$ is an $F$-Geometric mean labelling of the comb $P_n \circ K_1$. Thus, the comb $P_n \circ K_1$ is an $F$-Geometric mean graph for any positive integer $n$.

An $F$-Geometric mean labelling of $P_n \circ K_1$ is shown in Figure 6.

![Figure 6](image_url)

**Theorem 2.8.**

Every ladder graph is an $F$-Geometric mean graph.

Proof:

Let $G = P_2 \times P_n$ be a ladder graph for any positive integer $n$ having $2n$ vertices and $3n - 2$ edges. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of $G$. Then the edge set of $G$ is

$$\{u_i u_{i+1}, v_i v_{i+1} ; 1 \leq i \leq n-1\} \cup \{u_i v_i ; 1 \leq i \leq n\}.$$ 

We define $f : V(G) \to \{1, 2, 3, \ldots, 3n-1\}$ as follows:

$$f(u_i) = 3i - 1, \text{ for } 1 \leq i \leq n \text{ and } f(v_i) = 3i - 2, \text{ for } 1 \leq i \leq n.$$
The induced edge labelling is as follows:

\[ f^*(u_1u_{i+1}) = 3i, \text{ for } 1 \leq i \leq n-1, \quad f^*(v_i v_{i+1}) = 3i-1, \text{ for } 1 \leq i \leq n-1, \text{ and } \]
\[ f^*(u_i v_i) = 3i-2, \text{ for } 1 \leq i \leq n. \]

Hence, \( f \) is an F-Geometric mean labelling of the ladder \( P_2 \times P_n \). Thus, the ladder \( P_2 \times P_n \) is an F-Geometric mean graph for any positive integer \( n \).

An F-Geometric mean labelling of \( P_2 \times P_6 \) is shown in Figure 7.

![Figure 7. An F-Geometric mean labelling of \( P_2 \times P_6 \) and its edge labelling](image-url)

**Theorem 2.9.**

The middle graph of a path is an F-Geometric mean graph.

**Proof:**

Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(P_n) = \{e_i = v_i v_{i+1}; 1 \leq i \leq n-1\} \) be the vertex set and edge set of the path \( P_n \). Then,

\[ V\left(M\left(P_n\right)\right) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\} \text{ and } \]
\[ E\left(M\left(P_n\right)\right) = \{v_i e_i, e_i v_{i+1}; 1 \leq i \leq n-1\} \cup \{e_i e_{i+1}; 1 \leq i \leq n-2\}. \]

We define \( f : V\left(M\left(P_n\right)\right) \rightarrow \{1, 2, 3, \ldots, 3n-3\} \) as follows:

\[ f(v_i) = 3i-2, \text{ for } 1 \leq i \leq n-1, \quad f(v_n) = 3n-3 \text{ and } f(e_i) = 3i-1, \text{ for } 1 \leq i \leq n-1. \]

The induced edge labelling is as follows:

\[ f^*(v_i e_i) = 3i-2, \text{ for } 1 \leq i \leq n-1, \quad f^*(e_i v_{i+1}) = 3i-1, \text{ for } 1 \leq i \leq n-1 \]
and \( f^*(e_i e_{i+1}) = 3i, \text{ for } 1 \leq i \leq n-2. \)
Hence, \( f \) is an \( F \)-Geometric mean labelling of the middle graph \( M\left( P_n \right) \). Thus, the middle graph \( M\left( P_n \right) \) is an \( F \)-Geometric mean graph.

An \( F \)-Geometric mean labelling of \( M\left( P_6 \right) \) is shown in Figure 8.

![Figure 8. An \( F \)-Geometric mean labelling of \( M\left( P_6 \right) \) and its edge labelling](image)

**Theorem 2.10.**

For any vertex \( v \) of the cycle \( C_n \), the duplicate graph \( D\left( C_n, v' \right) \) is an \( F \)-Geometric mean graph, for \( n \geq 3 \).

**Proof:**

Let \( v_1, v_2, ..., v_n \) be the vertices of the cycle \( C_n \) and let \( v = v_1 \) and its duplicated vertex is \( v'_1 \).

**Case (i).** \( n \geq 5 \)

We define \( f : V\left( D\left( C_n, v' \right) \right) \rightarrow \{1, 2, 3, ..., n+3\} \) as follows:

\[
f(v_1) = n - 1, \quad f(v'_1) = n + 1, \quad f(v_2) = n + 2, \quad f(v_3) = n + 3, \quad \text{and} \quad f(v_i) = \begin{cases} i - 3, & 4 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor + 2, \\ i - 2, & \left\lfloor \frac{n+3}{2} \right\rfloor + 3 \leq i \leq n. \end{cases}
\]

The induced edge labelling is as follows:

\[
f^*(v_1v_2) = n, \quad f^*(v_1v_n) = n - 2, \quad f^*(v_1v_2) = n + 1, \quad f^*(v_1v_n) = n - 1, \quad f^*(v_2v_3) = n + 2, \quad f^*(v_3v_4) = \left\lfloor \sqrt{n+3} \right\rfloor \quad \text{and} \quad f^*(v_iv_{i+1}) = \begin{cases} i - 3, & 4 \leq i \leq \left\lfloor \frac{n+3}{2} \right\rfloor + 2, \\ i - 2, & \left\lfloor \frac{n+3}{2} \right\rfloor + 3 \leq i \leq n - 1. \end{cases}
\]

Hence, \( f \) is an \( F \)-Geometric mean labelling of the graph \( D\left( C_n, v' \right) \).

**Case (ii).** \( n = 3, 4 \)

The \( F \)-Geometric mean labelling of \( D\left( C_3, v'_1 \right) \) and \( D\left( C_4, v'_1 \right) \) are given in Figure 9.
An $F$-geometric mean labelling of the graph $G$ obtained by duplicating the vertex $v_1$ of the cycle $C_8$ is shown in Figure 10.

**Theorem 2.11.**

For any edge $e$ of the cycle $C_n$, the duplicate graph $D(C_n, e')$ is an $F$-Geometric mean graph, for $n \geq 3$.

**Proof:**

Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$ and let $e = v_1v_2$ and its duplicated edge is $e' = v_1'v_2'$. 
Case (i). \( n \geq 6 \)

We define \( f : V(D(C_n, e')) \rightarrow \{1,2,3,\ldots,n+4\} \) as follows:

\[
\begin{align*}
f(v_1) &= n - 1, \quad f(v'_1) = n + 1, \quad f(v_2) = n + 2, \quad f(v'_2) = n + 3, \\
f(v_3) &= n + 4 \quad \text{and} \quad f(v_i) = \begin{cases} i - 3, & 4 \leq i \leq \left\lfloor \sqrt{n+4} \right\rfloor + 2, \\
-3, & \left\lfloor \sqrt{n+4} \right\rfloor + 3 \leq i \leq n.
\end{cases}
\end{align*}
\]

The induced edge labelling is as follows:

\[
\begin{align*}
f^*(v_1v_2) &= n, \quad f^*(v_1v'_n) = n - 2, \quad f^*(v_1v'_n) = n - 1, \quad f^*(v'_1v_2) = n + 1, \\
f^*(v'_2v_3) &= n + 3, \quad f^*(v_2v'_3) = n + 2, \quad f^*(v_3v_4) = \left\lfloor \frac{n+4}{\sqrt{n+4}} \right\rfloor \\
\text{and} \quad f^*(v_{i+1}) &= \begin{cases} i - 3, & 4 \leq i \leq \left\lfloor \frac{n+4}{\sqrt{n+4}} \right\rfloor + 2, \\
-3, & \left\lfloor \frac{n+4}{\sqrt{n+4}} \right\rfloor + 3 \leq i \leq n - 1.
\end{cases}
\end{align*}
\]

Hence, \( f \) is an F-Geometric mean labelling of the graph \( D(C_n, e') \).

Case (ii). \( n = 3,4,5 \)

The F-Geometric mean labelling of \( D\left(C_3, v'_1v_2\right) \), \( D\left(C_4, v'_1v'_2\right) \) and \( D\left(C_5, v'_1v'_2\right) \) are given in Figure 11.

Figure 11. An F-Geometric mean labelling of \( D\left(C_3, v'_1v_2\right) \), \( D\left(C_4, v'_1v'_2\right) \) and \( D\left(C_5, v'_1v'_2\right) \) and its edge labelling

An F-geometric mean labelling of the graph \( G \) obtained by duplicating an edge \( v_1v_2 \) of the cycle \( C_9 \) is shown in Figure 12.
Theorem 2.12.

The triangular ladder $TL_n$ is an $F$-Geometric mean graph, for $n \geq 2$.

Proof:

Let $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ be the vertex set of $TL_n$ and let $\{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{u_iv_i; 1 \leq i \leq n\}$ be the edge set of $TL_n$. Then $TL_n$ have $2n$ vertices and $4n - 3$ edges.

We define $f : V(TL_n) \rightarrow \{1, 2, 3, ..., 4n - 2\}$ as follows:

\[
f(u_i) = 4i - 1, \text{ for } 1 \leq i \leq n - 1, \ f(u_n) = 4n - 2 \text{ and } f(v_i) = 4i - 3, \text{ for } 1 \leq i \leq n.
\]

The induced edge labelling is as follows:

\[
f^*(u_iu_{i+1}) = 4i, \text{ for } 1 \leq i \leq n - 1, \ f^*(u_iv_i) = 4i - 3, \text{ for } 1 \leq i \leq n,
\]

\[
f^*(v_iv_{i+1}) = 4i - 1, \text{ for } 1 \leq i \leq n - 1 \text{ and } f^*(v_iv_{i+1}) = 4i - 2, \text{ for } 1 \leq i \leq n - 1.
\]

Hence, $f$ is an $F$-Geometric mean labelling of the $TL_n$. Thus the triangular ladder $TL_n$ is an $F$-Geometric mean graph, for $n \geq 2$.

An $F$-Geometric mean labelling of $TL_8$ is shown in Figure 13.
Theorem 2.13.

$S(P_n \circ K_2)$ is an $F$-Geometric mean graph, for $n \geq 2$.

**Proof:**

Let $V(P_n \circ K_2) = \{u_i, v_i ; 1 \leq i \leq n\}$ and $E(P_n \circ K_1) = \{u_iu_{i+1} ; 1 \leq i \leq n-1\} \cup \{u_iv_i ; 1 \leq i \leq n\}$. Let $x_i$ be the vertex which divides the edge $u_iv_i$, for $1 \leq i \leq n$ and $y_i$ be the vertex which divides the edge $u_{i+1}u_i$, for $1 \leq i \leq n-1$. Then $V(S(P_n \circ K_2)) = \{u_i, v_i, x_i, y_j ; 1 \leq i \leq n, 1 \leq j \leq n-1\}$ and $E(S(P_n \circ K_2)) = \{u_ix_i, y_jx_j ; 1 \leq i \leq n\} \cup \{u_iy_i, y_ju_{i+1} ; 1 \leq i \leq n-1\}$.

We define $f: V(S(P_n \circ K_2)) \rightarrow \{1,2,3,...,4n-1\}$ as follows:

$$f(u_i) = 4i - 1, \text{ for } 1 \leq i \leq n, \quad f(y_i) = 4i + 1, \text{ for } 1 \leq i \leq n-1,$$

$$f(x_i) = 4i - 2, \text{ for } 1 \leq i \leq n \quad \text{and} \quad f(v_i) = \begin{cases} 1, & i = 1, \\ 4i - 4, & 2 \leq i \leq n. \end{cases}$$

The induced edge labelling is as follows:

$$f^*(u_ix_i) = 4i - 1, \text{ for } 1 \leq i \leq n-1, \quad f^*(y_ju_{i+1}) = 4i + 1, \text{ for } 1 \leq i \leq n-1,$$

$$f^*(u_iy_i) = 4i - 2, \text{ for } 1 \leq i \leq n \quad \text{and} \quad f^*(v_ix_i) = \begin{cases} 1, & i = 1, \\ 4i - 4, & 2 \leq i \leq n. \end{cases}$$

Hence, $f$ is an F-Geometric mean labelling of $S(P_n \circ K_2)$. Thus, the graph $S(P_n \circ K_2)$ is an F-Geometric mean graph, for $n \geq 2$.

An F-Geometric mean labelling of $S(P_n \circ K_2)$ is shown in Figure 14.
**Theorem 2.14.**

Any arbitrary subdivision of $S_3$ is an $F$-Geometric mean graph.

**Proof:**

Let $v_0,v_1,v_2,v_3$ be the vertices of $S_3$ in which $v_0$ is the central vertex and $v_1,v_2$ and $v_3$ are the pendant vertices of $S_3$. Let the edges $v_0v_1,v_0v_2$ and $v_0v_3$ of $S_3$ be subdivided by $p_1,p_2$ and $p_3$ number of new vertices respectively. Let $G$ be a graph of arbitrary subdivision of $S_3$.

Let $v_0,v_1^{(1)},v_2^{(1)},v_3^{(1)},...,v_p^{(1)}+1(=v_1)$, $v_0,v_1^{(2)},v_2^{(2)},v_3^{(2)},...,v_p^{(2)}+1(=v_2)$ and $v_0,v_1^{(3)},v_2^{(3)},v_3^{(3)},...,v_p^{(3)}+1(=v_3)$ be the vertices of $G$ and $v_0=v_0^{(i)}$, for $1 \leq i \leq 3$.

Let $e_j^{(i)}=v_j^{(i)}v_{j+1}^{(i)}$, $1 \leq j \leq p_i +1$ and $1 \leq i \leq 3$ be the edges of $G$ and $G$ has $p_1+p_2+p_3+4$ vertices and $p_1+p_2+p_3+3$ edges with $p_1 \leq p_2 \leq p_3$.

We define $f:V(G) \rightarrow \{1,2,3,...,p_1+p_2+p_3+4\}$ as follows:

$$f(v_0) = p_1+p_2+3, \quad f(v_i^{(1)}) = p_1+p_2+4-2i, \quad \text{for} \ 1 \leq i \leq p_1+1,$$

$$f(v_i^{(2)}) = \begin{cases} p_1+p_2+3-2i, & 1 \leq i \leq p_1 +1, \\ p_2 +2-i, & p_1 +2 \leq i \leq p_2 +1 \quad \text{and} \quad p_1 \neq p_2 \end{cases},$$

and
$$f(v_i^{(3)}) = p_1+p_2+3+i, \quad \text{for} \ 1 \leq i \leq p_1+1.$$  

The induced edge labelling is as follows:

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = p_1+p_2+2-2i, \quad \text{for} \ 1 \leq i \leq p_1,$$

$$f^*(v_i^{(2)}v_{i+1}^{(2)}) = \begin{cases} p_1+p_2+1-2i, & 1 \leq i \leq p_1, \\ p_2 +1-i, & p_1 +1 \leq i \leq p_2 \quad \text{and} \quad p_1 \neq p_2, \end{cases},$$

$$f^*(v_i^{(3)}v_{i+1}^{(3)}) = p_1+p_2+3+i, \quad \text{for} \ 1 \leq i \leq p_1, \quad f^*(v_0v_1^{(1)}) = p_1+p_2+2,$$

$$f^*(v_0v_1^{(2)}) = p_1+p_2+1 \quad \text{and} \quad f^*(v_0v_3^{(1)}) = p_1+p_2+3.$$

Hence, $f$ is an $F$-Geometric mean labelling of $G$. Thus, the arbitrary subdivision of $S_3$ is an $F$-Geometric mean graph.

An $F$-Geometric mean labelling of $G$ with $p_1=6, p_2=9$ and $p_3=10$ is as shown in Figure 15.
3. Conclusion

In this paper, we analysed the F-Geometric meanness of some standard graphs. We propose the following open problems to the readers for further research work.

Open Problem 1.

Find a sub graph of a graph in which the graph is not an $F$-Geometric mean graph.

Open Problem 2.

Find a necessary condition for a graph to be an $F$-Geometric mean graph.

By Theorem 2.6, we observe that $G + e$ is not necessarily an $F$-geometric mean graph when $G$ is an $F$-geometric mean graph and $e$ is an additional edge. Also from Theorem 2.2, $G - e$ is not necessarily an $F$-geometric mean graph when $e$ is a cut edge and $G$ is an $F$-geometric mean graph. So it is possible to discuss the remaining case.

Figure 15. An $F$-Geometric mean labelling of arbitrary subdivision of $S_3$ and its edge labelling
Open Problem 3.

For a non-cut edge $e$, characterize the $F$-geometric mean graph $G$ in which $G - e$ is also an $F$-geometric mean graph.

Acknowledgement

The authors would like to thank the editor and three anonymous reviewers for helpful suggestions which improved the presentation of the paper.

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